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The Rate of Convergence of Expansions in Freud Polynomials

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For a function f of bounded variation on compact intervals, satisfying certain growth conditions, we estimate the rate of convergence of its expansion in a series of polynomials orthogonal on the whole real axis with respect to a weight function, now known as a Freud weight. The case where f has higher order derivatives of bounded variation is also studied. The principal techniques include the finite-infinite range inequalities due to the author and Saff, and Freud's theorems on one-sided weighted L^1-approximation. Our theorem holds, in particular, when the weight function is $\exp(-x^m)$, m a positive even integer. © 1988 Academic Press, Inc.

1. INTRODUCTION

In 1979, R. Bojanic [1] obtained an estimate on the rate at which the trigonometric Fourier series of a 2π-periodic function of bounded variation converges. His theorem can be stated as follows.

THEOREM 1.1 ([1]). *Let f be a 2π-periodic function having bounded variation on $[-\pi, \pi]$ and, for $n \geqslant 1$, let $S_n(f)$ denote the* nth *partial sum of the Fourier series of f. Then, for $x \in [-\pi, \pi]$ and $n \geqslant 1$,*

$$\left| S_n(f, x) - \frac{1}{2}(f(x+) + f(x-)) \right| \leqslant \frac{3}{n} \sum_{k=1}^{n} V\left(\left[0, \frac{\pi}{k}\right], g_x^*\right), \tag{1.1}$$

where

$$g_x^*(t) := \begin{cases} f(x+t) + f(x-t), & t \neq 0 \\ f(x+) + f(x-), & t = 0 \end{cases} \tag{1.2}$$

and $V([a, b]), g_x^*)$ denotes the total variation of g_x^* on $[a, b]$.

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This theorem seems to have inspired a great deal of research in which analogous results have been obtained (e.g., [2, 3, 4, 6, 7, 9, 10, 11, 12, 16, 17]). All of these papers deal with processes defined for functions of bounded variation on compact intervals. Various technical difficulties prevent a straightforward extension of these results to the case of functions supported on the whole real line.

In 1974, G. Freud [13] proved a theorem of Dirichlet–Jordan type for expansions in polynomials orthogonal on the whole real line with respect to a weight function satisfying some mild conditions. We postpone the detailed statement of his results to Section 2 (cf. Theorem 2.1) but make a few comments here. The usual integration-by-parts argument used for obtaining such a theorem for Fourier series did not work because of the lack of detailed information about the orthogonal polynomials. Freud used, instead, a Tauberian argument, involving the comparison of the partial sums of the expansion with their shifted artithmetic averages. The novelty of his ideas in [13] together with the connections between his result and the weighted analogues of Jackson–Favard-type estimates, in our opinion, make [13] an important landmark in the theory of weighted polynomial approximation. Nevertheless, in order for these ideas to work, it is very important that the function being expanded be continuous on the whole real line.

Recent progress in the theory of weighted approximation, particularly [19], has made it possible to use the old integration-by-parts argument again to obtain the rate of convergence of the orthogonal polynomial expansions on the whole real line for functions of bounded variation which are not necessarily continuous. In [5], we studied the case of the expansions in Hermite polynomials, using heavily the special properties of Hermite polynomials. In this paper, we continue these investigations for a more general class of weight functions and also in the case when the function being expanded has higher derivatives. The core of the argument is similar to that in [17]; other key ingredients being the finite–infinite range inequalities of [19] and the one-sided approximation theorem of Freud [14].

In the next section, we state Freud's theorem in [13] as well as our theorem. The proofs are given in Section 3.

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2. Main Results

We consider weight functions of the form $w_Q(x) := \exp(-Q(x))$ which satisfy the following conditions:

(W1) Q is an even, convex, positive function on $\mathbb{R}$ and is differentiable and increasing on $(0, \infty)$.

(W2) $Q'(x) \to \infty$ as $x \to \infty$.

(W3) $1 + c_1 < Q'(2x)/Q'(x) < 1 + c_2$, $(x > c_3)$.

(W4) For each sufficiently large n, let q_n be the smallest number for which

$$q_n Q'(q_n) = n.$$

Then there exists a constant $B > 1$ such that

$$Bq_n \leqslant q_{2n} \leqslant cq_n.$$

Here and elsewhere in the paper we adopt the following convention concerning the various constants: we shall denote the constants depending on Q alone by c, c_1, c_2, etc. However, the same symbol may denote different constants in different formulas. Constants denoted by capital letters, however, retain their value when referred to in different formulas.

Let Π_n be the class of all polynomials of degree at most n, $\{p_k(x) := \gamma_k x^k + \cdots \in \Pi_k, \gamma_k > 0\}$ the system of orthogonal polynomials with respect to w_Q^2:

$$\int p_k(x)\, p_j(x)\, w_Q^2(x)\, dx = \delta_{kj}. \tag{2.1}$$

If $fw_Q \in L^1(\mathbb{R})$, put

$$a_k := a_k(f) := \int f p_k w_Q^2\, dx \tag{2.2a}$$

$$s_n(w_Q^2, f, x) := s_n(f, x) := \sum_{k=0}^{n-1} a_k p_k(x). \tag{2.2b}$$

Freud's theorem can now be formulated as follows.

THEOREM 2.1 ([13]). *Let f be a continuous function on $\mathbb{R}$ which is of bounded variation over every compact interval and*

$$\int w_Q |df| < \infty. \tag{2.3}$$

Then

$$\lim_{n \to \infty} \|w_Q(x)[f(x) - s_n(f, x)]\|_\infty = 0. \tag{2.4}$$

Freud's theorem is, in fact, true under much weaker conditions on the weight function.

