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Characterizations of some function spaces associated with Bloch type spaces on the unit ball of C^n

Jineng Dai[1] and Baobin Wang[2*]

*Correspondence:
wbb1818@126.com
[2]School of Mathematics and Statistics, South Central University for Nationalities, Wuhan, 430074, China
Full list of author information is available at the end of the article

Abstract
Motivated by characterizations of Bloch type spaces, we define some related function spaces and characterize them in this paper. Our results reveal the reason in theory that some equivalent characterizations of the Bloch type space B^α require extra conditions for α.

MSC: 32A18

Keywords: Bloch type space; unit ball

1 Introduction

Let B_n be the unit ball of the complex Euclidean space C^n and ∂B_n be the unit sphere. The class of all holomorphic functions on B_n is denoted by $H(B_n)$. For $0 \leq \alpha < \infty$, let H_α^∞ be the space of holomorphic functions f on B_n satisfying $\sup_{z\in B_n}(1-|z|^2)^\alpha|f(z)| < \infty$. When $\alpha = 0$, we write H^∞ for H_0^∞.

For $f \in H(B_n)$, its complex gradient is defined by

$$\nabla f(z) = \left(\frac{\partial f}{\partial z_1}(z), \dots, \frac{\partial f}{\partial z_n}(z)\right).$$

As introduced by Timoney in [1], the Bloch space on the unit ball B_n is the space of all $f \in H(B_n)$ such that $\sup_{z\in B_n} Q_f(z) < \infty$, where

$$Q_f(z) = \sup\left\{\frac{(1-|z|^2)|\langle\nabla f(z), \overline{w}\rangle|}{\sqrt{(1-|z|^2)|w|^2 + |\langle z, w\rangle|^2}} : 0 \neq w \in C^n\right\}.$$

Let $0 < \alpha < \infty$. The Bloch type space B^α consists of functions $f \in H(B_n)$ satisfying

$$\sup_{z\in B_n}\left(1-|z|^2\right)^\alpha\left|\nabla f(z)\right| < \infty.$$

It is well known that $B^\alpha = H_{\alpha-1}^\infty$ for $\alpha > 1$. Moreover, when $\alpha > 1/2$, a function $f \in B^\alpha$ if and only if

$$\sup_{z\in B_n}\left(1-|z|^2\right)^{\alpha-1}Q_f(z) < \infty. \tag{1}$$

But it is not true for the case $0 < \alpha \leq 1/2$ in the setting of several complex variables. For example (see [2]), when $0 < \alpha < 1/2$, $f \in B^\alpha$ if and only if $\sup_{z\in B_n}(1-|z|^2)^{\alpha-1}G_f(z) < \infty$, where

$$G_f(z) = \sup\left\{\frac{(1-|z|^2)|\langle\nabla f(z), \overline{w}\rangle|}{\sqrt{(1-|z|^2)^{2\alpha}|w|^2 + |\langle z, w\rangle|^2}} : 0 \neq w \in C^n\right\}.$$

Thus an interesting question arises naturally: what is the property for a holomorphic function f on B_n satisfying (1) when $0 < \alpha \leq 1/2$. Below we define the space

$$T_\alpha = \left\{f \in H(B_n) : \sup_{z\in B_n}\left(1-|z|^2\right)^{\alpha-1}Q_f(z) < \infty\right\}.$$

In Section 2 of this paper, we prove that $f \in T_{1/2}$ if and only if the directional derivatives of f in the directions perpendicular to the radial direction are uniformly bounded. In particular, T_α is a trivial space consisting of constants for $\alpha < 1/2$.

In 1986 Holland and Walsh [3] gave another characterization for the Bloch space on the unit disc D, namely, f belongs to the Bloch space if and only if

$$\sup_{\substack{z,w\in D\\ z\neq w}}\left(1-|z|^2\right)^{1/2}\left(1-|w|^2\right)^{1/2}\frac{|f(z)-f(w)|}{|z-w|} < \infty.$$

Ren and Tu [4] extended the above form to the unit ball B_n. Zhao [5] generalized these results as follows.

