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Fourth-order differential equations satisfied by the generalized co-recursive of all classical orthogonal polynomials. A study of their distribution of zeros

A. Ronveaux [a], A. Zarzo [b,*], E. Godoy [c]

[a] *Mathematical Physics, Facultés Universitaires Notre-Dame de la Paix, B-5000 Namur, Belgium*

[b] *Departamento de Matemática Aplicada, Escuela Técnica Superior de Ingenieros Industriales, Universidad Politécnica de Madrid, José Gutiérrez Abascal, 2, 28006 Madrid, Spain*

[c] *Departamento de Matemática Aplicada, Escuela Técnica Superior de Ingenieros Industriales, Universidad de Vigo, Apartado de correos 62, 36280 Vigo, Spain*

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Abstract

The unique fourth-order differential equation satisfied by the generalized co-recursive of all classical orthogonal polynomials is given for any (but fixed) level of recursivity. Up to now, these differential equations were known only for each classical family separately and also for a specific recursivity level. Moreover, we use this unique fourth-order differential equation in order to study the distribution of zeros of these polynomials via their Newton sum rules (i.e., the sums of powers of their zeros) which are closely related with the moments of such distribution. Both results are obtained with the help of two programs built in *Mathematica* symbolic language.

Keywords: Orthogonal polynomials; Differential equations; Zeros; Special functions

1. Introduction

We start by considering the well-known sequences $\{P_n(x)\}_{n=0}^{\infty}$ of orthogonal polynomials belonging to the classical class (Jacobi, Laguerre, Hermite, Bessel). Therefore, they satisfy a three-term recurrence relation (TTRR, in short) which for monic families can be written as [5]

$$\begin{aligned} &P_{n+1}(x) = (x-\beta_n)P_n(x) - \gamma_n P_{n-1}(x), \quad n \geqslant 1, \\ &P_0(x) = 1, \quad P_1(x) = x - \beta_0, \qquad \gamma_n \neq 0, \end{aligned} \tag{1}$$

* Corresponding author. e-mail: c0550001@vec.ccupm.upm.es.

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and they are solutions of a second-order differential equation of hypergeometric type [23]:

$$L_2 P_n \equiv \sigma P_n'' + \tau P_n' + \lambda_n P_n = 0, \tag{2}$$

where $\sigma \equiv \sigma(x)$ is a polynomial in x of degree at most 2, $\tau \equiv \tau(x)$ is also a polynomial in x of degree 1 and $\lambda_n = -\frac{1}{2}n[(n-1)\sigma'' + 2\tau']$ is a constant.

Also, they can be expressed by means of a Rodrigues formula [23]

$$P_n(x) = \frac{C_n}{\omega(x)} \frac{\mathrm{d}^n}{\mathrm{d}x^n}[\sigma(x)^n \omega(x)], \quad [\sigma(x)\omega(x)]' = \tau(x)\omega(x), \tag{3}$$

where, for monic polynomials,

$$C_n = \frac{1}{\prod_{j=0}^{n-1}\{\tau' + \frac{1}{2}(n+j-1)\sigma''\}}.$$

Here $\omega(x)$ denotes the symmetrization factor of the differential equation (2) which transforms it in its self-adjoint form. This function $\omega(x)$ is also the usual (positive) weight when normalized to one ($\int_I w(x)\,\mathrm{d}x = 1$).

The corresponding generalized co-recursive polynomials at level k ($\{P_n^*(x;\nu;k)\}_{n=0}^{\infty}$, $k \in \mathbb{N}$, $\nu \in \mathbb{R}$) are the polynomials which satisfy a TTRR obtained from (1) by replacing β_k by $\beta_k + \nu$ [21]. So, the TTRR which completely characterizes the modified family is

$$\begin{aligned} &P_{n+1}^*(x;\nu;k) = (x - \beta_n^*)P_n^*(x;\nu;k) - \gamma_n^* P_{n-1}^*(x;\nu;k), \quad n \geqslant 0, \\ &P_{-1}^*(x;\nu;k) = 0, \qquad P_0^*(x;\nu;k) = 1, \end{aligned} \tag{4}$$

where the coefficients β_n^*, γ_n^* are related to the starting ones by

$$\begin{aligned} &\beta_j^* = \beta_j + \nu, \quad j = k, \quad k \in \mathbb{N}, \nu \in \mathbb{R}, \\ &\beta_j^* = \beta_j, \quad \forall j \neq k, \\ &\gamma_j^* = \gamma_j, \quad j \geqslant 0. \end{aligned}$$

Then, the following relations between the old and new polynomials hold:

$$\begin{aligned} &P_{n+1}^*(x;\nu;k) = P_{n+1}(x), \quad 0 \leqslant n \leqslant k-1, \\ &P_{k+1}^*(x;\nu;k) = P_{k+1}(x) - \nu P_k(x), \quad n = k. \end{aligned}$$

Notice that, with the notation we have used here, the case $k = 0$ corresponds to the usual co-recursive first defined by Chihara [4].

On the other hand, it is well known that these polynomials belong to the Laguerre–Hahn class [9,10,21,30,32]. In particular this means that they are solutions of a fourth-order linear differential equation of the following form:

$$\sum_{i=0}^{4} G_i Y^{(i)} = 0, \tag{5}$$

where $Y \equiv P_n^*(x;\nu;k)$ and the coefficients $G_i = G_i(x;\nu;k;n)$ are polynomials of degree independent of n, k and ν.

Due to the relevant role that these modified polynomials play in several branches of mathematics and physics (see, e.g., [4,16,21,34]), the research activity on them has been very intensive during the last few years [4,10,18,19,21,25,29–32,34]. The generalized co-recursive at level $k = 0$ (commonly known as co-recursive polynomials) were introduced for the first time by Chihara [4] in order to generalize works on Lommel and Bessel polynomials [8]. Extensions to the co-recursive of the semi-classical and Laguerre–Hahn classes were given in [9,10,31] and in [21], where co-recursivity at any level k was introduced. Concerning the fourth-order differential equation (5), an algorithm to obtain it for each classical orthogonal polynomial family has been given in [30,32]. Moreover, when $k = 0$, an alternative approach based on the Orr method [24,35] can be found in [18] (where co-recursive associated polynomials were introduced) for the associated and co-recursive Laguerre polynomials and in [19] for the associated and co-recursive Jacobi polynomials. Extensions to the semi-classical class are in [9,10,31] where the Stieltjes function is also investigated. About the orthogonality measures, they could be obtained as in [25] for the generalized co-recursive Jacobi. In the case $k = 0$ it is given in [18] for co-recursive Laguerre and in [19] for co-recursive Jacobi as particular cases of the so-called co-recursive associated [18,19].

In this work we have two different aims. Firstly, in Section 2 the unique fourth-order differential equation satisfied by the generalized co-recursive polynomials at level k of the classical class is given for any previously specified k. The algorithm presented in [30,32] together with the Rodrigues formula (3) allow to obtain the coefficients G_i of (5) in terms of the polynomials σ and τ in (2), which completely characterize the classical class. From this algorithm a program built in *Mathematica* symbolic language [36] is introduced. It is able to give the aforementioned unique fourth-order differential equation. Also it can be used to obtain (5) for each classical orthogonal polynomial family separately.