Before we state our result, we need to introduce additional notation. If ϕ is a function having bounded variation on compact intervals, and $x \in \mathbb{R}$, we set

$$g_x(\phi, t) := \begin{cases} \phi(t) - \phi(x-), & -\infty < t < x \\ 0, & t = x \\ \phi(t) - \phi(x+), & x < t < \infty. \end{cases} \tag{2.5}$$

If $[a, b] \subset \mathbb{R}$, we set

$$V_Q([a, b], \phi) := \int_a^b w_Q(t)\, |d\phi(t)|. \tag{2.6}$$

Finally, denoting the quantity $\max(y^r, 0)$ by y^r_+, we put

$$\Gamma_r(t, x) := \begin{cases} (t-x)^r_+, & r > 0 \\ (t-x)^0_+ - \frac{1}{2}, & r = 0 \end{cases} \tag{2.7}$$

and

$$\sigma_{n,r}(x) := s_n(w_Q^2, \Gamma_r(\cdot, x), x). \tag{2.8}$$

Our main theorem can now be stated as follows.

THEOREM 2.2. *Let $r \geqslant 0$ be an integer, f an r-times iterated integral of a function ϕ $(= f$ if $r = 0)$ having bounded variation on compact intervals. Suppose that*

$$\int_{-\infty}^{\infty} w_Q(t)\, |t|^r\, |d\phi(t)| < \infty. \tag{2.9}$$

Assume that the system of orthogonal polynomials satisfies the condition

$$|p_n(t)\, w_Q(t)| \leqslant c \cdot q_n^{-1/2}, \qquad (|t| \leqslant A q_n). \tag{PB}$$

Then, for $x \in \mathbb{R}$ *and* $n = 1, 2, \ldots,$

$$\left| s_n(w_Q^2, f, x) - \frac{1}{2}(f(x+) + f(x-)) - \frac{[\phi(x+) - \phi(x-)]}{r!} \sigma_{n,r}(x) \right|$$
$$\leqslant c_1(x) \left\{ \frac{q_n^r}{n^{r+1}} \sum_{k=1}^{n} V_Q\left(\left[x - \frac{q_n}{k}, x + \frac{q_n}{k} \right], g_x(\phi, \cdot) \right) \right.$$
$$\left. + \int_{c_2 q_n}^{\infty} |t|^r w_Q(t) \, |d\phi(t)| \right\}. \tag{2.10}$$

In order to see why the first term on the right hand side of (2.10) tends to 0 as $n \to \infty$, observe that $q_n/n \to 0$ as $n \to \infty$. Hence, denoting $g_x(\phi, \cdot)$ by g,

$$\frac{1}{n} \sum_{k=1}^{[\sqrt{nq_n}]+1} V_Q\left(\left[x - \frac{q_n}{k}, x + \frac{q_n}{k} \right], g \right) \leqslant c \cdot \sqrt{\frac{q_n}{n}} \int w_Q \, |dg| \tag{2.11}$$

while, if $k \geqslant \sqrt{nq_n}$ then

$$V_Q\left(\left[x - \frac{q_n}{k}, x + \frac{q_n}{k} \right], g \right)$$
$$\leqslant V_Q\left(\left[x - \sqrt{\frac{q_n}{n}}, x + \sqrt{\frac{q_n}{n}} \right], g \right).$$

Thus,

$$\frac{1}{n} \sum_{k=[\sqrt{nq_n}]+1}^{n} V_Q\left(\left[x - \frac{q_n}{k}, x + \frac{q_n}{k} \right], g \right)$$
$$\leqslant V_Q\left(\left[x - \sqrt{\frac{q_n}{n}}, x + \sqrt{\frac{q_n}{n}} \right], g \right). \tag{2.12}$$

Since g is continuous at x, so is its variation function. An integration by parts in the formulas for V_Q then shows that the right hand side of (2.12) tends to zero as $n \to \infty$. Estimates (2.11) and (2.12) then show that the first term on the right hand side of (2.10) tends to 0 as $n \to \infty$.

The Condition (PB) is perhaps unduly stringent. Currently, it is known to be true only in the case when $Q(x) = x^m$, m being an even, positive integer [8]. In [20], it is conjectured that such an estimate is true also when $Q(x) = |x|^\alpha$, $\alpha > 1$.

When $Q(x) = x^2$, then the orthogonal polynomials are precisely the Hermite polynomials. In this case, we found an asymptotic expression for $\sigma_{n,r}(x)$ in [5]. In the general case, it is easy to check that $\sigma_{n,r}(x) = \mathcal{O}(q_n/n)^r$; but a better estimate would be desirable.

The proof of our theorem depends upon the relationship between various constants. For the convenience of the reader, we give a list of a few constants which will be used often along with the location where they first occur.

Symbol	Location
A	Condition (PB) in Theorem 2.2
A^*	Formula (3.42)
B	Condition (W4) on the weight function
B_1	Lemma 3.1(b), Formula (3.6)
B_2	Lemma 3.1(c)
B_3	Lemma 3.1(c)
γ	Formula (3.17)
D	Lemma 3.1(d)
K	Formula (3.17)
L	Formula (3.44)

3. Proof

Before we proceed to prove Theorem 2.2, we summarize some known estimates as well as certain computational aspects of our proof in the following lemma.

LEMMA 3.1. (a) *Let*

$$K_n(x, t) := \sum_{k=0}^{n-1} p_k(x)\, p_k(t). \tag{3.1}$$

Then

$$K_n(x, t) = \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x)\, p_{n-1}(t) - p_{n-1}(x)\, p_n(t)}{x - t}. \tag{3.2}$$

For $K_n(x, x)$, we have the following:

$$K_n(x, x) \leqslant c \cdot \frac{n}{q_n} w_Q^{-2}(x), \qquad x \in \mathbb{R}. \tag{3.3}$$

(b) *Let $m \geqslant 1$ be an integer, a_m be defined by the formula*

$$m = \frac{2}{\pi} \int_0^1 \frac{a_m x Q'(a_m x)}{\sqrt{1 - x^2}}\, dx. \tag{3.4}$$

If $P \in \Pi_m$ *satisfies*

$$\int_{|t| \geqslant 2a_m} |P(t)\, w_Q(t)|\, dt \leqslant 1 \tag{3.5}$$

then

$$\int_{|t| \geqslant 2a_m} |P(t)\, w_Q(t)|^2\, dt \leqslant c \exp(-B_1 m). \tag{3.6}$$

(c) *There exist positive constants* B_2, B_3 *with* $B_2 < 1$ *such that whenever* $P \in \Pi_{[B_2 n]}$ *and*

$$\int_{|t| \leqslant Aq_n} |P(t)\, w_Q(t)|\, dt \leqslant 1 \tag{3.7}$$

we have

$$\int_{|t| \geqslant Aq_n} |P(t)\, w_Q(t)|^2\, dt \leqslant c \cdot \exp(-B_3 n), \tag{3.8}$$

where A *is the constant appearing in* (PB).

