Two-weight norm inequalities for commutators of potential type integral operators

Wenming Li

College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang, 050016, Hebei, PR China

Received 25 January 2005
Available online 20 March 2006
Submitted by L. Grafakos

Abstract

We give a condition which is sufficient for the two-weight \((p, q)\) inequalities for commutators of potential type integral operators.

© 2005 Elsevier Inc. All rights reserved.

Keywords: Potential type integral operator; Riesz potential; Commutator; Two-weight inequality

1. Introduction

For a non-negative, locally integrable function \(\Phi\) on \(\mathbb{R}^n\), define the potential type operator \(T_\Phi\) by

\[
T_\Phi f(x) = \int_{\mathbb{R}^n} \Phi(x - y) f(y) \, dy.
\]

Although the basic example is provided by the Riesz potentials or fractional integrals \(I_\alpha\), defined by the kernel \(\Phi(x) = |x|^{\alpha-n}, 0 < \alpha < n\), there are other important examples such as the Bessel potentials. They are denoted by \(J_{\beta,\lambda}\), \(\beta, \lambda > 0\), and the kernel \(\Phi(x) = K_{\beta,\lambda}(x)\) is best defined by means of its Fourier transform \(\widehat{K}_{\beta,\lambda}(\xi) = (\lambda^2 + |\xi|^2)^{-\beta/2}\).
Now assume that the kernel $\Phi$ satisfies the following weak growth condition: there are constants $\delta, c > 0$, $0 \leq \varepsilon < 1$ with the property that for all $k \in \mathbb{Z}$,

$$\sup_{2^k < |x| \leq 2^{k+1}} \Phi(x) \leq \frac{c}{2^{kn}} \int_{\delta(1-\varepsilon)2^k < |y| \leq 2\delta(1+\varepsilon)2^k} \Phi(y) \, dy. \quad (1)$$

Associated to any kernel $\Phi$, we denote by $\tilde{\Phi}$ the positive function defined for $t \geq 0$,

$$\tilde{\Phi}(t) = \int_{|z|<t} \Phi(z) \, dz.$$

The study of weighted inequalities for the potential type operators is relevant to many applications in partial differential equations and quantum mechanics. For the weighted inequalities for Riesz potentials $I_\alpha$ and their commutators with $BMO$ functions, we refer the readers to [3,5,9,10] and the related papers for further information. For the two-weight problem of general potential operator $T_\Phi$ with $\Phi$ satisfying the condition (1), in [7] Pérez gave some sufficient conditions on weights $(u,v)$ such that the strong type $(p,q)$, $1 < p \leq q < \infty$, inequality

$$\left( \int_{\mathbb{R}^n} \left( \frac{\left| T_\Phi f(y) \right| |u(y)|^q}{\lambda} \, dy \right)^{1/q} \right)^{1/p} \leq C \left( \int_{\mathbb{R}^n} \left( \frac{\left| f(y) \right| |v(y)|^p}{\lambda} \, dy \right)^{1/p} \right)^{1/q}$$

holds for appropriate $f$. His results sharpen and unify the results in [2,11].

Let $1 < p \leq q < \infty$, $b \in BMO$, and $\Phi$ satisfy (1), the higher order commutators of $T_\Phi$ are defined by

$$T^{b,m}_\Phi f(x) = \int_{\mathbb{R}^n} (b(x) - b(y))^m \Phi(x-y) f(y) \, dy, \quad (2)$$

where $m = 0, 1, 2, \ldots$.

In this paper, using the technique developed by Pérez [7], we give a sufficient condition in terms of Orlicz bumps for the two-weight strong type $(p,q)$ inequalities for the commutators $T^{b,m}_\Phi$.