Theorem A *Let* $0 < \alpha \leq 2$. *Let* λ *be any real number satisfying the following properties*:
(1) $0 \leq \lambda \leq \alpha$ *if* $0 < \alpha < 1$;
(2) $0 < \lambda < 1$ *if* $\alpha = 1$;
(3) $\alpha - 1 \leq \lambda \leq 1$ *if* $1 < \alpha \leq 2$.
Then a holomorphic function f *on* B_n *is in* B^α *if and only if*

$$\sup_{\substack{z,w\in B_n\\ z\neq w}}\left(1-|z|^2\right)^{\lambda}\left(1-|w|^2\right)^{\alpha-\lambda}\frac{|f(z)-f(w)|}{|z-w|} < \infty. \quad (2)$$

Zhao gave some examples showing that the conditions on α and λ in Theorem A cannot be improved.

Motivated by Theorem A, we denote the following function space:

$$S_{\alpha,\lambda} = \left\{f \in H(B_n) : \sup_{\substack{z,w\in B_n\\ z\neq w}}\left(1-|z|^2\right)^{\lambda}\left(1-|w|^2\right)^{\alpha-\lambda}\frac{|f(z)-f(w)|}{|z-w|} < \infty\right\},$$

where α and λ are real numbers. Our purpose is to characterize the space $S_{\alpha,\lambda}$ for α and λ without satisfying those conditions in Theorem A. If $\lambda < 0$ or $\lambda > \alpha$, by the maximum modulus principle, it is easy to see that $S_{\alpha,\lambda}$ consists of constants. So we always assume $0 \leq \lambda \leq \alpha$. In Section 3 we show $S_{\alpha,\lambda} \subset B^\alpha$ when α and λ do not satisfy the conditions of Theorem A. More explicitly, $S_{\alpha,\lambda}$ coincides with the bounded space H^∞, or the Bloch type space $B^{\lambda+1}$ or $B^{\alpha-\lambda+1}$ in terms of different numbers α and λ. Our results reveal in theory instead of examples that the conditions on α and λ in Theorem A cannot be improved.

Throughout this paper, constants are denoted by C, and they are positive finite quantities and not necessarily the same in each occurrence.

2 Characterizations of T_α

Theorem 2.1 *The following statements are equivalent*:

(i) $f \in T_{1/2}$;

(ii) *There exists a constant* $C > 0$ *such that*

$$\left|\left\langle \nabla f(z), \overline{\zeta}\right\rangle\right| \le C \tag{3}$$

for all $z \in B_n$ *and* $\zeta \in \partial B_n$ *with* $\langle z, \zeta\rangle = 0$;

(iii) *For all* $1 \le i, j \le n$,

$$\sup_{z\in B_n}\left|\overline{z_i}\frac{\partial f}{\partial z_j}(z) - \overline{z_j}\frac{\partial f}{\partial z_i}(z)\right| < \infty. \tag{4}$$

Proof (i) ⇔ (ii): For $z \in B_n$ and $\zeta \in \partial B_n$ with $\langle z, \zeta\rangle = 0$, we have

$$\left(1-|z|^2\right)^{-1/2} Q_f(z) \ge \frac{(1-|z|^2)^{1/2}|\langle \nabla f(z), \overline{\zeta}\rangle|}{\sqrt{(1-|z|^2)|\zeta|^2 + |\langle z, \zeta\rangle|^2}} = \left|\left\langle \nabla f(z), \overline{\zeta}\right\rangle\right|. \tag{5}$$

This shows that (i) ⇒ (ii).

For the converse, if (ii) holds, we have $(1-|z|^2)^{1/2}|\langle \nabla f(z), \overline{z}\rangle| \le C$ for all $z \in B_n$ (see [6]). When $1/2 \le |z| < 1$ and $0 \ne w \in C^n$, by the projection theorem, there exists $\zeta \in \partial B^n$ such that $\langle \zeta, z\rangle = 0$ and

$$w = w_1 z + w_2 \zeta,$$

where $w_1 = \langle w, z\rangle / |z|^2$ and $|w|^2 = |w_1|^2|z|^2 + |w_2|^2$. Thus,

$$\begin{aligned}\left|\left\langle \nabla f(z), \overline{w}\right\rangle\right| &\le \left|w_1\left\langle \nabla f(z), \overline{z}\right\rangle\right| + \left|w_2\left\langle \nabla f(z), \overline{\zeta}\right\rangle\right| \\ &\le C\left|\langle z, w\rangle\right|\left(1-|z|^2\right)^{-1/2} + C|w|.\end{aligned}$$