(d) *There exists a constant* D *such that for* $|y| \leqslant Dq_n$,

$$\exp\left(-\frac{B_3 n}{8}\right) \leqslant c \cdot w_Q(y). \tag{3.9}$$

(e) *For any* $D_1 > 0$, *if* $z \geqslant D_1 q_n$ *then*

$$\int_z^\infty v^r w_Q(v)\, dv \leqslant c \cdot \frac{q_n}{n} z^r w_Q(z). \tag{3.10}$$

(f) *For any* $D_1 > 0$, *if* $|x| \leqslant \frac{1}{4} D_1 q_n$ *and* $0 \leqslant u \leqslant \frac{1}{2} D_1 q_n$, *we have*

$$w_Q(x + D_1 q_n) \leqslant c_1 \exp(-c_2 n)\, w_Q(x+u). \tag{3.11}$$

(g) *For* $x \in \mathbb{R}$, $t > 0$, *and integers* $n \geqslant r \geqslant 0$, *let*

$$G(t) := \frac{1}{r!\, t} \int_0^t (t-u)^r\, dg_x(\phi, x+u) \tag{3.12}$$

$$A_n(t) := \frac{1}{r!} \int_t^\infty (t-u)^r\, p_n(x+u)\, w_Q^2(x+u)\, du. \tag{3.13}$$

Then, for integer k, $0 \leqslant k \leqslant r$, *and* $|x+t| \leqslant Dq_n$

$$|G^{(k)}(t)| \leqslant c \cdot t^{r-k-1} \int_0^t |dg_x(\phi, x+u)| \tag{3.14}$$

$$|A_n^{(k)}(t)| \leqslant c \left(\frac{q_n}{n}\right)^{r-k+1} q_n^{-1/2} w_Q(x+t), \qquad n \geqslant c_1. \tag{3.15}$$

Proof of Lemma 3.1. Part (a) was proved in [15]. Part (b) is a special case of Theorem 2.7(b) of [19] (cf. Example 3, pp. 76–78 in [18]). Since $Q'(x)$ is increasing, (3.4) implies that

$$q_m \leqslant a_m \leqslant 3q_m. \tag{3.16}$$

In view of Assumption (W4) on the weight function, we may find B_2, $0 < B_2 < 1$, such that with $m = [B_2 n]$,

$$2a_m \leqslant 6q_m \leqslant Aq_n.$$

Part (c) now follows with $B_3 := B_2 B_1/2$. In view of Assumption (W3), we see that for any $D_2 > 0$

$$D_2 q_n Q'(D_2 q_n) \leqslant K \cdot D_2^{\gamma} n \tag{3.17}$$

for some constants K and γ independent of D_2 or n. Hence, for $D := (B_3/8K)^{1/\gamma}$,

$$Dq_n Q'(Dq_n) \leqslant B_3 n/8. \tag{3.17a}$$

So, if $|y| \leqslant Dq_n$

$$\begin{aligned} Q(y) = Q(|y|) \leqslant Q(Dq_n) &= Q(0) + \int_0^{Dq_n} Q'(t)\, dt \\ &\leqslant Dq_n Q'(Dq_n) + Q(0) \leqslant Q(0) + B_3 n/8. \end{aligned} \tag{3.18}$$

This yields (3.9).

In order to prove part (e), observe that for $v \geqslant c$,

$$[v^r w_Q^{1/2}(v)]' = \tfrac{1}{2} v^{r-1} [2r - vQ'(v)] \cdot w_Q^{1/2}(v) < 0. \tag{3.19}$$

Also, if $v \geqslant z \geqslant D_1 q_n$, then $Q'(v) \geqslant c \cdot n/q_n$. So, for $v \geqslant z$,

$$v^r w_Q(v) \leqslant z^r w_Q^{1/2}(z) \cdot c \frac{q_n}{n} Q'(v)\, w_Q^{1/2}(v). \tag{3.20}$$

Integrating both sides of (3.20) with respect to v between z and ∞, we get (3.10). To prove part (f), we first prove that

$$w_Q\left(x+\frac{D_1 q_n}{2}\right) \leqslant w_Q(x+u). \tag{3.21}$$

Indeed, this is obvious when $x+u \geqslant 0$. If $x+u<0$ then

$$|x+u| = -u-x \leqslant |x| \leqslant \frac{D_1}{4} q_n \leqslant \frac{D_1}{2} q_n - |x| = \left|x+\frac{D_1 q_n}{2}\right|. \tag{3.22}$$

This gives (3.21) in this case also. Now,

$$\left|Q(x+D_1 q_n) - Q\left(x+\frac{D_1 q_n}{2}\right)\right| \leqslant \int_{D_1 q_n/2}^{D_1 q_n} Q'(x+t)\, dt$$
$$\leqslant c \cdot q_n Q'(D_1 q_n/4) \leqslant c \cdot n. \tag{3.23}$$

The Estimate (3.11) follows from (3.23) and (3.22).