2. Preliminaries and main results

We will need the following facts about Orlicz spaces, for further information see [1]. A function $B : [0, \infty) \rightarrow [0, \infty)$ is a Young function if it is convex and increasing, and if $B(0) = 0$ and $B(t) \rightarrow \infty$ as $t \rightarrow \infty$. A Young function $B$ is doubling if there exists a positive constant $C$ such that $B(2t) \leq C B(t)$ for all $t > 0$. If $A, B$ are two Young functions, we write $A(t) \approx B(t)$ if there are constants $c_1, c_2 > 0$ with $c_1 A(t) \leq B(t) \leq c_2 A(t)$ for $t > c$. Given a Young function $B$, define the mean Luxemburg norm of $f$ on a cube $Q$ by

$$\| f \|_{B,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q B \left( \frac{|f(y)|}{\lambda} \right) \, dy \leq 1 \right\}.$$

For a given Young function $B$, there exists a complementary Young function $\overline{B}$ such that $t \leq B^{-1}(t) \overline{B}^{-1}(t) \leq 2t$. 

Now assume that the kernel $\Phi$ satisfies the following weak growth condition: there are constants $\delta, c > 0$, $0 \leq \varepsilon < 1$ with the property that for all $k \in \mathbb{Z}$,
There is another characterization of the Luxemburg norm, which we will need:

$$\|f\|_{B,Q} \leq \inf_{s>0} \left\{ s + \frac{s}{|Q|} \int_{Q} B\left( \frac{|f(x)|}{s} \right) \, dx \right\} \leq 2 \|f\|_{B,Q}. \quad (3)$$

Given three Young functions $A, B$ and $C$ such that for all $t > 0$,

$$A^{-1}(t)C^{-1}(t) \leq B^{-1}(t),$$

then we have the following generalized Hölder inequality due to O’Neil [6]: for all functions $f$ and $g$ and for any cube $Q$,

$$\|fg\|_{B,Q} \leq 2 \|f\|_{A,Q} \|g\|_{C,Q}. \quad (4)$$

In particular, given any Young function $B$,

$$\frac{1}{|Q|} \int_{Q} |f(x)g(x)| \, dx \leq 2 \|f\|_{B,Q} \|g\|_{B,Q}. \quad (5)$$

Given a Young function $B$, define the associated Orlicz maximal operator by

$$M_Bf(x) = \sup_{Q \ni x} \|f\|_{B,Q}.$$

The dyadic maximal operator $M^d_B$ is defined similarly, except the supremum is restricted to dyadic cubes containing $x$.

**Lemma 2.1.** [8] For $1 < p < \infty$, let $B$ be a doubling Young function. Then

$$\int_{\infty}^{c} \frac{B(t) \, dt}{t^p} \frac{1}{t} < \infty$$

if and only if $M_B$ is $L^p(\mathbb{R}^n)$ bounded operator.

The main example that we are going to use is $B_m(t) = t \log(e + t)^m$, $m = 1, 2, \ldots$. For the Young function $B_m$, we denote the mean Luxemburg norm of $f$ on a cube $Q$ by $\|f\|_{L((\log L)^m),Q}$, and the corresponding Orlicz maximal function by $M_{L((\log L)^m)}$. The complementary Young function of $B_m$ is given by $\overline{B}_m(t) \approx e^{1/m}$, with the corresponding mean Luxemburg norm and maximal function being denoted by $\|f\|_{\exp L^{1/m}}$ and $M_{\exp L^{1/m}}$, respectively.

Our main result is the following theorem.

**Theorem 2.2.** Given $1 < p \leq q < \infty$ and $m \geq 0$ an integer, let $A_m, \tilde{A}_m, C_m$ and $\tilde{C}_m$ be Young functions such that $A_m$ and $\tilde{A}_m$ satisfy conditions

$$\int_{\infty}^{c} \frac{A_m(t) \, dt}{t^p} \frac{1}{t} < \infty, \quad \int_{\infty}^{c} \frac{\tilde{A}_m(t) \, dt}{t^q} \frac{1}{t} < \infty, \quad (6)$$