Secondly, in Section 3 we study the distribution of zeros of the generalized co-recursive polynomials of each classical sequence. If we denote by $\{x_{n,i}^{(\nu,k)}\}_{i=1}^{n}$ the zeros of $P_n^*(x;\nu;k)$, this density function is defined as follows:

$$\rho_n^*(x;\nu;k) = \frac{1}{n}\sum_{i=1}^{n}\delta(x - x_{n,i}^{(\nu,k)}). \tag{6}$$

More precisely, starting from the differential equation satisfied by $P_n^*(x;\nu;k)$, we can generate in an exact and recurrent way [2,13,37,38] the Newton sum rules of their zeros $N_j^*(n,\nu,k)$, $j = 0,1,2,\ldots,n$, or, equivalently, the moments around the origin of the distribution (6)

$$\mu_j^*(n,\nu,k) = \frac{1}{n}N_j^*(n,\nu,k) = \frac{1}{n}\sum_{i=1}^{n}x^j\delta(x - x_{n,i}^{(\nu,k)}) = \frac{1}{n}\sum_{i=1}^{n}[x_{n,i}^{(\nu,k)}]^j, \quad j = 0,1,\ldots,n. \tag{7}$$

In doing this we have also used a program built in *Mathematica* symbolic language [36] which was introduced in [38].

The moments (7) were also obtained in [21] by using a completely different approach based on the TTRR (4) [6,7]. Here we reproduce the same results for the generalized co-recursive of the classical class, but using as starting point the differential equation (5).

2. Fourth-order differential equations

2.1. Algorithm

Following the ideas in [30,32], we first consider the associated polynomials of order r corresponding to the classical class. Denoting them by $\{P_n^{(r)}(x)\}_{n\in\mathbb{N}}$, they satisfy the TTRR

$$P_{n+1}^{(r)}(x) = (x - \beta_{n+r})P_n^{(r)}(x) - \gamma_{n+r}P_{n-1}^{(r)}(x), \quad n \geqslant 1,$$
$$P_0^{(r)}(x) = 1, \qquad P_1^{(r)}(x) = x - \beta_r,$$

where β_j, γ_j are the coefficients of the TTRR (1).

Then, the generalized co-recursive polynomials (solutions of the TTRR (4)) can be expressed in terms of the associated ones in the following form [21]:

$$\begin{aligned} &P_n^*(x;\nu;k) = P_n(x), \quad n \leqslant k, \\ &P_n^*(x;\nu;k) = P_n(x) - \nu P_k(x)P_{n-(k+1)}^{(k+1)}(x), \quad n \geqslant k+1, \end{aligned} \tag{8}$$

where $P_{n-r}^{(r)}(x)$ is the rth associated of $P_n(x)$ of degree $n-r$.

Since the associated polynomial $P_n^{(r)}(x)$ of order r is available [9] from $P_{n+r-1}^{(1)}(x)$ and $P_{n+r}(x)$ in the form

$$P_n^{(r)}(x) = M(r,x)P_{n+r-1}^{(1)}(x) + N(r,x)P_{n+r}(x), \quad n \geqslant 0, \ r \geqslant 2,$$

where

$$M(r,x) = \left(\prod_{j=1}^{r-1}\gamma_j\right)^{-1} P_{r-1}(x), \quad N(r,x) = -\left(\prod_{j=1}^{r-1}\gamma_j\right)^{-1} P_{r-2}^{(1)}(x), \qquad r \geqslant 2,$$

it follows from (8) that

$$P_n^*(x;\nu;k) = A(x)P_{n-1}^{(1)}(x) + B(x)P_n(x), \quad n \geqslant k+1, \tag{9}$$

where

$$A(x) = -\nu\left(\prod_{j=1}^{k}\gamma_j\right)^{-1} P_k^2(x), \qquad B(x) = 1 + \nu\left(\prod_{j=1}^{k}\gamma_j\right)^{-1} P_k(x)P_{k-1}^{(1)}(x). \tag{10}$$

These two polynomials $A(x)$ and $B(x)$ are essentially the same as the polynomials $C(x)$ and $D(x)$ of [25, Eq. (4.8)] from which the absolutely continuous part of the orthogonality measure of the $\{P_n^*\}$ family can be computed (see [25, Eq. (3.33)]) when the starting family $\{P_n\}$ is the Jacobi one.

On the other hand, notice that when $k = 0$, Eq. (9) gives rise to the well-known relation [5,21]

$$P_n^*(x;\nu;0) = P_n(x) - \nu P_{n-1}^{(1)}(x), \quad k = 0.$$

The first associated polynomials satisfy [14,15,28]

$$L_2^*[P_{n-1}^{(1)}(x)] = KP_n'(x),$$

where L_2^* is the formal adjoint of the differential operator L_2 defined in (2) and $K = \sigma'' - 2\tau'$. Then, if we denote by $y(x) = AP_{n-1}^{(1)}(x)$, $A = A(x)$ being the function defined in (10), we have the following identity:

$$L_2^*\left(\frac{y}{A}\right) = KP_n'(x),$$

and from it [1,30]

$$R_2(y) \equiv R_2[A(x)P_n^{(1)}(x)] = KA(x)^3P_n'(x), \tag{11}$$

where

$$R_2 \equiv \sigma A^2D^2 + (A^2(2\sigma' - \tau) - 2AA'\sigma)D$$
$$+ (2\sigma(A')^2 - \sigma AA'' - (2\sigma' - \tau)AA' + A^2(\sigma'' - \tau' + \lambda_n))I_d.$$

We can now apply the operator R_2 to $P_n^*(x;\nu;k)$ as given in (9). Taking into account (11), we obtain

$$R_2[P_n^*(x;\nu;k)] = R_2[P_n(x)] + KA^3P_n'(x) + \nu\left(\prod_{j=1}^{k}\gamma_j\right)^{-1} R_2[P_k(x)P_{k-1}^{(1)}(x)P_n(x)].$$

From this it follows that when k is specified, the polynomial $P_k(x)P_{k-1}^{(1)}(x)$ can be computed and the right-hand side of the above equation can be written as a combination of $P_n(x)$ and $P_n'(x)$ in the form

$$R_2[P_n^*(x;\nu;k)] = A_0(x)P_n(x) + B_0(x)P_n'(x), \quad n \geqslant k+1, \tag{12}$$

being

$$A_0(x) = (2\sigma(A')^2 - \sigma AA'' - (2\sigma' - \tau)AA' + A^2(\sigma'' - \tau' + \lambda_n))$$
$$- \lambda_n A^2 + \nu\left(\prod_{j=1}^{k}\gamma_j\right)^{-1} U(x,n),$$

$$B_0(x) = A^2(2\sigma' - 2\tau) - 2AA'\sigma + KA^3 + \nu\left(\prod_{j=1}^{k}\gamma_j\right)^{-1} V(x,n),$$

with

$$U(x,n) = \sigma A^2H''(x) + (A^2(2\sigma' - \tau) - 2AA'\sigma)H'(x)$$
$$+ ((2\sigma(A')^2 - \sigma AA'' - (2\sigma' - \tau)AA' + A^2(\sigma'' - \tau' + \lambda_n)) - \lambda_n A^2)H(x),$$

$$V(x,n) = (A^2(2\sigma' - 2\tau) - 2AA'\sigma)H(x) + 2\sigma A^2H'(x).$$

Here the notation $H(x) = P_k(x)P_{k-1}^{(1)}(x)$ has been introduced and the differential equation $L_2[P_n] = 0$ has been used in order to eliminate $P_n''(x)$ in $R_2[P_n(x)]$ and also in $R_2[P_k(x)P_{k-1}^{(1)}(x)P_n(x)]$.