It follows that

$$\frac{(1-|z|^2)^{1/2}|\langle \nabla f(z), \overline{w}\rangle|}{\sqrt{(1-|z|^2)|w|^2 + |\langle z, w\rangle|^2}} \le C$$

for all $1/2 \le |z| < 1$ and $0 \ne w \in C^n$. On the other hand, for $0 \le |z| < 1/2$, we have

$$\frac{(1-|z|^2)^{1/2}|\langle \nabla f(z), \overline{w}\rangle|}{\sqrt{(1-|z|^2)|w|^2 + |\langle z, w\rangle|^2}} \le \frac{(1-|z|^2)^{1/2}|\langle \nabla f(z), \overline{w}\rangle|}{\sqrt{(1-|z|^2)|w|^2}} = \left|\nabla f(z)\right| \le C.$$

This proves (i).

(ii) ⇔ (iii): Without loss of generality, we only need to show that

$$\sup_{z\in B_N}\left|\overline{z_2}\frac{\partial f}{\partial z_1}(z)-\overline{z_1}\frac{\partial f}{\partial z_2}(z)\right|<\infty.$$

If $z_1 = z_2 = 0$, there is nothing to prove. If $|z_1|^2 + |z_2|^2 \neq 0$, put

$$\zeta=\left(\frac{\overline{z_2}}{\sqrt{|z_1|^2+|z_2|^2}},\frac{-\overline{z_1}}{\sqrt{|z_1|^2+|z_2|^2}},0,\ldots,0\right).$$

Obviously, $\zeta \in \partial B_n$ and $\langle z,\zeta\rangle = 0$. Therefore,

$$\left|\overline{z_2}\frac{\partial f}{\partial z_1}(z)-\overline{z_1}\frac{\partial f}{\partial z_2}(z)\right|=\sqrt{|z_1|^2+|z_2|^2}\left|\left\langle\nabla f(z),\overline{\zeta}\right\rangle\right|\le C.$$

Conversely, suppose (iii) holds. When $|z| \le 1/2$, it is clear that (3) holds. For $|z| = \sqrt{|z_1|^2+\cdots+|z_n|^2} > 1/2$, there exists z_i $(1 \le i \le n)$ such that $|z_i| > 1/(2\sqrt{n})$. We may assume that $|z_1| > 1/(2\sqrt{n})$. Let $V = \{w \in C^n : \langle z,w\rangle = 0\}$. Then V is a subspace of C^n and a basis of V is $\{v_1,\ldots,v_{n-1}\}$, where

$$v_i=(-\overline{z_{i+1}},\underbrace{0,\ldots 0}_{i-1},\overline{z_1},\underbrace{0,\ldots,0}_{n-(i+1)}),\quad i=1,\ldots,n-1. \tag{6}$$

Therefore, for $\zeta = (\zeta_1,\ldots,\zeta_n) \in \partial B_n$ with $\langle z,\zeta\rangle = 0$, there exist scalars $k_1,\ldots,k_{n-1}$ such that ζ is expressed as a linear combination of $v_1,\ldots,v_{n-1}$ in only one way. That is,

$$\zeta=k_1v_1+k_2v_2+\cdots+k_{n-1}v_{n-1}. \tag{7}$$

By (6) and (7), we get

$$\zeta_1=-k_1\overline{z_2}-k_2\overline{z_3}-\cdots-k_{n-1}\overline{z_n}$$

and

$$\zeta_{i+1}=k_i\overline{z_1},\quad i=1,\ldots,n-1. \tag{8}$$

Note that $|\zeta| = 1$ and $|z_1| > 1/(2\sqrt{n})$. Hence, it follows from (8) that

$$|k_i|=\frac{|\zeta_{i+1}|}{|z_1|}<2\sqrt{n},\quad i=1,\ldots,n-1. \tag{9}$$

Thus we have

$$\left\langle\nabla f(z),\overline{\zeta}\right\rangle=\sum_{i=1}^{n-1}k_i\left\langle\nabla f(z),\overline{v_i}\right\rangle=\sum_{i=1}^{n-1}k_i\left(-\overline{z_{i+1}}\frac{\partial f}{\partial z_1}(z)+\overline{z_1}\frac{\partial f}{\partial z_{i+1}}(z)\right).$$