and

$$A_m^{-1}(t)C_m^{-1}(t) \leq B_m^{-1}(t), \quad \tilde{A}_m^{-1}(t)\tilde{C}_m^{-1}(t) \leq B_m^{-1}(t), \quad (7)$$
where $B_m(t) = t \log(e + t)^m$. Let $T_\Phi$ be the potential type operator with $\Phi$ satisfies (1) and $b \in BMO$. If the pair of weights $(u, v)$ is such that, for some positive constant $K$ and for any cube $Q$,

$$\tilde{\Phi}(l(Q)) |Q|^{1/q-1/p} \| u \|_{\tilde{C}_m \cdot Q} \| v^{-1} \|_{\tilde{C}_m \cdot Q} \leq K, \quad (8)$$

then the commutator $T_{\Phi}^{b,m}$ satisfies the strong type $(p, q)$ inequality

$$\left( \int_{\mathbb{R}^n} \left| T_{\Phi}^{b,m} f(y) \right|^q u(y)^q \ dy \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} \left| f(y) \right|^p v(y)^p \ dy \right)^{1/p}. \quad (9)$$

For $0 < \alpha < n$ the case $\Phi(x) = |x|^{\alpha-n}$ corresponds to the Riesz potential of order $\alpha$. In this case $\tilde{\Phi}(t) \equiv t^\alpha$. By Theorem 2.2, we have the following corollary:

**Corollary 2.3.** Given $1 < p \leq q < \infty$, $0 < \alpha < n$ and $m \geq 0$ an integer, let $A_m$, $\tilde{A}_m$, $C_m$ and $\tilde{C}_m$ be Young functions such that $A_m$ and $\tilde{A}_m$ satisfy conditions

$$\int_c^{\infty} \frac{\tilde{A}_m(t) \ dt}{t^{q'}} < \infty, \quad \int_c^{\infty} \frac{A_m(t) \ dt}{t^p} < \infty,$$

and

$$A_m^{-1}(t) C_m^{-1}(t) \leq B_m^{-1}(t), \quad \tilde{A}_m^{-1}(t) \tilde{C}_m^{-1}(t) \leq B_m^{-1}(t),$$

where $B_m(t) = t \log(e + t)^m$. Let

$$I_{\alpha}^{b,m} f(x) = \int_{\mathbb{R}^n} \frac{(b(x) - b(y))^m}{|x - y|^{n-\alpha}} f(y) \ dy$$

be the commutator of $I_{\alpha}$ and $b \in BMO$. If the pair of weights $(u, v)$ is such that, for some positive constant $K$ and for any cube $Q$,

$$|Q|^{\alpha/n+1/q-1/p} \| u \|_{\tilde{C}_m \cdot Q} \| v^{-1} \|_{\tilde{C}_m \cdot Q} \leq K,$$

then

$$\left( \int_{\mathbb{R}^n} \left| I_{\alpha}^{b,m} f(y) u(y) \right|^q \ dy \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} \left| f(y) \right|^p v(y)^p \ dy \right)^{1/p}.$$

For $1 < p < \infty$, an example of Young functions $A_m$ and $C_m$ which satisfy the conditions (6) and (7) given by Cruz-Uribe and Pérez [4] is

$$A_m(t) = t^p \log(e + t)^{(1+\delta(p-1))}, \quad \delta > 0,$$

and

$$C_m(t) = t^p' \log(e + t)^{(m+1)p'-1+\delta}.$$
3. The proof of Theorem 2.2

In order to prove Theorem 2.2, we need the following generalized Calderón–Zygmund decomposition.