From (12) and using again $L_2[P_n] = 0$, the following two expressions are obtained:

$$\sigma \frac{\mathrm{d}}{\mathrm{d}x}\{R_2[P_n^*(x;\nu;k)]\} = A_1(x)P_n(x) + B_1(x)P_n'(x), \quad n \geqslant k+1, \tag{13}$$

where

$$A_1(x) = \sigma A_0'(x) - \lambda_n B_0(x), \qquad B_1(x) = \sigma A_0(x) + \sigma B_0'(x) - \tau B_0(x),$$

and

$$\sigma \frac{\mathrm{d}}{\mathrm{d}x}\left\{\sigma \frac{\mathrm{d}}{\mathrm{d}x}\{R_2[P_n^*(x;\nu;k)]\}\right\} = A_2(x)P_n(x) + B_2(x)P_n'(x), \quad n \geqslant k+1, \tag{14}$$

where

$$A_2(x) = \sigma A_1'(x) - \lambda_n B_1(x), \qquad B_2(x) = \sigma A_1(x) + \sigma B_1'(x) - \tau B_1(x).$$

Finally, (12)–(14) give the searched fourth-order differential equation which can be expressed in the following determinantal form:

$$\begin{vmatrix} A_0 & B_0 & R_2[P_n^*(x;\nu;k)] \\ A_1 & B_1 & \sigma(R_2[P_n^*(x;\nu;k)])' \\ A_2 & B_2 & \sigma(\sigma(R_2[P_n^*(x;\nu;k)])')' \end{vmatrix} = 0, \quad n \geqslant k+1. \tag{15}$$

On the other hand, the first associated $P_{n-1}^{(1)}(x)$ is available from $P_n(x)$ [5] in the form

$$P_n^{(1)}(x) = \frac{P_{n+1}(x)P_{n-1}^{(1)}(x) + \prod_{j=1}^{n}\gamma_j}{P_n(x)}, \qquad P_{-1}^{(1)}(x) = 0, \qquad P_0^{(1)}(x) = 1 \tag{16}$$

and the γ-coefficients of the starting TTRR (1) have been obtained in [20] in terms of σ and τ as follows:

$$\gamma_n = \frac{n(n\pi_2 + \eta_1)(n(\pi_1^2 - 4\pi_0\pi_2)(n\pi_2 + \eta_1) + \pi_1\eta_0\eta_1 - \pi_2\eta_0^2 - \pi_0\eta_1^2)}{((2n-1)\pi_2 + \eta_1)(2n\pi_2 + \eta_1)^2((2n+1)\pi_2 + \eta_1)}, \tag{17}$$

with

$$\pi_0 = \tfrac{1}{2}\sigma''x^2 - \sigma'x + \sigma, \qquad \pi_1 = -\sigma''x + \sigma', \qquad \pi_2 = \tfrac{1}{2}\sigma'',$$

$$\eta_0 = (\sigma'' - \tau')x - \sigma' + \tau, \qquad \eta_1 = \tau' - \sigma''.$$

So the only thing which is left in order to give the coefficients of the differential equation (15) in terms of σ and τ is to obtain the classical orthogonal polynomials $P_n(x)$ as a function of them. This expression is provided by the Rodrigues formula (3) which can be rewritten by using the so-called Bell polynomials $F_n(x) = Y_n(y_1, y_2, \ldots, y_n)$ [27, p. 37], in order to compute $\omega^{(k)}/\omega$ in terms of $\omega'/\omega = (\tau - \sigma')/\sigma$. Choosing

$$y_i = \frac{\mathrm{d}^{i-1}}{\mathrm{d}x^{i-1}}\left[\frac{\omega'}{\omega}\right] = \frac{\mathrm{d}^{i-1}}{\mathrm{d}x^{i-1}}\left[\frac{\tau - \sigma'}{\sigma}\right], \quad i = 1, 2, \ldots n,$$

one has the following recurrence relation [27]:

$$F_0(x) = 1, \qquad F_1(x) = \frac{\tau(x) - \sigma'(x)}{\sigma(x)},$$
$$F_n(x) = \sum_{j=0}^{n-1} \binom{n-1}{j} \left(\frac{\mathrm{d}^{n-1-j}}{\mathrm{d}x^{n-1-j}} (F_1(x)) \right) F_j(x)$$

and the Rodrigues formula (3) becomes

$$\begin{aligned} P_n(x) = {} & \frac{1}{\prod_{j=0}^{n-1} \{\tau' + \frac{1}{2}(n+j-1)\sigma''\}} \\ & \times \left\{ \sum_{j=0}^{n-1} \binom{n}{j} \left(\frac{\mathrm{d}^j}{\mathrm{d}x^j} (\sigma^{n-1}(x)) \right) (\tau(x) F_{n-j-1}(x) + (n-j-1)\tau'(x) F_{n-j-2}(x)) \right. \\ & \left. + \left(\frac{\mathrm{d}^n}{\mathrm{d}x^n} (\sigma^{n-1}(x)) \right) \sigma(x) \right\}. \end{aligned} \tag{18}$$

It is then clear that (16)–(18) allow to write explicitly the differential equation (15) (for a specific k) in terms of the polynomials σ and τ which completely characterize the classical class of orthogonal polynomials. So we are able to give *the unique fourth-order differential equation* (5) *satisfied by the generalized co-recursive at any* (*but previously specified*) *level of the classical class.*

Of course, the calculations involved are very difficult, but the algorithm has been devised in such a way that it can be implemented in any computer algebra system just by using (15) together with (16)–(18). In fact, we have constructed a *Mathematica* [36] program, we have called DiffEqCorecursive.m, which is able to give the above mentioned fourth-order differential equations and also it can calculate the same equations but for each classical family separately.

This is done by means of the *Mathematica* function

CoeffEq[i_,x_,k_,ν_,ST_List] $\equiv G_i(x;\nu;k;n)$,

where G_i, $i = 0, 1, \ldots, 4$, are the coefficients of the fourth-order differential equation (5) and the last argument is: ST_List $\equiv \{\sigma(x), \tau(x)\}$.

In the next paragraph, the way in which this function works into a *Mathematica* session and the fourth-order differential equations are given.

2.2. *Results*

Once a *Mathematica* session is started, the first instruction we have to type in is:

In[1]:= << DiffEqCorecursive.m.

This instruction makes available every function contained in the program. Then, the coefficients G_i, $i = 0, 1, \ldots, 4$, can be obtained for specific values of the level k of co-recursivity. As an illustration we consider here the cases $k = 0$ and $k = 1$. The same procedure could be used for any k, but this will be done elsewhere.