The desired result follows from (4) and (9). This finishes the proof of Theorem 2.1. □

From Theorem 2.1 and the result of [6], we see that $T_{1/2} \subset B^{1/2}$. Meanwhile, it is evident that $B^\alpha \subset T_{1/2}$ for $0 < \alpha < 1/2$. Below we give two examples to show that these inclusions are strict.

Example 1 Let

$$f(z_1,z_2)=(1-z_1)^{1/2}.$$

Note that $|z_1|^2+|z_2|^2<1$ for $z=(z_1,z_2)\in B_2$. Then

$$\left|\overline{z_2}\frac{\partial f}{\partial z_1}(z)-\overline{z_1}\frac{\partial f}{\partial z_2}(z)\right|=\frac{|z_2|}{2|1-z_1|^{1/2}}<\frac{(1-|z_1|^2)^{1/2}}{2(1-|z_1|)^{1/2}}<\frac{\sqrt{2}}{2}.$$

On the other hand, when $0<\alpha<1/2$ and $z=(r,0)\to(1,0)$ $(0<r<1)$,

$$\left(1-|z|^2\right)^{\alpha}\left|\nabla f(z)\right|=\frac{(1-|z|^2)^{\alpha}}{2|1-z_1|^{1/2}}=\frac{(1-r^2)^{\alpha}}{2(1-r)^{1/2}}\to\infty.$$

Therefore $f\in T_{1/2}$ by Theorem 2.1, but $f\overline{\in}B^{\alpha}$ $(0<\alpha<1/2)$.

Example 2 Let

$$g(z_1,z_2)=z_2\log(1-z_1).$$

Then, for $z=(z_1,z_2)\in B_2$,

$$\begin{aligned}\left(1-|z|^2\right)^{1/2}\left|\nabla g(z)\right|&\le\left(1-|z|^2\right)^{1/2}\left(\left|\frac{z_2}{1-z_1}\right|+\left|\log(1-z_1)\right|\right)\\&\le\left(1-|z|^2\right)^{1/2}\left(\frac{(1-|z_1|^2)^{1/2}}{1-|z_1|}+\left|\log(1-z_1)\right|\right)\le C.\end{aligned}$$

Meanwhile, let $z=(r,0)\to(1,0)$ $(0<r<1)$, we have

$$\left|\overline{z_2}\frac{\partial g}{\partial z_1}(z)-\overline{z_1}\frac{\partial g}{\partial z_2}(z)\right|=-r\log(1-r)\to\infty.$$

Thus $g\overline{\in}T_{1/2}$ but $g\in B^{1/2}$.

Lemma 2.1 *Let $n>1$, $1\le i,j\le n$ and $i\ne j$. Suppose that $f\in H(B_n)$, $g\in H(B_n)$ and*

$$\lim_{|z|\to1}\left|\overline{z_i}f(z)-\overline{z_j}g(z)\right|=0.$$

Then $f\equiv0$ and $g\equiv0$.

Proof We may assume $i=1$, $j=2$ without loss of generality. Let $h(z)=\overline{z_1}f(z)-\overline{z_2}g(z)$. For each fixed $\zeta=(\zeta_1,\ldots,\zeta_n)\in\partial B_n$, define the slice function $h_\zeta(\lambda)=h(\lambda\zeta)$ on the unit disk $D=\{\lambda:|\lambda|<1\}$. Then

$$\lim_{|\lambda|\to1}\left|\overline{\lambda\zeta_1}f(\lambda\zeta_1,\ldots,\lambda\zeta_n)-\overline{\lambda\zeta_2}g(\lambda\zeta_1,\ldots,\lambda\zeta_n)\right|=0.$$

That is,

$$\lim_{|\lambda|\to1}\left|\overline{\zeta_1}f(\lambda\zeta_1,\ldots,\lambda\zeta_n)-\overline{\zeta_2}g(\lambda\zeta_1,\ldots,\lambda\zeta_n)\right|=0.$$