The fact that (3.12) implies (3.14) is a simple application of Leibnitz's formula. Further

$$A_n^{(k)}(t) = \int_{-\infty}^{\infty} (x+t-v)_+^{r-k} p_n(v)\, w_Q^2(v)\, dv. \tag{3.24}$$

Let $m := [B_2 n]$. Then, in view of a theorem of G. Freud on one-sided weighted L^1-approximation, there exists a polynomial $P \in \Pi_m$ such that

$$\int_{-\infty}^{\infty} |(x+t-v)_+^{r-k} - P(v)|\, w_Q(v)\, dv \leqslant c \cdot \left(\frac{q_m}{m}\right)^{r-k+1} w_Q(x+t). \tag{3.25}$$

(cf. [14], Lemmas 2.1, 5.1, and the Estimate (5.15). Note, however, the difference in notation.) Since p_n is orthogonal to P, we see that

$$|A_n^{(k)}(t)| = \left|\int_{-\infty}^{\infty} [(x+t-v)_+^{r-k} - P(v)]\, p_n(v)\, w_Q^2(v)\, dv\right| \leqslant J_1 + J_2, \tag{3.26}$$

where

$$J_1 := \int_{|v| \leqslant Aq_n} |(x+t-v)_+^{r-k} - P(v)|\, w_Q(v)\, |p_n(v)\, w_Q(v)|\, dv \tag{3.27a}$$

$$J_2 := \int_{|v| \geqslant Aq_n} |(x+t-v)_+^{r-k} - P(v)|\, w_Q(v) \cdot |p_n(v)\, w_Q(v)|\, dv. \tag{3.27b}$$

In view of (3.25) and Assumption (PB) on orthogonal polynomials, we have

$$J_1 \leqslant c \cdot q_n^{-1/2}(q_n/n)^{r-k+1}\, w_Q(x+t). \tag{3.28}$$

Also, if $|x+t| \leqslant Dq_n$, $P_1(v) := (x+t-v)^{r-k} - P(v) \in \Pi_m$ satisfies, in view of (3.25),

$$\int_{|v| \leqslant Aq_n} |P_1(v)|\, w_Q(v)\, dx \leqslant c \cdot q_n^{r-k}. \tag{3.29}$$

So, part (c) of this Lemma implies that

$$\int_{|v| \geqslant Aq_n} |P_1(v)|\, w_Q(v)\, dv \leqslant c \cdot \exp\left(-\frac{1}{2} B_3 n\right). \tag{3.30a}$$

Similarly,

$$\int_{|v| \geqslant Aq_n} |P(v)|\, w_Q(v)\, dv \leqslant c \cdot \exp\left(-\frac{1}{2} B_3 n\right). \tag{3.30b}$$

Thus

$$\int_{|v| \geqslant Aq_n} |(x+t-v)_+^{r-k} - P(v)|\, w_Q(v)\, dv \leqslant c \cdot \exp\left(-\frac{1}{2} B_3 n\right). \tag{3.30c}$$

The part (a) of this Lemma implies that $|p_n(v)|\, w_Q(v) \leqslant c \cdot (n/q_n)^{1/2}$ for all $v \in \mathbb{R}$. So, (3.30c) yields

$$J_2 \leqslant c \cdot \exp(-\tfrac{1}{4} B_3 n). \tag{3.31}$$

Since $|x+t| \leqslant Dq_n$, part (d) of this lemma now gives

$$J_2 \leqslant c \cdot \exp\left(-\frac{1}{8} B_3 n\right) w_Q(x+t) \leqslant c \left(\frac{q_n}{n}\right)^{r-k+1} q_n^{-1/2} w_Q(x+t). \tag{3.32}$$

Substituting from (3.32) and (3.28) into (3.26), we get (3.15). ∎

In the proof of (3.15), we did not really use the fact that p_n is a polynomial; merely that it is orthogonal to Π_{n-1}. Thus, the same proof gives us the following analogue of H. Bohr-type inequality. We shall not need it in this paper, but record it here for a possible future reference.

Proposition 3.2. *Let* $w_Q F \in L^\infty(\mathbb{R})$, $n \geqslant c$ *be an integer*, $0 < \beta < 1$,

$$\operatorname{ess\,sup}\{|w_Q(t)F(t)|: |t| \leqslant \beta q_n\} =: M_{1,n} \tag{3.33a}$$

$$\operatorname{ess\,sup}\{|w_Q(t)F(t)|: |t| \geqslant \beta q_n\} =: M_{2,n}. \tag{3.33b}$$

Suppose that for every $P \in \Pi_n$,

$$\int_{-\infty}^{\infty} P(t) F(t)\, w_Q^2(t)\, dt = 0. \tag{3.34}$$

Let, for an integer $r \geqslant 0$,

$$\mathscr{F}_r(y) := \frac{1}{r!} \int_{-\infty}^{y} (y-u)^r F(u)\, w_Q^2(u)\, du. \tag{3.35}$$

Then, for $|y| \leqslant c(\beta, r)\, q_n$,

$$|w_Q^{-1}(y)\, \mathscr{F}_r(y)| \leqslant c_1 \left\{ \left(\frac{q_n}{n}\right)^{r+1} M_{1,n} + \exp(-c_2 n)\, M_{2,n} \right\}, \tag{3.36}$$

where c_1 *and* c_2 *are positive constants depending only upon* Q, β, *and* r.

For $r = 0$, a cruder form of this proposition was proved by G. Freud in [15]. From this proposition (in its cruder form), Freud then obtained the direct theorems of weighted polynomial approximation. We shall not pursue this line of thought here.

We now return to the proof of Theorem 2.2. We observe that when $r = 0$ and hence $f = \phi$, we have (cf. (2.5), (2.7)), for $t \neq x$,

$$f(t) = \frac{f(x+) + f(x-)}{2} + [f(x+) - f(x-)]\, \Gamma_0(t, x) + g_x(t). \tag{3.37}$$

When $r > 0$ then $[f(x+) + f(x-)]/2 = f(x)$ and the fact that f is an r-times iterated integral of ϕ can be reformulated to state that for $t \neq x$,

$$f(t) = P(x, t) + \frac{1}{(r-1)!} \int_x^t (t-u)^{r-1} \phi(u)\, du, \tag{3.38}$$

where $P(x, \cdot) \in \Pi_r$ and $P(x, x) = f(x)$. A simple computation now gives for $t \neq x$,

$$f(t) = P(x, t) + \frac{[\phi(x+) - \phi(x-)]}{r!}\, \Gamma_r(t, x) + F(x, t), \tag{3.39}$$

where

$$\begin{aligned} F(x, t) &= \frac{1}{(r-1)!} \int_x^t (t-u)^{r-1} g_x(\phi, u)\, du \\ &= \frac{1}{r!} \int_x^t (t-u)^r\, dg_x(\phi, u). \end{aligned} \tag{3.40}$$