Co-recursive at level $k=0$ of the classical class

The *Mathematica* function

$$\text{CoeffEq[i,x,0,}\nu\text{,}\{\sigma(x),\tau(x)\}] \equiv G_i(x;\nu;0;n), \quad i=0,1,2,3,4,$$

gives the searched coefficients of the differential equation (5). For writing them down we introduce the coefficients $K_{pq}^{(t)}$ which appear in G_p and the indexes q and t are related to the powers of ν and σ, respectively. After eliminating the common factor $\frac{1}{8}\nu^6\sigma$ of every G-coefficient, their expressions are:

$$G_4(x;\nu;0;n) = 4\sigma^2\{K_{42}^{(0)}\nu^2 + K_{41}^{(0)}\nu + (K_{40}^{(0)} + K_{40}^{(1)}\sigma)\},$$

where

$$K_{42}^{(0)} = n(\sigma'' - 2\tau')^2((1-n)\sigma'' - 2\tau'),$$

$$K_{41}^{(0)} = 2(\sigma'' - 2\tau')(4n\sigma'\tau' + (1+2n-2n^2)\tau\sigma'' + 2n(n-1)\sigma'\sigma'' - (4n+1)\tau\tau'),$$

$$K_{40}^{(0)} = 4(\sigma' - \tau)((2n+1)\tau\tau' - 2n\sigma'\tau' + (n^2-n-1)\tau\sigma'' + n(1-n)\sigma'\sigma''),$$

$$K_{40}^{(1)} = 6(\sigma'' - \tau')^2;$$

$$G_3(x;\nu;0;n) = 4\sigma\{K_{32}^{(0)}\nu^2 + 2(2\tau' - \sigma'')[K_{31}^{(0)} + K_{31}^{(1)}\sigma]\nu + 4(K_{30}^{(0)} + K_{30}^{(1)}\sigma)\},$$

where

$$K_{32}^{(0)} = 5n\sigma'(\sigma'' - 2\tau')^2((1-n)\sigma'' - 2\tau'),$$

$$K_{31}^{(0)} = 5\sigma'((4n+1)\tau\tau' - 4n\sigma'\tau' + (2n^2-2n-1)\tau\sigma'' + 2n(1-n)\sigma'\sigma''),$$

$$K_{31}^{(1)} = (\sigma'' - \tau')[(4n+1)\tau' + 2n(n-1)\sigma''],$$

$$K_{30}^{(0)} = 5\sigma'(\sigma' - \tau)((2n+1)\tau\tau' - 2n\sigma'\tau' + (n^2-n-1)\tau\sigma'' + n(1-n)\sigma'\sigma''),$$

$$K_{30}^{(1)} = (\sigma'' - \tau')((4n-5)\sigma'\tau' - 2(2n+1)\tau\tau' + (1+2n-2n^2)\tau\sigma'' + 2(3-n+n^2)\sigma'\sigma'');$$

$$G_2(x;\nu;0;n) = 4\{K_{22}^{(0)}\nu^2 + (\sigma'' - 2\tau')[K_{21}^{(0)} + K_{21}^{(1)}\sigma]\nu + 2[K_{20}^{(0)} + K_{20}^{(1)}\sigma + K_{20}^{(2)}\sigma^2]\},$$

where

$$K_{22}^{(0)} = n(2\tau' - \sigma'')^2(2\tau' + (n-1)\sigma'')(\tau^2 - 2\tau\sigma' - 3\sigma'^2 + 2(n+1)\sigma\tau' + (n^2-n-6)\sigma\sigma''),$$

$$K_{21}^{(0)} = 2(3\sigma' - \tau)(\sigma' + \tau)(-(4n+1)\tau\tau' + 4n\sigma'\tau' + (1+2n-2n^2)\tau\sigma'' + 2n(n-1)\sigma'\sigma''),$$

$$\begin{aligned} K_{21}^{(1)} = {} & 4(1+4n+4n^2)\tau\tau'^2 + 2(3+7n-8n^2)\sigma'\tau'^2 + 8(2n^3-n^2-7n-2)\tau\tau'\sigma'' \\ & + (-16n^3+23n^2+11n-6)\sigma'\tau'\sigma'' + 4(n^4-2n^3-5n^2+6n+3)\tau\sigma''^2 \\ & + (-4n^4+8n^3+5n^2-9n)\sigma'\sigma''^2, \end{aligned}$$

$$K_{20}^{(0)} = 2(\tau - 3\sigma')(\tau^2 - \sigma'^2)((2n+1)\tau\tau' - 2n\sigma'\tau' + (n^2-n-1)\tau\sigma'' + n(1-n)\sigma'\sigma''),$$

$$\begin{aligned} K_{20}^{(1)} = {} & (8n^2+8n-1)\tau^2\tau'^2 + (17+14n-16n^2)\tau\sigma'\tau'^2 + 2(4n^2-11n-3)\sigma'^2\tau'^2 \\ & + 2(4n^3-2n^2-14n-3)\tau^2\tau'\sigma'' + (-16n^3+23n^2+11n-20)\tau\sigma'\tau'\sigma'' \\ & + (8n^3-19n^2+17n+6)\sigma'^2\tau'\sigma'' + (2n^4-4n^3-10n^2+12n+7)\tau^2\sigma''^2 \\ & + (3-9n+5n^2+8n^3-4n^4)\tau\sigma'\sigma''^2 + (2n^4-4n^3+5n^2-3n)\sigma'^2\sigma''^2, \end{aligned}$$

$$K_{20}^{(2)} = 9(\sigma''-\tau')^2(-(2n+1)\tau' + (2+n-n^2)\sigma'');$$

$$G_1(x;\nu;0;n) = 2\{K_{12}^{(0)}\nu^2 + 2(\sigma''-2\tau')[K_{11}^{(0)} + K_{11}^{(1)}\sigma]\nu + 4[K_{10}^{(0)} + K_{10}^{(1)}\sigma]\},$$

where

$$K_{12}^{(0)} = 3n(2\tau'-\sigma'')^2(2\tau' + (n-1)\sigma'')(2\tau\tau' + 2n\sigma'\tau' - 2\tau\sigma'' + (n^2-n-2)\sigma'\sigma''),$$

$$\begin{aligned} K_{11}^{(0)} = {} & 2(2(1+5n)\tau^2\tau'^2 + 2(1-4n+6n^2)\tau\sigma'\tau'^2 + 6n(1-2n)\sigma'^2\tau'^2 \\ & + 5(n^2-3n-1)\tau^2\tau'\sigma'' + (12n^3-16n^2-5)\tau\sigma'\tau'\sigma'' + 3n(1+5n-4n^2)\sigma'^2\tau'\sigma'' \\ & + (3+5n-5n^2)\tau^2\sigma''^2 + (3+2n+n^2-6n^3+3n^4)\tau\sigma'\sigma''^2 \\ & + 3n(-1+2n^2-n^3)\sigma'^2\sigma''^2), \end{aligned}$$

$$K_{11}^{(1)} = (2+n)(\sigma''-\tau')(2\tau' + (n-3)\sigma'')((4n+1)\tau' + 2n(n-1)\sigma''),$$

$$\begin{aligned} K_{10}^{(0)} = {} & (\sigma'-\tau)(-4(1+2n)\tau^2\tau'^2 + 4(-1+n-3n^2)\tau\sigma'\tau'^2 + 12n(n-1)\sigma'^2\tau'^2 \\ & + 2(5+6n-2n^2)\tau^2\tau'\sigma'' + 2(5+3n+7n^2-6n^3)\tau\sigma'\tau'\sigma'' \\ & + 6n(1-3n+2n^2)\sigma'^2\tau'\sigma'' + 2(2n^2-2n-3)\tau^2\sigma''^2 \\ & + (-6-4n+n^2+6n^3-3n^4)\tau\sigma'\sigma''^2 + 3n^2(1-2n+n^2)\sigma'^2\sigma''^2), \end{aligned}$$