Since the function $\overline{\zeta_1}f(\lambda\zeta_1,\dots,\lambda\zeta_n) - \overline{\zeta_2}g(\lambda\zeta_1,\dots,\lambda\zeta_n)$ is holomorphic on the unit disk $|\lambda| < 1$, by the maximum modulus principle of one complex variable, it follows that

$$\overline{\zeta_1}f(\lambda\zeta_1,\dots,\lambda\zeta_n) - \overline{\zeta_2}g(\lambda\zeta_1,\dots,\lambda\zeta_n) = 0$$

for all $\lambda \in D$. Therefore, for any $\zeta \in \partial B_n$ and $\lambda \in D$,

$$h(\lambda\zeta) = \overline{\lambda\zeta_1}f(\lambda\zeta_1,\dots,\lambda\zeta_n) - \overline{\lambda\zeta_2}g(\lambda\zeta_1,\dots,\lambda\zeta_n) = 0.$$

When $0 \neq z \in B_n$, let $\lambda = |z|$ and $\zeta = z/|z|$. Then we get

$$h(z) = h(\lambda\zeta) = \overline{z_1}f(z) - \overline{z_2}g(z) = 0.$$

Hence, for all $z \in B_n$,

$$\overline{z_1}f(z) = \overline{z_2}g(z).$$

Since f and g are holomorphic on B_n, we can conclude $f(z) = g(z) \equiv 0$. The proof is finished. □

An immediate consequence of Lemma 2.1 is the following theorem.

Theorem 2.2 *Let $n > 1$ and $\alpha < 1/2$. If $f \in T_\alpha$, then f is constant.*

Proof By (5) it follows that $|\langle\nabla f(z),\overline{\zeta}\rangle| \leq C(1-|z|)^{1/2-\alpha}$ for $z \in B_n$ and $\zeta \in \partial B_n$ with $\langle z,\zeta\rangle = 0$. Since $\alpha < 1/2$, we get $\lim_{|z|\to 1}|\langle\nabla f(z),\overline{\zeta}\rangle| = 0$. Using a similar argument as in the proof of Theorem 2.1, we have $\lim_{|z|\to 1}|\overline{z_j}\frac{\partial f}{\partial z_i}(z) - \overline{z_i}\frac{\partial f}{\partial z_j}(z)| = 0$ for all $1 \leq i,j \leq n$. Thus the desired result follows from Lemma 2.1. □

3 Characterizations of $S_{\alpha,\lambda}$

In the section we will characterize $S_{\alpha,\lambda}$ explicitly for real numbers α and λ in several cases. For this, we need the following lemma which plays an important role in the proof of Theorem 3.1.

Lemma 3.1 *Let $\alpha > 1$. Let λ be any real number satisfying the following properties*:

(1) $0 < \lambda < \alpha - 1$ *if* $1 < \alpha \leq 2$;
(2) $0 < \lambda \leq \alpha/2$ *if* $\alpha > 2$.

Let

$$H(x,y) = \frac{x^\lambda y^{\alpha-\lambda}}{y-x}\int_x^y \frac{d\tau}{\tau^{\lambda+1}}.$$

Then there exists a constant $C > 0$ such that $H(x,y) \leq C$ for any x and y satisfying $0 < x,y \leq 1$ and $x \neq y$.

Proof Let $t = x/y$. Then $t \in (0,1)\cup(1,\infty)$, and

$$H(x,y) = H(ty,y) = \frac{t^\lambda y^{\alpha-1}}{(1-t)}\int_{ty}^y \frac{d\tau}{\tau^{\lambda+1}}.$$

Let $s = \tau/y$, we have

$$H(x,y) = \frac{t^\lambda y^{\alpha-(\lambda+1)}}{(1-t)} \int_t^1 \frac{ds}{s^{\lambda+1}} = y^{\alpha-(\lambda+1)} G(t),$$

where

$$G(t) = \frac{t^\lambda}{(1-t)} \int_t^1 \frac{ds}{s^{\lambda+1}} = \frac{1-t^\lambda}{\lambda(1-t)}.$$