If $r=0$, we let $F(x, t) := g_x(\phi, t)$. Since $s_n(w_Q^2, P(x, \cdot), x) = P(x, x) = f(x)$, we see from (3.37) and (3.39) that

$$s_n(w_Q^2, f, x) - \frac{f(x+)+f(x-)}{2} - \frac{[\phi(x+)-\phi(x-)]}{r!}\,\sigma_{n,r}(x)$$
$$= s_n(w_Q^2, F(x, \cdot), x). \tag{3.41}$$

Thus, the proof of Theorem 2.2 consists of an estimation of $s_n(w_Q^2, F(x, \cdot), x)$. We now fix x, and for simplicity of notation, write $F(t)$ instead of $F(x, t)$ and $g(t)$ instead of $g_x(\phi, t)$. Set

$$A^* := \min(A, \tfrac{1}{90}D), \tag{3.42}$$

where A is the constant appearing in Condition (PB) on orthogonal polynomials and D is the constant given by Lemma 3.1(d). Assume that n is so large that

$$|x| \leqslant A^* q_n. \tag{3.43}$$

Further, let

$$L := \tfrac{1}{2}D. \tag{3.44}$$

Next, we observe that (cf. [21], p. 39),

$$s_n(w_Q^2, F, x) = \int_{-\infty}^{\infty} K_n(x, x+t)\, F(x+t)\, w_Q^2(x+t)\, dt$$
$$= I_1 + I_2 + I_3 + I_4 + I_5, \tag{3.45}$$

where, with $H(t) := K_n(x, x+t)\, F(x+t)\, w_Q^2(x+t)$,

$$I_1 := \int_{|t| \leqslant q_n/n} H(t)\, dt \tag{3.46a}$$

$$I_2 := \int_{-\infty}^{-Lq_n} H(t)\, dt, \qquad I_3 := \int_{Lq_n}^{\infty} H(t)\, dt \tag{3.46b}$$

$$I_4 := \int_{-Lq_n}^{-q_n/n} H(t)\, dt, \qquad I_5 := \int_{q_n/n}^{Lq_n} H(t)\, dt. \tag{3.46c}$$

We shall estimate I_1, I_3, I_5, the estimation of I_2 and I_4 being similar to that of I_3 and I_5, respectively. The estimation of I_1 is the simplest. From Lemma 3.1(a), we see that

$$K_n(x, x+t) \leqslant \sqrt{K_n(x, x)} \cdot \sqrt{K_n(x+t, x+t)}$$
$$\leqslant c \cdot \frac{n}{q_n} w_Q^{-1}(x)\, w_Q^{-1}(x+t). \tag{3.47}$$

Hence,

$$|I_1| \leqslant c \cdot \frac{n}{q_n} w_Q^{-1}(x) \int_{|t| \leqslant q_n/n} |F(x+t)|\, w_Q(x+t)\, dt. \tag{3.48}$$

But, from (3.40), if $|t| \leqslant q_n/n$, then

$$|F(x+t)| = \left| \frac{1}{r!} \int_0^t (t-u)^r\, dg(x+u) \right|$$
$$\leqslant c \left(\frac{q_n}{n}\right)^r \int_{-q_n/n}^{q_n/n} |dg(x+u)|. \tag{3.49}$$

Also, if $|t-u| \leqslant 2q_n/n$, (3.43) shows that

$$|Q(x+t) - Q(x+u)| \leqslant c \cdot \frac{q_n}{n} Q'(2A^* q_n) \leqslant c. \tag{3.50}$$

Hence,

$$|F(x+t)|\, w_Q(x+t) \leqslant c \cdot \left(\frac{q_n}{n}\right)^{r+1} \int_{-q_n/n}^{q_n/n} w_Q(x+u)\, |dg(x+u)|$$
$$= c \cdot \left(\frac{q_n}{n}\right)^{r+1} V_Q\left(\left[x - \frac{q_n}{n}, x + \frac{q_n}{n}\right], g\right).$$

Hence, (3.48) yields that

$$|I_1| \leqslant c \cdot w_Q^{-1}(x) \cdot \left(\frac{q_n}{n}\right)^r V_Q\left(\left[x - \frac{q_n}{n}, x + \frac{q_n}{n}\right], g\right). \tag{3.51}$$

Next, we estimate I_3. In view of (3.47),

$$|I_3| = \frac{1}{r!} \left| \int_{Lq_n}^{\infty} K_n(x, x+t)\, w_Q^2(x+t) \int_0^t (t-u)^r\, dg(x+u)\, dt \right|$$
$$\leqslant c \cdot \frac{n}{q_n} w_Q^{-1}(x)(|I_{3,1}| + |I_{3,2}| + |I_{3,3}|), \tag{3.52}$$

where

$$I_{3,1} := \int_{Lq_n}^{\infty} w_Q(x+t) \int_0^{Lq_n/2} |t-u|^r \, |dg(x+u)| \, dt \tag{3.53a}$$

$$I_{3,2} := \int_{Lq_n}^{\infty} w_Q(x+t) \int_{Lq_n/2}^{Lq_n} |t-u|^r \, |dg(x+u)| \, dt \tag{3.53b}$$

$$I_{3,3} := \int_{Lq_n}^{\infty} w_Q(x+t) \int_{Lq_n}^{t} |t-u|^r \, |dg(x-u)| \, dt. \tag{3.53c}$$

If we interchange the order of integration in $I_{3,1}$ and $I_{3,2}$ and then put $v := x+t$, we get

$$I_{3,1} = \int_0^{Lq_n/2} \int_{x+Lq_n}^{\infty} w_Q(v) \, |v-x-u|^r \, dv |dg(x+u)| \tag{3.54a}$$

$$I_{3,2} = \int_{Lq_n/2}^{Lq_n} \int_{x+Lq_n}^{\infty} w_Q(v) \, |v-x-u|^r \, dv |dg(x+u)|. \tag{3.54b}$$