$$\begin{aligned} K_{10}^{(1)} = {} & (\sigma''-\tau')(2(7+4n+4n^2)\tau\tau'^2 + 4(-1+3n-2n^2)\sigma'\tau'^2 \\ & + (8n^3-4n^2-18n-31)\tau\tau'\sigma'' + 2(3-6n+7n^2-4n^3)\sigma'\tau'\sigma'' \\ & + (15+7n-5n^2-4n^3+2n^4)\tau\sigma''^2 + n(3-5n+4n^2-2n^3)\sigma'\sigma''^2); \end{aligned}$$

$$\begin{aligned} G_0(x;\nu;0;n) = {} & n(n+1)(2\tau' + (n-2)\sigma'')(2\tau' + (n-1)\sigma'') \\ & \times \{K_{02}^{(0)}\nu^2 + 2(\sigma''-2\tau')K_{01}^{(0)}\nu + 2[K_{00}^{(0)} + K_{00}^{(1)}\sigma]\}, \end{aligned}$$

where

$$K_{02}^{(0)} = (1-n)(\sigma''-2\tau')^2(2\tau' + n\sigma''),$$

$$K_{01}^{(0)} = (1-4n)\tau\tau' + (4n-7)\sigma'\tau' + (3+2n-2n^2)\tau\sigma'' + (3-2n+2n^2)\sigma'\sigma'',$$

$$K_{00}^{(0)} = 2(-\tau+\sigma')((2n+1)\tau\tau' + (5-2n)\sigma'\tau' + (n^2-n-3)\tau\sigma'' + (-3+n-n^2)\sigma'\sigma''),$$

$$K_{00}^{(1)} = 15(\sigma''-\tau')^2.$$

Thus, the explicit expression of a unique fourth-order differential equation satisfied by the usual co-recursive ($k=0$) of the classical class has been obtained in terms of the polynomials σ and τ (see (2)), which completely characterize this class. Let us mention here that in [32] a method to obtain this differential equation in a factorized $(2+2)$ form was outlined. This factorized form is given in [32, Eq. (27)], where there is a mistake in the first element of the determinant (it should be $(\sigma''-\tau')(3\sigma'-2\tau)$) coming from a misprint in Eq. (25) of the same paper. The explicit form of this factorized fourth-order differential equation in terms of σ and τ is

$$\{\sigma\mathcal{A}D^2 + [(\tau+\sigma')\mathcal{A} - \sigma\mathcal{B}]D + (1-n)[\tfrac{1}{2}n\sigma'' + \tau']\mathcal{A} - \tau\mathcal{B} + \mathcal{C}\}$$
$$\times\{\sigma D^2 + (2\sigma'-\tau)D - (n+1)[\tau' + \tfrac{1}{2}(n-2)\sigma'']\}P_n^*(x;\nu;0) = 0 \quad \left(D \equiv \frac{\mathrm{d}}{\mathrm{d}x}\right),$$

where

$$\begin{aligned}\mathcal{A} \equiv \mathcal{A}(x;n;\nu) = {} & 4n(\sigma''-2\tau')^2((1-n)\sigma''-2\tau')\nu^2 \\ & + 8(\sigma''-2\tau')\{\tau(\sigma''-\tau') + 2(\sigma'-\tau)(2\tau'-\sigma'')n + 2(\sigma'-\tau)\sigma''n^2\}\nu \\ & + 8(\sigma''-\tau')[2\tau(\tau-\sigma') - 3\sigma(\tau'-\sigma'')] \\ & + 16(\sigma'-\tau)^2(\sigma''-2\tau')n - 16(\sigma'-\tau)^2\sigma''n^2,\end{aligned}$$

$$\begin{aligned}\mathcal{B} \equiv \mathcal{B}(x;n;\nu) = {} & 8(\sigma''-\tau')(\sigma''-2\tau')[(4n+1)\tau' + 2n(n-1)\sigma'']\nu \\ & + 8(\sigma''-\tau')\{2\tau(2\tau'-\sigma'') + \sigma'(3\sigma''-5\tau') + 4(\sigma'-\tau)(\sigma''-2\tau')n \\ & \quad + 4(\sigma'-\tau)\sigma''n^2\},\end{aligned}$$

$$\begin{aligned}\mathcal{C} \equiv \mathcal{C}(x;n;\nu) = {} & 4n(2\tau-3\sigma')(\sigma''-2\tau')(\sigma''-\tau')(2\tau' + (n-1)\sigma'')\nu \\ & + 8(\sigma''-\tau')\{(\tau'-\sigma'')[\tau(2\tau-3\sigma') + 3\sigma\tau'] \\ & \quad + (\sigma''-2\tau')[\tau(5\sigma'-2\tau) - 3\sigma'^2 + 6\sigma(\sigma''-\tau')]n \\ & \quad + \sigma''[\tau(2\tau-5\sigma') + 3\sigma'^2 - 6\sigma(\sigma''-\tau')]n^2\}.\end{aligned}$$

By using *Mathematica*, it has been verified that the latter factorized differential equation exactly coincides with the one given by the program DiffEqCorecursive.m we have written down above.

Co-recursive at level $k=1$ of the classical class

The *Mathematica* function

CoeffEq[i,x,1,ν,{$\sigma(x)$,$\tau(x)$}] $\equiv G_i(x;\nu;1;n)$, $\quad i = 0,1,2,3,4,$

gives the searched coefficients of the differential equation (5). After eliminating the factor

$$-\frac{\nu^6\tau^8\sigma(2\tau'+\sigma'')^6}{8[2\sigma\tau'^2+\tau^2\sigma''-2\tau\sigma'\tau']^6}$$

appearing in every G-coefficient, their expressions are

$$\begin{aligned}
G_4(x;\nu;1;n) &= 4\sigma^2\{(n-1)\tau^4(2\tau'+\sigma'')^2(2\tau'+n\sigma'')\nu^2 - 2\tau^3(2\tau'+\sigma'')[C_{41}^{(0)}+C_{41}^{(1)}\sigma]\nu \\
&\quad + C_{40}^{(0)}+C_{40}^{(1)}\sigma+C_{40}^{(2)}\sigma^2+C_{40}^{(3)}\sigma^3\},\\
G_3(x;\nu;1;n) &= 4\sigma\{(n-1)\tau^3(2\tau'+\sigma'')^2(2\tau'+n\sigma'')(5\tau\sigma'-4\tau'\sigma)\nu^2\\
&\quad - 2\tau^2(2\tau'+\sigma'')[C_{31}^{(0)}+C_{31}^{(1)}\sigma+C_{31}^{(2)}\sigma^2]\nu\\
&\quad + C_{30}^{(0)}+C_{30}^{(1)}\sigma+C_{30}^{(2)}\sigma^2+C_{30}^{(3)}\sigma^3\},\\
G_2(x;\nu;1;n) &= -4\{(n-1)\tau^2(2\tau'+\sigma'')^2(2\tau'+n\sigma'')[C_{22}^{(0)}+C_{22}^{(1)}\sigma+C_{22}^{(2)}\sigma^2]\nu^2\\
&\quad - \tau(2\tau'+\sigma'')[C_{21}^{(0)}+C_{21}^{(1)}\sigma+C_{21}^{(2)}\sigma^2+C_{21}^{(3)}\sigma^3]\nu\\
&\quad + [C_{20}^{(0)}+C_{20}^{(1)}\sigma+C_{20}^{(2)}\sigma^2+C_{20}^{(3)}\sigma^3+C_{20}^{(4)}\sigma^4]\},\\
G_1(x;\nu;1;n) &= 2\{(1-n)\tau(2\tau'+\sigma'')^2(2\tau'+n\sigma'')[C_{12}^{(0)}+C_{12}^{(1)}\sigma+C_{12}^{(2)}\sigma^2]\nu^2\\
&\quad + 2(2\tau'+\sigma'')[C_{11}^{(0)}+C_{11}^{(1)}\sigma+C_{11}^{(2)}\sigma^2+C_{11}^{(3)}\sigma^3]\nu\\
&\quad + [C_{10}^{(0)}+C_{10}^{(1)}\sigma+C_{10}^{(2)}\sigma^2+C_{10}^{(3)}\sigma^3]\},\\
G_0(x;\nu;1;n) &= (n-1)(2\tau'+\sigma'')^2(2\tau'+n\sigma'')[C_{02}^{(0)}+C_{02}^{(1)}\sigma+C_{02}^{(2)}\sigma^2]\nu^2\\
&\quad + 2(2\tau'+\sigma'')[C_{01}^{(0)}+C_{01}^{(1)}\sigma+C_{01}^{(2)}\sigma^2]\nu\\
&\quad + 2n(2\tau'+(n-1)\sigma'')[C_{00}^{(0)}+C_{00}^{(1)}\sigma+C_{00}^{(2)}\sigma^2+C_{00}^{(3)}\sigma^3].
\end{aligned}$$