It is evident that $G(t)$ is continuous on $(0,1) \cup (1,\infty)$, and

$$\lim_{t\to 0} G(t) = \frac{1}{\lambda}, \qquad \lim_{t\to 1} G(t) = 1.$$

For the case (1), since $0 < \lambda < 1$, we get

$$\lim_{t\to\infty} G(t) = 0.$$

Noticing that $y^{\alpha-(\lambda+1)} \leq 1$ for $y \in (0,1]$, we conclude that $H(x,y)$ is bounded in this case. For the case (2), we easily see that $0 < \lambda < \alpha - 1$. Write

$$H(x,y) = H(x,x/t) = \frac{x^{\alpha-(\lambda+1)}}{t^{\alpha-(\lambda+1)}} G(t).$$

It is clear that

$$\lim_{t\to\infty} \frac{G(t)}{t^{\alpha-(\lambda+1)}} = \lim_{t\to\infty} \frac{t^{2\lambda-\alpha}}{\lambda}.$$

The above limit is $1/\lambda$ for $\lambda = \alpha/2$ and 0 for $0 < \lambda < \alpha/2$. Therefore $H(x,y)$ is also bounded in the case (2) since $x^{\alpha-(\lambda+1)} \leq 1$ for $x \in (0,1]$. The proof is complete. □

Theorem 3.1 *Let* $\alpha > 1$. *Let* λ *be any real number satisfying the following properties*:
(1) $0 < \lambda < \alpha - 1$ *if* $1 < \alpha \leq 2$;
(2) $0 < \lambda \leq \alpha/2$ *if* $\alpha > 2$.
Then $S_{\alpha,\lambda} = B^{\lambda+1}$.

Proof Let $f \in B^{\lambda+1}$. For any $z, w \in B_n$, since

$$\begin{aligned} f(z) - f(w) &= \int_0^1 \frac{d[f(tz+(1-t)w)]}{dt}\,dt \\ &= \sum_{k=1}^n (z_k - w_k) \int_0^1 \frac{\partial f}{\partial z_k}\bigl(tz+(1-t)w\bigr)\,dt, \end{aligned}$$

we get

$$\begin{aligned} \bigl|f(z)-f(w)\bigr| &\leq C|z-w| \int_0^1 \bigl|(\nabla f)\bigl(tz+(1-t)w\bigr)\bigr|\,dt \\ &\leq C|z-w| \int_0^1 \frac{dt}{(1-|tz+(1-t)w|)^{\lambda+1}} \end{aligned}$$

$$\begin{aligned}&\leq C|z-w|\int_0^1 \frac{dt}{(1-t|z|-(1-t)|w|)^{\lambda+1}}\\&= C|z-w|\int_0^1 \frac{dt}{[t(1-|z|)+(1-t)(1-|w|)]^{\lambda+1}}.\end{aligned}$$

If $|z| = |w|$, noting that $\lambda + 1 < \alpha$, we get

$$\begin{aligned}\int_0^1 \frac{dt}{[t(1-|z|)+(1-t)(1-|w|)]^{\lambda+1}} &= \frac{1}{(1-|z|)^{\lambda+1}} \leq \frac{C}{(1-|z|^2)^{\lambda+1}}\\&\leq \frac{C}{(1-|z|^2)^{\lambda}(1-|w|^2)^{\alpha-\lambda}}.\end{aligned}$$

If $|z| \neq |w|$, let $\tau = t(1-|z|) + (1-t)(1-|w|)$. By Lemma 3.1, we have

$$\begin{aligned}\int_0^1 \frac{dt}{[t(1-|z|)+(1-t)(1-|w|)]^{\lambda+1}} &= \frac{1}{(1-|z|)-(1-|w|)}\int_{1-|w|}^{1-|z|} \frac{dt}{\tau^{\lambda+1}}\\&\leq \frac{C}{(1-|z|^2)^{\lambda}(1-|w|^2)^{\alpha-\lambda}}.\end{aligned}$$

Therefore,

$$\bigl|f(z)-f(w)\bigr| \leq \frac{C|z-w|}{(1-|z|^2)^{\lambda}(1-|w|^2)^{\alpha-\lambda}},$$

which shows that $f \in S_{\alpha,\lambda}$.