We estimate the inner integral in both of these integrals first. In both $I_{3,1}$ and $I_{3,2}$ we have $|x| \leqslant A^* q_n \leqslant (D/90)\, q_n = (L/45)\, q_n \leqslant (L/8)\, q_n$, $u \leqslant Lq_n$, and $v \geqslant x + Lq_n$, So,

$$\begin{aligned} |v-(x+u)| &\leqslant |v| + |x+u| \leqslant |v| + (A^* + L)\, q_n \\ &\leqslant v + \frac{9}{8} Lq_n = v + \frac{9}{7}\left(Lq_n - \frac{Lq_n}{8}\right) \\ &\leqslant v + \frac{9}{7}(Lq_n + x) \leqslant \frac{16}{7} v. \end{aligned} \tag{3.55}$$

Moreover, $x + Lq_n \geqslant Lq_n - |x| \geqslant \frac{7}{8} Lq_n$. Thus, we may use Lemma 3.1(e) with $\frac{7}{8}L$ in place of D_1 to get

$$\begin{aligned} &\int_{x+Lq_n}^{\infty} |v-(x+u)|^r \, w_Q(v)\, dv \\ &\leqslant c \cdot \int_{x+Lq_n}^{\infty} |v|^r \, w_Q(v)\, dv \\ &\leqslant c \cdot \frac{q_n}{n} (x+Lq_n)^r \, w_Q(x+Lq_n). \end{aligned} \tag{3.56}$$

Substituting this in (3.54), we have

$$|I_{3,1}| \leqslant c \cdot \frac{q_n}{n} |x+Lq_n|^r w_Q(x+Lq_n) \int_0^{Lq_n/2} |dg(x+u)| \tag{3.57a}$$

$$|I_{3,2}| \leqslant c \cdot \frac{q_n}{n} |x+Lq_n|^r w_Q(x+Lq_n) \int_{Lq_n/2}^{Lq_n} |dg(x+u)|. \tag{3.57b}$$

Our next objective is to bring the w_Q term inside the integral. In $I_{3,1}$, $u \leqslant \frac{1}{2}Lq_n$ and $|x| \leqslant \frac{1}{4}Lq_n$. So, we apply Lemma 3.1(f) to get

$$w_Q(x+Lq_n) \leqslant c_1 \exp(-c_2 n)\, w_Q(x+u).$$

Since $|x+Lq_n| \leqslant cq_n$, we now obtain that

$$|I_{3,1}| \leqslant c \cdot \exp(-c_2 n) \int_0^{Lq_n/2} w_Q(x+u)\, |dg(x+u)|. \tag{3.58}$$

In $I_{3,2}$ $x+u \geqslant u-|x| \geqslant (\frac{1}{2}L-(L/8))\, q_n > 0$. So, $w_Q(x+Lq_n) \leqslant w_Q(x+u)$. Also,

$$|x+Lq_n|^r \leqslant cq_n^r \leqslant c(x+u)^r.$$

Hence, (3.57b) gives

$$\begin{aligned} |I_{3,2}| &\leqslant c \cdot \frac{q_n}{n} \int_{Lq_n/2}^{Lq_n} |x+u|^r w_Q(x+u)\, |dg(x+u)| \\ &\leqslant c \cdot \frac{q_n}{n} \int_{x+(L/2)\,q_n}^{\infty} |t|^r w_Q(t)\, |dg(t)|. \end{aligned} \tag{3.59}$$

Next, we estimate $I_{3,3}$. Interchanging the order of integration in (3.53c),

$$I_{3,3} = \int_{Lq_n}^{\infty} \int_{x+u}^{\infty} (v-x-u)^r w_Q(v)\, dv\, |dg(x+u)|. \tag{3.60}$$

Since $x+u \geqslant Lq_n - |x| \geqslant \frac{7}{8}Lq_n$, we may use Lemma 3.1(e) again with $\frac{7}{8}L$ in place of D_1 to get

$$\int_{x+u}^{\infty} (v-x-u)^r w_Q(v)\, dv \leqslant c \cdot \frac{q_n}{n} (x+u)^r w_Q(x+u).$$

Substituting this into (3.60), we get

$$\begin{aligned} |I_{3,3}| &\leqslant c \cdot \frac{q_n}{n} \int_{Lq_n}^{\infty} |x+u|^r w_Q(x+u)\, |dg(x+u)| \\ &\leqslant c \cdot \frac{q_n}{n} \int_{x+(L/2)\,q_n}^{\infty} |t|^r w_Q(t)\, |dg(t)|. \end{aligned} \tag{3.61}$$

In view of (3.58), (3.59), (3.61), and (3.52), we have

$$|I_3| \leqslant c \cdot w_Q^{-1}(x) \left\{ e^{-c_1 n} V_Q \left(\left[x, x + \frac{L}{2} q_n \right], g \right) + \int_{x + (L/2) q_n}^{\infty} t^r w_Q(t) \, |dg(t)| \right\}. \tag{3.62}$$

Next, we turn to the estimation of I_5. This estimation is done using an integration by parts argument similar to the one in [17]. The details are, however, more complicated partly because we need an estimate involving V_Q instead of the ordinary total variation. In view of (3.2), we may write

$$I_5 = \frac{\gamma_{n-1}}{\gamma_n} \{ p_{n-1}(x) \, I_{5,1} - p_n(x) \, I_{5,2} \}, \tag{3.63}$$

where, with the notation (3.12) and (3.13),

$$I_{5,1} := \int_{q_n/n}^{Lq_n} p_n(x+t) G(t) \, w_Q^2(x+t) \, dt \tag{3.64a}$$

$$I_{5,2} := \int_{q_n/n}^{Lq_n} p_{n-1}(x+t) G(t) \, w_Q^2(x+t) \, dt. \tag{3.64b}$$

Now, using the inequality [15]

$$\gamma_{n-1}/\gamma_n \leqslant c q_n \tag{3.65}$$

and Condition (PB), for $|x| \leqslant A^* q_n$,

$$|I_5| \leqslant c \cdot q_n^{1/2} w_Q^{-1}(x) \{ |I_{5,1}| + |I_{5,2}| \}. \tag{3.66}$$