Then, with these values of the G-coefficients, (5) is the unique fourth-order differential equation satisfied by the co-recursive at first level $P_n^*(x;\nu;1)$ (for $n \geqslant 2$) of the classical. For the sake of completeness, the coefficients $C_{pq}^{(t)}$ are listed in Appendix A. The notation used is the same as above, that is, in $C_{pq}^{(t)}$, p is related to the $G_p(x;\nu;1;n)$-coefficient, q is related to the power of ν and t is related to the power of σ.

The question whether this differential equation can be factorized in a $(2+2)$ form as it happens when $k=0$ has not been solved. Several trials have been performed with *Mathematica* (including the simplest case which could be Hermite) with no success. However, the fact that the fourth-order differential equation for the associated of the classical class of order bigger than one cannot probably be factorized [30,38] would make it impossible to factorize the differential equation for the co-recursive of the classical class for $k \geqslant 1$.

2.3. *Reliability of the results*

Since the two fourth-order differential equations just given are very complicated, it is important to have some ways of testing their reliability. One of them is to get cross-checks with already known results. When $k = 0$, this has been done by using [32, Eq. (27)] in the general situation (i.e., in terms of σ and τ) and it is also done below for some particular cases (Laguerre and Jacobi polynomials). However, for $k = 1$ the corresponding differential equation is not known.

In spite of this, from the recurrence relation (4) it is possible to obtain the polynomial solutions $P_n^*(x;\nu;k)$ in terms of σ and τ. Notice that γ_n is given in (17) and for β_n one has [30,38]

$$\beta_n = \frac{n[(n-1)\sigma' + \tau]}{(n-1)\sigma'' + \tau'} - \frac{(n+1)[n\sigma' + \tau]}{n\sigma'' + \tau'} + x.$$

Thus, one can check whether the polynomials obtained in this way are in fact solutions of the corresponding differential equations. This test has been successfully performed by using *Mathematica* for $k = 0, 1$ and for the polynomial solutions of degrees $1, 2, \ldots, 5$.

On the other hand, when $\nu = 0$, these two fourth-order differential equations become a linear combination of the L_2 operator defined in (2) and its first and second derivatives. So, in this case ($\nu = 0$) a fourth-order differential equation satisfied by the starting classical families is found. This property provides another way of getting confidence in the results, because the classical orthogonal polynomials can be expressed in terms of σ and τ as indicated in (18). It has been tested with *Mathematica* that the classical orthogonal polynomials of degree $1, 2, \ldots, 5$ are solutions of the two ($k = 0, 1$) fourth-order differential equations obtained from the ones given above by putting $\nu = 0$.

These two tests have been also successfully performed in all the examples shown below for specific classical orthogonal polynomial families.

2.4. Particular cases

As pointed out before, the DiffEqCorecursive.m program is also able to give the above fourth-order differential equations for each classical family separately. This is done by means of the same *Mathematica* function

$$\text{CoeffEq}[\text{i,x,1,}\nu,\{\sigma(x),\ \tau(x)\}] \equiv G_i(x;\nu;1;n), \quad i = 0, 1, 2, 3, 4,$$

by replacing $\sigma(x)$ and $\tau(x)$ by their specific values corresponding to each classical orthogonal polynomial sequence. As an illustration we consider here the following cases.

Jacobi polynomials: $\sigma(x) = 1 - x^2$, $\tau(x) = b - a - (a+b+2)x$
Co-recursive at level $k = 0$

$$G_4(x;\nu;0;n) = 4\sigma^2\{T_{42}^{(0)}\nu^2 + T_{41}^{(0)}\nu + (T_{40}^{(0)} + T_{40}^{(1)}\sigma)\},$$

where

$$T_{42}^{(0)} = 8(1+a+b)^2 n(1+a+b+n),$$

$$T_{41}^{(0)} = 4(1+a+b)[(b-a)(a+b+4n+4an+4bn+4n^2) - (a+b)(2+a+b+4n+4an+4bn+4n^2)x],$$

$$T_{40}^{(0)} = 4(a-b+(a+b)x)[(a-b)(a+b+2n+2an+2bn+2n^2) + (a+b)(2+a+b+2n+2an+2bn+2n^2)x],$$

$$T_{40}^{(1)} = 6(a+b)^2;$$

$$G_3(x;\nu;0;n) = 4\sigma\{T_{32}^{(0)}\nu^2 + 4(a+b+1)[T_{31}^{(0)} + T_{31}^{(1)}\sigma]\nu + 4(T_{30}^{(0)} + T_{30}^{(1)}\sigma)\},$$

where

$$T_{32}^{(0)} = -80(1+a+b)^2 n(1+a+b+n)x,$$

$$T_{31}^{(0)} = 10x[(a-b)(a+b+4n+4an+4bn+4n^2) + (a+b)(2+a+b+4n+4an+4bn+4n^2)x],$$

$$T_{31}^{(1)} = (a+b)(2+a+b+4n+4an+4bn+4n^2),$$

$$T_{30}^{(0)} = 10x(b-a-(a+b)x)[(a-b)(a+b+2n+2an+2bn+2n^2) + (a+b)(2+a+b+2n+2an+2bn+2n^2)x],$$

$$T_{30}^{(1)} = 2(a+b)[(b-a)(1+a+b+2n+2an+2bn+2n^2) - (a+b)(8+a+b+2n+2an+2bn+2n^2)x];$$

$$G_2(x;\nu;0;n) = 4\{T_{22}^{(0)}\nu^2 + 2(a+b+1)[T_{21}^{(0)} + T_{21}^{(1)}\sigma]\nu + 2[T_{20}^{(0)} + T_{20}^{(1)}\sigma + T_{20}^{(2)}\sigma^2]\},$$

where

$$T_{22}^{(0)} = 8(1+a+b)^2 n(1+a+b+n) \times [-8+2a-a^2+2b+2ab-b^2+2n+2an+2bn+2n^2+2(b^2-a^2)x + (24-2a-a^2-2b-2ab-b^2-2n-2an-2bn-2n^2)x^2],$$

$$T_{21}^{(0)} = 2(a-b-(4-a-b)x)(a-b+(4+a+b)x) \times [(a-b)(a+b+4n+4an+4bn+4n^2) + (a+b)(2+a+b+4n+4an+4bn+4n^2)x],$$