Conversely, if $f \in S_{\alpha,\lambda}$, it follows that

$$\sup_{\substack{z\in B_n\\ z\neq 0}} \bigl(1-|z|^2\bigr)^{\lambda} \frac{|f(z)-f(0)|}{|z|} < \infty.$$

Thus we get

$$\sup_{z\in B_n} \bigl(1-|z|^2\bigr)^{\lambda} \bigl|f(z)\bigr| < \infty,$$

namely, $f \in H_\lambda^\infty$ and so $f \in B^{\lambda+1}$. This finishes the proof of the theorem. □

Theorem 3.2 *Let* $\alpha > 1$. *Let* λ *be any real number satisfying the following properties*:

(1) $1 < \lambda < \alpha$ *if* $1 < \alpha \leq 2$;

(2) $\alpha/2 < \lambda < \alpha$ *if* $\alpha > 2$.

Then $S_{\alpha,\lambda} = B^{\alpha-\lambda+1}$.

Proof If $1 < \lambda < \alpha$ for $1 < \alpha \leq 2$, then $0 < \alpha - \lambda < \alpha - 1$. If $\alpha/2 < \lambda < \alpha$ for $\alpha > 2$, then $0 < \alpha - \lambda < \alpha/2$. Applying Theorem 3.1, we immediately conclude that $S_{\alpha,\lambda} = B^{\alpha-\lambda+1}$. □

For any point $w \in B_n - \{0\}$, we recall that the bi-holomorphic mapping φ_w of B_n, which interchanges the points 0 and w, is defined by

$$\varphi_w(z) = \frac{w - P_w(z) - s_w Q_w(z)}{1 - \langle z, w\rangle}, \quad z \in B_n,$$

where $s_w = \sqrt{1-|w|^2}$, $P_w(z) = \frac{\langle z,w\rangle}{|w|^2}w$ and $Q_w(z) = z - P_w(z)$. When $w = 0$, let $\varphi_w(z) = -z$. The pseudo-hyperbolic distance between w and z is denoted by $\rho(w,z) = |\varphi_w(z)|$.

Theorem 3.3 *Let* $\alpha \geq 1$. *Then* $S_{\alpha,\lambda} = H^\infty$ *for* $\lambda = 0$ *or* $\lambda = \alpha$.

Proof It suffices to prove for $\alpha \geq 1$ and $\lambda = 0$. If $f \in S_{\alpha,0}$, that is,

$$\sup_{\substack{z,w\in B_n\\ z\neq w}} \left(1-|w|^2\right)^\alpha \frac{|f(z)-f(w)|}{|z-w|} < \infty. \tag{10}$$

Then

$$\sup_{\substack{z\in B_n\\ z\neq 0}} \frac{|f(z)-f(0)|}{|z|} < \infty,$$

which implies that $f \in H^\infty$.

Conversely, assume $f \in H^\infty$. Then we have (see Lemma 1 in [7])

$$\left|f(z)-f(w)\right| \leq C\rho(z,w) \tag{11}$$

for all $z, w \in B_n$. On the other hand,

$$\left|w - P_w(z) - s_w Q_w(z)\right|^2 = |z-w|^2 + \left|\langle z,w\rangle\right|^2 - |z|^2|w|^2 \leq |z-w|^2,$$

and so

$$\rho(w,z) \leq \frac{|z-w|}{|1-\langle z,w\rangle|} \leq \frac{|z-w|}{1-|w|}. \tag{12}$$

By (11) and (12), we get (10) since $\alpha \geq 1$. Thus $f \in S_{\alpha,0}$. The proof is complete. □

Remark 3.1 We easily see that $\lambda + 1 < \alpha$ in Theorem 3.1, and $\lambda > 1$ in Theorem 3.2. Combining with Theorem 3.3, we conclude that $S_{\alpha,\lambda} \subset B^\alpha$ and the inclusion is strict for real numbers α and λ which do not satisfy the conditions of Theorem A.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

BW proposed the problem. JD and BW together finished the proof. All authors read and approved the final manuscript.

Author details

[1] Department of Mathematics, Wuhan University of Technology, Wuhan, 430070, China. [2] School of Mathematics and Statistics, South Central University for Nationalities, Wuhan, 430074, China.

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