Lemma 3.1. [8] Given a doubling Young function \( B \), suppose \( f \) is a non-negative function such that \( \| f \|_{B, Q} \) tends to zero as \( |Q| \) tends to infinity. Then for each \( t > 0 \) there exists a disjoint collection of maximal dyadic cubes \( \{ Q_{t,j} \} \) such that for each \( j, t < \| f \|_{B, Q_{t,j}} \leq 2^n t \), and

\[
\{ x \in \mathbb{R}^n : M_{B}^d f (x) > t \} = \bigcup_j Q_{t,j}, \quad \{ x \in \mathbb{R}^n : M_{B} f (x) > 4^n t \} \subset \bigcup_j 3 Q_{t,j}.
\]

The collection of dyadic cubes \( \{ Q_{t,j} \} \) is referred to as the Calderón–Zygmund decomposition of \( f \) with respect to \( B \) at height \( t \).

We fix a constant \( a > 2^n \), and for each integer \( k \) we let

\[
\Omega_k = \{ x \in \mathbb{R}^n : M_{B} f (x) > 4^n a^k \}, \quad D_k = \{ x \in \mathbb{R}^n : M_{B}^d f (x) > a^k \}.
\]

Hence, by Lemma 3.1 with \( t = a^k \) there is a family of maximal non-overlapping dyadic cubes \( \{ Q_{k,j} \} \) for which \( D_k = \bigcup_j Q_{k,j}, \Omega_k \subset \bigcup_j 3 Q_{k,j} \), and \( a^k < \| f \|_{B, Q_{k,j}} \leq 2^n a^k \).

Lemma 3.2. [8] Suppose \( a > 2^n \). For all integers \( k,j \) we let \( E_{k,j} = Q_{k,j} \setminus Q_{k,j} \cap D_{k+1} \). Then \( \{ E_{k,j} \} \) is a disjoint family of sets which satisfies

\[
|Q_{k,j}| \leq \frac{1}{1 - 2^n / a} |E_{k,j}|.
\]

Proof of Theorem 2.2. Since the set of bounded functions with compact support is dense in \( L^p(u^p) \), it is enough to show that there is a constant \( C \) such that

\[
\left( \int_{\mathbb{R}^n} \left| T_{\Phi}^{b,m} f(y) u(y) \right|^q \, dy \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} \left| f(y) v(y) \right|^p \, dy \right)^{1/p}
\]

for each bounded function with compact support \( f \). By duality, this is equivalent to

\[
\int_{\mathbb{R}^n} u(y) \left| T_{\Phi}^{b,m} f(y) g(y) \right| \, dy \leq C \left( \int_{\mathbb{R}^n} \left| v(y) f(y) \right|^p \, dy \right)^{1/p} \left( \int_{\mathbb{R}^n} |g(y)|^{q'} \, dy \right)^{1/q'}, \quad (10)
\]

for all bounded functions with compact support \( f, g \).

We set for each \( t > 0 \),

\[
\Phi (t) = \sup_{r < |y| \leq 2t} \Phi (y),
\]

and

\[
\Phi (t) = \frac{1}{r^n} \int_{\delta (1 - \varepsilon) r < |y| \leq 2 \delta (1 + \varepsilon) r} \Phi (y) \, dy,
\]

where \( \delta, \varepsilon \) are the numbers in (1). Following [7], we can discretize the operator \( T_{\Phi}^{b,m} \) as follows:
\[ |T_{\phi}^{b,m} f(y)| = \left| \sum_{v \in \mathbb{Z}} \left( \frac{1}{2^{v-1}} \right) \int_{|z-y| \leq 2^{-v}} (b(y) - b(z))^m \Phi(y - z) f(z) dz \right| \]
\[ \leq \sum_{v \in \mathbb{Z}} \Phi(2^{-v-1}) \int_{|z-y| \leq 2^{-v}} |b(y) - b(z)|^m f(z) dz \]
\[ = \sum_{v \in \mathbb{Z}} \sum_{Q: l(Q) = 2^{-v}} \chi_Q(y) \Phi(2^{-v-1}) \int_{|z-y| \leq 2^{-v}} |b(y) - b(z)|^m f(z) dz. \]