$$T_{21}^{(1)} = 4[(b-a)((a+b)(a+b-4)-16n-12an+4n(a+b)^2-12bn-12n^2 + 12an^2+4a^2n^2+12bn^2+8abn^2+4b^2n^2+8n^3+8an^3+8bn^3+4n^4) + (a+b)(2-a-a^2-b-2ab-b^2+n-3an-4a^2n-3bn - 8abn-4b^2n-3n^2-12an^2-4a^2n^2-12bn^2-8abn^2 - 4b^2n^2-8n^3-8an^3-8bn^3-4n^4)x],$$

$$T_{20}^{(0)} = 2(b-a-(4+a+b)x)(a-b+(a+b)x)(a-b-(4-a-b)x)$$
$$\times [(a-b)(a+b+2n+2an+2bn+2n^2)$$
$$+ (a+b)(2+a+b+2n+2an+2bn+2n^2)x],$$

$$T_{20}^{(1)} = (b-a)^2(-16a-a^2-16b-2ab-b^2-32n-24an+8a^2n-24bn+16abn$$
$$+8b^2n-24n^2+24an^2+8a^2n^2+24bn^2+16abn^2+8b^2n^2+16n^3$$
$$+16an^3+16bn^3+8n^4)$$
$$+2(b^2-a^2)(4+(a+b)(a+b+1)+2n-6n(a+b)-8n(a+b)^2-6n^2-24an^2$$
$$-8a^2n^2-24bn^2-16abn^2-8b^2n^2-16n^3-16an^3-16bn^3-8n^4)x$$
$$+(a+b)^2(32+(a+b)(14-a-b)+28n+36an+8a^2n+36bn+16abn$$
$$+8b^2n+36n^2+24an^2+8a^2n^2+24bn^2+16abn^2+8b^2n^2+16n^3+16an^3$$
$$+16bn^3+8n^4)x^2,$$

$$T_{20}^{(2)} = 9(a+b)^2(-2+a+b+2n+2an+2bn+2n^2);$$

$$G_1(x;\nu;0;n) = 2\{T_{12}^{(0)}\nu^2 + 4(a+b+1)[T_{11}^{(0)} + T_{11}^{(1)}\sigma]\nu + 4[T_{10}^{(0)} + T_{10}^{(1)}\sigma]\},$$

where

$$T_{12}^{(0)} = 48(1+a+b)^2 n(1+a+b+n)$$
$$\times [(b^2-a^2) + (4-(a+b)(a+b+2)-2n(a+b+n+1))x],$$

$$T_{11}^{(0)} = 4[(b-a)^2(a+b)(-1+a+b+5n+5an+5bn+5n^2)+2(b-a)$$
$$\times ((a+b)(3-2(a+b)-(a+b)^2)+12n+n(a+b)$$
$$\times (6-11(a+b)-5(a+b)^2)+6n^2-n^2(a+b)(18+a+b)$$
$$-12n^3(a+b+1)-6n^4)x+(a+b)$$
$$\times (-8+2a+5a^2+a^3+2b+10ab+3a^2b+5b^2+3ab^2+b^3-8n+4an+17a^2n$$
$$+5a^3n+4bn+34abn+15a^2bn+17b^2n+15ab^2n+5b^3n+4n^2+36an^2$$
$$+17a^2n^2+36bn^2+34abn^2+17b^2n^2+24n^3+24an^3+24bn^3+12n^4)x^2],$$

$$T_{11}^{(1)} = 2(a+b)(2+n)(a+b+n-1)[2+a+b+4n(a+b+1)+4n^2],$$

$$\begin{aligned}
T_{10}^{(0)} = {} & 4(b-a-(a+b)x)[(b-a)^2(a+b)(-1+a+b+2n+2an+2bn+2n^2) \\
& + 2(b-a)(3a-2a^2-a^3+3b-4ab-3a^2b-2b^2-3ab^2-b^3 \\
& + 6n+3an-5a^2n-2a^3n+3bn-10abn-6a^2bn \\
& - 5b^2n-6ab^2n-2b^3n+3n^2-9an^2-5a^2n^2-9bn^2 \\
& - 10abn^2-5b^2n^2-6n^3-6an^3-6bn^3-3n^4)x \\
& + (a+b)(-8+2a+5a^2+a^3+2b+10ab+3a^2b+5b^2 \\
& + 3ab^2+b^3+4n+10an+8a^2n+2a^3n+10bn \\
& + 16abn+6a^2bn+8b^2n+6ab^2n+2b^3n+10n^2 \\
& + 18an^2+8a^2n^2+18bn^2+16abn^2 \\
& + 8b^2n^2+12n^3+12an^3+12bn^3+6n^4)x^2],
\end{aligned}$$

$$\begin{aligned}
T_{10}^{(1)} = {} & 2(a+b)[(b-a)(-4-3a+7a^2-3b+14ab+7b^2-6n-2an+4a^2n-2bn \\
& + 8abn+4b^2n-2n^2+12an^2+4a^2n^2+12bn^2+8abn^2+4b^2n^2 \\
& + 8n^3+8an^3+8bn^3+4n^4) \\
& + (a+b)(14-7a-7a^2-7b-14ab-7b^2-14n \\
& - 18an-4a^2n-18bn-8abn-4b^2n-18n^2-12an^2-4a^2n^2 \\
& - 12bn^2-8abn^2-4b^2n^2-8n^3-8an^3-8bn^3-4n^4)x];
\end{aligned}$$

$$\begin{aligned}
G_0(x;\nu;0;n) = {} & 4n(n+1)(a+b+n)(a+b+n+1) \\
& \times \{T_{02}^{(0)}\nu^2+4(a+b+1)T_{01}^{(0)}\nu+2[T_{00}^{(0)}+T_{00}^{(1)}\sigma]\},
\end{aligned}$$

where

$$T_{02}^{(0)} = 8(1+a+b)^2(-1+n)(2+a+b+n),$$

$$\begin{aligned}
T_{01}^{(0)} = {} & (b-a)(-8-a-b+4n+4an+4bn+4n^2) \\
& + (a+b)(-4+a+b-4n-4an-4bn-4n^2)x,
\end{aligned}$$

$$\begin{aligned}
T_{00}^{(0)} = {} & 2(a-b+(a+b)x)[(b-a)(4-a-b-2n-2an-2bn-2n^2) \\
& + (a+b)(8+a+b+2n+2an+2bn+2n^2)x],
\end{aligned}$$

$$T_{00}^{(1)} = 15(a+b)^2.$$

In [19, Section 4.2] this differential equation has been given in a factorized (2 + 2) form. However, the comparison needs some attention due to few misprints (the correct factorized fourth-order differential equation can be easily obtained from the general expression given above).