We shall estimate $I_{5,1}$; the estimate for $I_{5,2}$ is similar. With the notation of (3.13),

$$I_{5,1} = \int_{q_n/n}^{Lq_n} (-1)^{r+1} \Lambda_n^{(r+1)}(t) G(t) \, dt. \tag{3.67}$$

Integrating by parts several times,

$$I_{5,1} = \sum_{k=0}^{r} (-1)^k \left[G^{(k)}(Lq_n) \, \Lambda_n^{(r-k)}(Lq_n) - G^{(k)}\left(\frac{q_n}{n}\right) \Lambda_n^{(r-k)}\left(\frac{q_n}{n}\right) \right] + \int_{q_n/n}^{Lq_n} \Lambda_n(t) \, dG^{(r)}(t). \tag{3.68}$$

To estimate the integrated terms, we recall that when $q_n/n \leqslant t \leqslant Lq_n$, $|x+t| \leqslant \frac{1}{8}Lq_n + Lq_n \leqslant Dq_n$. So, Lemma 3.1(g) can be applied to get

$$|G^{(k)}(Lq_n)\, A_n^{(r-k)}(Lq_n)| \leqslant c \cdot q_n^{-1/2} \frac{q_n^r}{n^{k+1}} w_Q(x+Lq_n) \int_0^{Lq_n} |dg(x+u)|. \tag{3.69}$$

$$\left|G^{(k)}\left(\frac{q_n}{n}\right) A_n^{(r-k)}\left(\frac{q_n}{n}\right)\right| \leqslant c \cdot q_n^{-1/2} \left(\frac{q_n}{n}\right)^r w_Q\left(x+\frac{q_n}{n}\right) \int_0^{q_n/n} |dg(x+u)|. \tag{3.70}$$

Now, when $0 \leqslant u \leqslant q_n/n$ and $|x| \leqslant A^* q_n$,

$$\left|Q\left(x+\frac{q_n}{n}\right) - Q(x+u)\right| \leqslant c \cdot \frac{q_n}{n} Q'(A^* q_n) \leqslant c$$

and so,

$$\left|G^{(k)}\left(\frac{q_n}{n}\right) A_n^{(r-k)}\left(\frac{q_n}{n}\right)\right| \leqslant c \cdot q_n^{-1/2} \left(\frac{q_n}{n}\right)^r V_Q\left(\left[x, x+\frac{q_n}{n}\right], g\right). \tag{3.71}$$

The estimation of the right hand side of (3.69) is similar to that of $I_{3,1}$ and $I_{3,2}$ of (3.53). Thus, if $(L/2)\, q_n \leqslant u \leqslant Lq_n$ then

$$x + Lq_n \geqslant x + u \geqslant x + \frac{L}{2} q_n > 0.$$

Then $w_Q(x+Lq_n) \leqslant w_Q(x+u)$ and $q_n^r \leqslant c|x+u|^r$. Hence,

$$\begin{aligned} q_n^r w_Q(x+Lq_n) \int_{Lq_n/2}^{Lq_n} |dg(x+u)| &\leqslant c \int_{Lq_n/2}^{\infty} |x+u|^r\, w_Q(x+u)\, |dg(x+u)| \\ &= c \int_{x+Lq_n/2}^{\infty} |t|^r\, w_Q(t)\, |dg(t)|. \end{aligned} \tag{3.72}$$

If $0 \leqslant u \leqslant (L/2)\, q_n$, then we may apply Lemma 3.1(f) to get

$$w_Q(x+Lq_n) \leqslant c \cdot \exp(-c_1 n)\, w_Q(x+u).$$

So,

$$q_n^r w_Q(x+Lq_n)\int_0^{Lq_n/2} |dg(x+u)| \leqslant c_1 e^{-c_2 n} V_Q\left(\left[x, x+\frac{Lq_n}{2}\right], g\right). \quad (3.73)$$

Substituting from (3.72) and (3.73) in (3.69), we get

$$|G^{(k)}(Lq_n)\,\Lambda_n^{(r-k)}(Lq_n)| \leqslant c q_n^{-1/2}\left\{e^{-c_2 n} V_Q\left(\left[x, x+\frac{Lq_n}{2}\right], g\right)\right.$$
$$\left. + \int_{x+(L/2)q_n}^{\infty} |t|^r\, w_Q(t)\, |dg(t)|\right\}. \quad (3.74)$$

Next, we have to estimate the integral expression in (3.68). Using (3.15) with $k=0$, we see that

$$\left|\int_{q_n/n}^{Lq_n} \Lambda_n(t)\, dG^{(r)}(t)\right| \leqslant c q_n^{-1/2}\left(\frac{q_n}{n}\right)^{r+1}\int_{q_n/n}^{Lq_n} w_Q(x+t)\, |dG^{(r)}(t)|. \quad (3.75)$$

An application of Leibnitz's formula now yields that

$$\int_{q_n/n}^{Lq_n} w_Q(x+t)\, |dG^{(r)}(t)|$$
$$\leqslant c\int_{q_n/n}^{Lq_n} \frac{w_Q(x+t)}{t^2}\int_0^t |dg(x+u)|\, dt$$
$$+ c\int_{q_n/n}^{Lq_n} \frac{w_Q(x+t)\, |dg(x+t)|}{t}. \quad (3.76)$$

The last integral above can be estimated as usual. We integrate by parts and get

$$\int_{q_n/n}^{Lq_n} \frac{w_Q(x+t)\, |dg(x+t)|}{t}$$
$$\leqslant \frac{1}{Lq_n}\int_0^{Lq_n} w_Q(x+t)\, |dg(x+t)|$$
$$+ \frac{n}{q_n}\int_0^{q_n/n} w_Q(x+t)\, |dg(x+t)|$$
$$+ \int_{q_n/n}^{Lq_n} t^{-2}\int_0^t w_Q(x+u)\, |dg(x+u)|\, dt. \quad (3.77)$$