The ball \( B(y, 2^{-v}) \) is covered by the cube \( 3Q \) if \( y \in Q \) and \( l(Q) = 2^{-v} \). Hence

\[ |T_{\phi}^{b,m} f(y)| \leq \sum_{Q} \Phi \left( \frac{l(Q)}{2} \right) \sum_{l=0}^{m} \left( \frac{m}{l} \right) |b(y) - b_Q|^{m-l} \chi_Q(y) \int_{3Q} |b(z) - b_Q|^l f(z) dz, \]

where \( b_Q = \frac{1}{|Q|} \int_Q b(z) dz \). Then, by the Hölder inequality (5) and the John–Nirenberg theorem, we have

\[
\int_{\mathbb{R}^n} u(y) |T_{\phi}^{b,m} f(y) g(y)| dy \\
\leq \sum_{Q} \sum_{l=0}^{m} \left( \frac{m}{l} \right) \int_{Q} |b(y) - b_Q|^{m-l} |g(y)| u(y) dy \cdot \Phi \left( \frac{l(Q)}{2} \right) \int_{3Q} |b(z) - b_Q|^l f(z) dz \\
\leq \sum_{Q} \sum_{l=0}^{m} \left( \frac{m}{l} \right) |Q| |b - b_Q|^{m-l} \| u g \|_{L(\log L)^{m-l}, Q} \cdot \Phi \left( \frac{l(Q)}{2} \right) \int_{3Q} |b(z) - b_Q|^l f(z) dz \\
\leq \sum_{Q} \sum_{l=0}^{m} \left( \frac{m}{l} \right) |Q| |b - b_Q|^{m-l} \| u g \|_{L(\log L)^{m-l}, Q} \cdot \Phi \left( \frac{l(Q)}{2} \right) \int_{3Q} |b(z) - b_Q|^l f(z) dz \\
\leq C \| b \|_{BMO}^m \sum_{Q} \sum_{l=0}^{m} \| u g \|_{L(\log L)^{m-l}, Q} |Q| \cdot \Phi \left( \frac{l(Q)}{2} \right) \int_{3Q} |b(z) - b_Q|^l f(z) dz \\
\leq C \| b \|_{BMO}^m \sum_{Q} \| u g \|_{L(\log L)^m, Q} |Q| \cdot \Phi \left( \frac{l(Q)}{2} \right) \int_{3Q} |b(z) - b_Q|^l f(z) dz \\
\leq C \| b \|_{BMO}^m \sum_{Q} \| u g \|_{L(\log L)^m, Q} \cdot \Phi \left( \frac{l(Q)}{2} \right) |Q| \cdot \int_{3Q} |b(z) - b_Q|^l f(z) dz \\
\leq C \| b \|_{BMO}^m \sum_{Q} \| u g \|_{L(\log L)^m, Q} |Q| \cdot \Phi \left( \frac{l(Q)}{2} \right) |Q| \cdot \int_{3Q} |b(z) - b_Q|^l f(z) dz. \]

Since \( g \) has compact support, for the Young function \( B_m(t) = t \log(e + t)^m \), we have \( \| u g \|_{B_m, Q} \to 0 \) as \( l(Q) \to \infty \). For a constant \( a > 2^n \) and an integer \( k \), by Lemma 3.1 with \( t = a^k \) there is a family of maximal non-overlapping dyadic cubes \( \{ Q_{k,j} \} \) for which

\[ a^k < \| u g \|_{L(\log L)^m, Q_{k,j}} \leq 2^n a^k. \]

For each integer \( k \) we let

\[ C_k = \{ Q \in \Delta: a^k < \| u g \|_{L(\log L)^m, Q} \leq a^{k+1} \}. \]
Every dyadic cube \( Q \) for which \( \| u g \|_{L \log L, Q} \neq 0 \) belongs to exactly one \( C_k \). Furthermore, if \( Q \in C_k \), it follows that \( Q \subset Q_{k,j} \) for some \( j \). Then

\[
\int_{\mathbb{R}^n} u(y) \left| T_{\Phi}^b \right| \left| f(y) g(y) \right| dy 
\leq C \| b \|_{BMO} \sum_{k \in \mathbb{Z}} \sum_{Q \in C_k} \| u g \|_{L \log L, Q} |Q| \cdot \Phi \left( \frac{l(Q)}{2} \right) |3Q| \| f \|_{L \log L, 3Q}
\]

\[
\leq C \| b \|_{BMO} d \sum_{j \in \mathbb{Z}} \sum_{Q \in C_k: Q \subset Q_{k,j}} |Q| \cdot \Phi \left( \frac{l(Q)}{2} \right) |3Q| \| f \|_{L \log L, 3Q}
\]

\[
\leq C \| b \|_{BMO} \sum_{k, j} \| u g \|_{L \log L, Q_{k,j}} \sum_{Q: Q \subset Q_{k,j}} |Q| \cdot \Phi \left( \frac{l(Q)}{2} \right) |3Q| \| f \|_{L \log L, 3Q_{k,j}}
\]

where the last inequality will follow if we show that there is a constant \( C \) such that for any dyadic cube \( Q_0 \).

\[
\sum_{Q: Q \subset Q_0} |Q| \cdot \Phi \left( \frac{l(Q)}{2} \right) |3Q| \| f \|_{L \log L, 3Q} \leq C \tilde{\Phi} \left( \delta(1 + \varepsilon) l(Q_0) \right) |3Q_0| \| f \|_{L \log L, 3Q_0}.
\]  

(11)

However, if \( l(Q_0) = 2^{-\nu_0} \), then

\[
\sum_{Q: Q \subset Q_0} |Q| \cdot \Phi \left( \frac{l(Q)}{2} \right) |3Q| \| f \|_{L \log L, 3Q} = \sum_{\nu \geq \nu_0} \sum_{l(Q) = 2^{-\nu}} 2^{-\nu \mu} \Phi(2^{-\nu - 1}) |3Q| \| f \|_{L \log L, 3Q}
\]

\[
= \sum_{\nu \geq \nu_0} 2^{-\nu \nu} \Phi(2^{-\nu - 1}) \sum_{l(Q) = 2^{-\nu}} |3Q| \| f \|_{L \log L, 3Q}.
\]  

(12)

By inequality (3), we have

\[
\sum_{l(Q) = 2^{-\nu}} |3Q| \| f \|_{L \log L, 3Q} \leq \sum_{l(Q) = 2^{-\nu}} |3Q| \inf_{\mu > 0} \left\{ \mu + \frac{\mu}{|3Q|} \int_{3Q} B \left( \frac{|f(z)|}{\mu} \right) dz \right\}
\]

\[
\leq \inf_{\mu > 0} \left\{ |3Q_0| \mu + C \mu \int_{3Q_0} B \left( \frac{|f(z)|}{\mu} \right) dz \right\}
\]

\[
\leq C |3Q_0| \inf_{\mu > 0} \left\{ \mu + \frac{\mu}{|3Q_0|} \int_{3Q_0} B \left( \frac{|f(z)|}{\mu} \right) dz \right\}
\]

\[
\leq C |3Q_0| \| f \|_{L \log L, 3Q_0}.
\]
since the overlap is finite. Now, by condition (1), we have \( \tilde{\Phi}(2^{-\nu}) \leq C \Phi(2^{-\nu}) \) for \( \nu \in \mathbb{Z} \), and then

\[
\sum_{\nu \geq \nu_0} 2^{-\nu n} \Phi(2^{-\nu}) \leq C \sum_{\nu \geq \nu_0} 2^{-\nu n} \Phi(2^{-\nu})
\]

\[
= C \sum_{\nu \geq \nu_0} \int \Phi(y) \, dy \quad \delta(1-\epsilon)2^{-\nu-1}<|y| \leq \delta(1+\epsilon)2^{-\nu}
\]