The first integral on the right hand side of (3.76) is easy to estimate when $x \geqslant 0$. In this case, since w_Q is a decreasing function on $(0, \infty)$, $w_Q(x+t) \leqslant w_Q(x+u)$ for all u with $0 \leqslant u \leqslant t$. Then

$$\int_{q_n/n}^{Lq_n} w_Q(x+t)\, t^{-2} \int_0^t |dg(x+u)|\, dt$$

$$\leqslant \int_{q_n/n}^{Lq_n} t^{-2} \int_0^t w_Q(x+u)\, |dg(x+u)|\, dt. \tag{3.78}$$

When $x < 0$, the estimate is somewhat more difficult. If $0 \leqslant u \leqslant t \leqslant 2|x|$ then

$$|Q(x+t) - Q(x+u)| \leqslant cxQ'(x).$$

So,

$$\int_{q_n/n}^{2|x|} \frac{w_Q(x+t)}{t^2} \int_0^t |dg(x+u)|\, dt$$

$$\leqslant c_1 \exp(c_2 xQ'(x)) \int_{q_n/n}^{2|x|} \frac{1}{t^2} \int_0^t w_Q(x+u)\, |dg(x+u)|\, dt. \tag{3.79}$$

If $t \geqslant 2|x|$, but $0 \leqslant u \leqslant |x|$, then

$$0 \leqslant |x+u| = -x-u \leqslant -x = |x| \leqslant 2|x| + x \leqslant x+t.$$

So, $w_Q(x+t) \leqslant w_Q(x+u)$ and we get

$$\int_{2|x|}^{Lq_n} \frac{w_Q(x+t)}{t^2} \int_0^{|x|} |dg(x+u)|\, dt$$

$$\leqslant \int_{2|x|}^{Lq_n} t^{-2} \int_0^{|x|} w_Q(x+u)\, |dg(x+u)|\, dt. \tag{3.80}$$

Finally, if $t \geqslant 2|x|$ and $t \geqslant u \geqslant |x|$, then $0 \leqslant x+u \leqslant x+t$ and hence $w_Q(x+t) \leqslant w_Q(x+u)$. So,

$$\int_{2|x|}^{Lq_n} \frac{w_Q(x+t)}{t^2} \int_{|x|}^t |dg(x+u)|\, dt$$

$$\leqslant \int_{2|x|}^{Lq_n} t^{-2} \int_{|x|}^t w_Q(x+u)\, |dg(x+u)|\, dt. \tag{3.81}$$

Adding (3.79), (3.80), and (3.81), we get

$$\int_{q_n/n}^{Lq_n} \frac{w_Q(x+t)}{t^2} \int_0^t |dg(x+u)|\, dt$$

$$\leqslant c \cdot \exp(c_1 x Q'(x)) \int_{q_n/n}^{Lq_n} t^{-2} \int_0^t w_Q(x+u)\, |dg(x+u)|\, dt \quad (3.82)$$

when $x < 0$. In view of (3.78), we see that (3.82) holds for all $x \in \mathbb{R}$. We now substitute from (3.82) and (3.77) into (3.76) and then use the resulting estimate in (3.75) to get

$$\left| \int_{q_n/n}^{Lq_n} \Lambda_n(t)\, dG^{(r)}(t) \right|$$

$$\leqslant c \cdot q_n^{-1/2} \left(\frac{q_n}{n}\right)^r \exp(c_1 x Q'(x))$$

$$\cdot \left\{ \frac{1}{n} V_Q([x, x+Lq_n], g) + V_Q\left(\left[x, x+\frac{q_n}{n}\right], g\right) \right.$$

$$\left. + \frac{q_n}{n} \int_{q_n/n}^{Lq_n} \frac{V_Q([x, x+t], g)}{t^2}\, dt \right\}. \quad (3.83)$$

A routine computation now yields that

$$\left| \int_{q_n/n}^{Lq_n} \Lambda_n(t)\, dG^{(r)}(t) \right|$$

$$\leqslant c \cdot \exp(c_1 x Q'(x))\, q_n^{-1/2} \left(\frac{q_n}{n}\right)^r \cdot \frac{1}{n} \sum_{k=1}^{n} V_Q\left(\left[x, x+\frac{q_n}{k}\right], g\right). \quad (3.84)$$

In view of (3.71), (3.74), (3.75), (3.84), and (3.68),

$$|I_{5,1}| \leqslant c \cdot \exp(c_1 x Q'(x))\, q_n^{-1/2} \left(\frac{q_n}{n}\right)^r \cdot \frac{1}{n} \sum_{k=1}^{n} V_Q\left(\left[x, x+\frac{q_n}{k}\right], g\right)$$

$$+ c q_n^{-1/2} \int_{x+Lq_n/2}^{\infty} |t|^r\, w_Q(t)\, |dg(t)|. \quad (3.85)$$

We estimate $I_{5,2}$ in a similar fashion and use this estimate along with (3.85) in (3.66) to get

$$|I_5| \leqslant c(x) \left\{ \left(\frac{q_n}{n}\right)^r \frac{1}{n} \sum_{k=1}^{n} V_Q\left(\left[x, x+\frac{q_n}{k}\right], g\right) \right.$$

$$\left. + \int_{(L/4)\, q_n}^{\infty} |t|^r\, w_Q(t)\, |dg(t)| \right\}. \quad (3.86)$$

Here we have also used the fact that $x+(L/2)\,q_n \geqslant ((L/2)-A^*)\,q_n \geqslant (L/4)\,q_n$. The estimations for I_2 and I_4 are done in the same way as for I_3 and I_5, respectively. When we use these estimations along with (3.87), (3.62), and (3.51) in (3.45), we see that

$$|s_n(w_Q^2, F, x)| \leqslant c(x)\left\{\left(\frac{q_n}{n}\right)^r \frac{1}{n}\sum_{k=1}^{n} V_Q\left(\left[x-\frac{q_n}{k}, x+\frac{q_n}{k}\right], g\right) + \int_{|t| \geqslant (L/4)\,q_n} |t|^r\, w_Q(t)\, |dg(t)|\right\}. \tag{3.87}$$

In view of (3.41), this completes the proof of Theorem 2.2.

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