\[
\leq C \int \Phi(y) \, dy \quad |y| \leq \delta(1+\epsilon)2^{-\nu_0}
\]

\[
= C \Phi(\delta(1+\epsilon)l(Q_0)),
\]

again because the overlap is finite. Combining this and (12) yields inequality (11). Let \( \rho = \max\{\delta(1+\epsilon), 3\} \), then

\[
\int_{\mathbb{R}^n} u(y) \left| T_{b,m}^b f(y) g(y) \right| \, dy
\]

\[
\leq C \|b\|_{BMO}^m \sum_{k,j} \|u\|_{L(log L)^m, Q_{k,j}} \|c_{m,\rho Q_{k,j}} \tilde{\Phi}(\rho l(Q_{k,j}))\| \|f\|_{L(log L)^m, 3Q_{k,j}} \|3Q_{k,j} \|
\]

\[
\leq C \|b\|_{BMO}^m \sum_{k,j} \|u\|_{L(log L)^m, Q_{k,j}} \|c_{m,\rho Q_{k,j}} \tilde{\Phi}(\rho l(Q_{k,j}))\| \|f\|_{L(log L)^m, Q_{k,j}} \|Q_{k,j} \|
\]

The Hölder inequality (4) together with hypothesis (8) allows to estimate the last sum by a multiple of

\[
\sum_{k,j} \|g\|_{A_{m,\rho Q_{k,j}}} \|u\|_{C_{m,\rho Q_{k,j}}} \|f\|_{A_{m,\rho Q_{k,j}}} \|v^{-1}\|_C \|c_{m,\rho Q_{k,j}} \tilde{\Phi}(l(\rho Q_{k,j}))\| Q_{k,j} \|
\]

\[
\leq K \sum_{k,j} \|g\|_{A_{m,\rho Q_{k,j}}} \|f\|_{A_{m,\rho Q_{k,j}}} \|Q_{k,j} \|^{1/p+1/q'}.
\]

Now, by using Hölder inequality, Lemma 3.2 and then that \( p \leq q \), we can follow our chain of inequalities with

\[
\leq K \left( \sum_{k,j} \|g\|_{A_{m,\rho Q_{k,j}}} \|Q_{k,j} \|^{p'/q'} \right)^{1/p'} \left( \sum_{k,j} \|f\|_{A_{m,\rho Q_{k,j}}} \|Q_{k,j} \|^{p/q} \right)^{1/p}
\]

\[
\leq K \left( \sum_{k,j} \|g\|_{A_{m,\rho Q_{k,j}}} \|Q_{k,j} \|^{q'/q} \right)^{1/q'} \left( \sum_{k,j} \|f\|_{A_{m,\rho Q_{k,j}}} \|Q_{k,j} \|^{p/q} \right)^{1/p}
\]

\[
\leq K \left( \sum_{k,j} \|g\|_{A_{m,\rho Q_{k,j}}} \|E_{k,j} \| \right)^{1/q'} \left( \sum_{k,j} \|f\|_{A_{m,\rho Q_{k,j}}} \|E_{k,j} \| \right)^{1/p}
\]

\[
\leq K \left( \sum_{k,j} \int_{E_{k,j}} \|g\|_{A_{m,\rho Q_{k,j}}} \, dy \right)^{1/q'} \left( \sum_{k,j} \int_{E_{k,j}} \|f\|_{A_{m,\rho Q_{k,j}}} \, dy \right)^{1/p}
\]
\[
\begin{align*}
&\leq K \left( \sum_{k,j} \int_{E_{k,j}} M_{\tilde{\lambda}_m} g(y)^{q'} \, dy \right)^{1/q'} \\
&\times \left( \sum_{k,j} \int_{E_{k,j}} M_{A_m} (f v)(y)^p \, dy \right)^{1/p}
\end{align*}
\]

This concludes the proof of Theorem 2.2. \qed

Acknowledgment

The author thanks the referee for some valuable suggestions and corrections.

References