Co-recursive at level $k = 1$

$$G_4 = 4\sigma^2\{8(3+a+b)^2(1-n)(2+a+b+n)(b-a-(a+b+2)x)^4\nu^2 + 4(a+b+3)(b-a-(a+b+2)x)^3[J_{41}^{(0)} + J_{41}^{(1)}\sigma]\nu + [J_{40}^{(0)} + J_{40}^{(1)}\sigma + J_{40}^{(2)}\sigma^2 + J_{40}^{(3)}\sigma^3]\},$$

$$G_3 = 4\sigma\{16(3+a+b)^2(-1+n)(2+a+b+n)(a-b+2x+ax+bx)^3 \times (4+2a+2b+5ax-5bx+6x^2+3ax^2+3bx^2)\nu^2 + 4(a+b+3)(a-b+(a+b+2)x)^2[J_{31}^{(0)} + J_{31}^{(1)}\sigma + J_{31}^{(2)}\sigma^2]\nu + [J_{30}^{(0)} + J_{30}^{(1)}\sigma + J_{30}^{(2)}\sigma^2 + J_{30}^{(3)}\sigma^3]\},$$

$$G_2 = -4\{8(3+a+b)^2(1-n)(2+a+b+n)(a-b+2x+ax+bx)^2 \times [J_{22}^{(0)} + J_{22}^{(1)}\sigma + J_{22}^{(2)}\sigma^2]\nu^2 + 2(a+b+3)(b-a-(a+b+2)x)[J_{21}^{(0)} + J_{21}^{(1)}\sigma + J_{21}^{(2)}\sigma^2 + J_{21}^{(3)}\sigma^3]\nu + [J_{20}^{(0)} + J_{20}^{(1)}\sigma + J_{20}^{(2)}\sigma^2 + J_{20}^{(3)}\sigma^3 + J_{20}^{(4)}\sigma^4]\},$$

$$G_1 = 2\{8(3+a+b)^2(-1+n)(2+a+b+n)(b-a-(a+b+2)x) \times [J_{12}^{(0)} + J_{12}^{(1)}\sigma + J_{12}^{(2)}\sigma^2]\nu^2 - 4(a+b+3)[J_{11}^{(0)} + J_{11}^{(1)}\sigma + J_{11}^{(2)}\sigma^2 + J_{11}^{(3)}\sigma^3]\nu + [J_{10}^{(0)} + J_{10}^{(1)}\sigma + J_{10}^{(2)}\sigma^2 + J_{10}^{(3)}\sigma^3]\},$$

$$G_0 = 8(3+a+b)^2(1-n)(2+a+b+n)[J_{02}^{(0)} + J_{02}^{(1)}\sigma + J_{02}^{(2)}\sigma^2]\nu^2 - 4(a+b+3)[J_{01}^{(0)} + J_{01}^{(1)}\sigma + J_{01}^{(2)}\sigma^2]\nu - 4n(a+b+n+1)[J_{00}^{(0)} + J_{00}^{(1)}\sigma + J_{00}^{(2)}\sigma^2 + J_{00}^{(3)}\sigma^3].$$

These $J_{pq}^{(t)}$-coefficients are polynomials in the x-variable of degree at most eight. They are listed in Appendix B.

Hermite polynomials: $\sigma(x) = 1$, $\tau(x) = -2x$
Co-recursive at level $k = 0$

$$G_4(x;\nu;0;n) = [8n\nu^2 - 4x(1+4n)\nu + (3+4(1+2n)x^2)],$$

$$G_3(x;\nu;0;n) = 4[(1+4n)\nu - 2(1+2n)x],$$

$$G_2(x;\nu;0;n) = 2[16n(1+n-x^2)\nu^2 + 8x(-1-4n-4n^2+(1+4n)x^2)\nu + 9(1+2n) + 2(8n^2+8n-1)x^2 - 8(1+2n)x^4],$$

$$G_1(x;\nu;0;n) = 8[-12nx\nu^2 + (2+9n+4n^2+4(1+5n)x^2)\nu - x(7+4n+4n^2+4(1+2n)x^2)],$$

$$G_0(x;\nu;0;n) = 4n(1+n)[8(n-1)\nu^2 + 4x(1-4n)\nu + 4(1+2n)x^2 + 15].$$

This differential equation has been obtained in [32, Eqs. (30), (31)] where some misprints (coming from the already mentioned in Eq. (27) of the same reference) have been found.

Co-recursive at level $k=1$

$$G_4(x;\nu;1;n) = 8x^4(n-1)\nu^2 + 4x^3(-4+4n+3x^2-4nx^2)\nu + 3 + 2(4n-1)x^2 + (15-16n)x^4 + 4(2n-1)x^6,$$

$$G_3(x;\nu;1;n) = 4x\{8x^2(1-n)\nu^2 + x(12(1-n)+5(4n-3)x^2)\nu + 1 - 4n + (16n-15)x^2 + 6(1-2n)x^4\},$$

$$\begin{aligned} G_2(x;\nu;1;n) = 2\{&16x^2(n-1)(3+(n+2)x^2-x^4)\nu^2 \\ &+ 2x[24(n-1) + (13-44n+16n^2)x^2 + 4(9-8n-4n^2)x^4 \\ &\quad + 4(4n-3)x^6]\nu \\ &+ 18(n+1) + (43-20n+16n^2)x^2 + (11+42n-32n^2)x^4 \\ &+ 2(8n^2+24n-21)x^6 + 8(1-2n)x^8\},\end{aligned}$$

$$\begin{aligned} G_1(x;\nu;1;n) = 4\{&8x(1-n)(6+2(n+2)x^2+x^4)\nu^2 \\ &+ 2[12(1-n) - (21-48n+12n^2)x^2 \\ &\quad + (20n^2+13n-15)x^4 + 4(n-1)x^6]\nu \\ &+ x(-(29+2n+8n^2) + (32n^2-54n-17)x^2 - 4(1+2n+6n^2)x^4)\},\end{aligned}$$

$$\begin{aligned} G_0(x;\nu;1;n) = 4\{&8(n-1)(6+2(n+2)x^2+(1-2n+n^2)x^4)\nu^2 \\ &+ 2x[29-52n+8n^2 + (17+n-44n^2+8n^3)x^2 \\ &\quad + 2(2-7n+9n^2-4n^3)x^4]\nu \\ &+ n(24+15n+(49-18n+8n^2)x^2 - (23-75n+16n^2)x^4 \\ &\quad + 4(1-3n+2n^2)x^6)\}\end{aligned}$$

Generalized Laguerre polynomials: $\sigma(x) = x$, $\tau(x) = a + 1 - x$
Co-recursive at level $k = 0$

$$G_4(x;\nu;0;n) = x^2\{4n\nu^2 + 2(1 + a + 4an - x - 4nx)\nu + (2a + 2a^2 + 4a^2n + x - 4ax - 8anx + 2x^2 + 4nx^2)\},$$

$$G_3(x;\nu;0;n) = 2x\{10n\nu^2 + (5 + 5a + 20an - 4x - 16nx)\nu + (5a + 5a^2 + 10a^2n + 2x - 8ax - 16anx + 3x^2 + 6nx^2)\},$$

$$\begin{aligned}G_2(x;\nu;0;n) = \{&4n(4 - a^2 + 2x + 2ax + 2nx - x^2)\nu^2 \\ &+ 2(4 + 4a - a^2 - a^3 + 16an - 4a^3n + x + 4ax + 3a^2x - nx + 8anx \\ &+ 12a^2nx + 8an^2x - 3x^2 - 3ax^2 - 8nx^2 - 12anx^2 - 8n^2x^2 \\ &+ x^3 + 4nx^3)\nu \\ &+ (8a + 8a^2 - 2a^3 - 2a^4 + 16a^2n - 4a^4n + 2x - ax + 5a^2x + 8a^3x \\ &- 2anx + 8a^2nx + 16a^3nx + 8a^2n^2x + 2x^2 - 4ax^2 - 12a^2x^2 \\ &+ 4nx^2 - 16anx^2 - 24a^2nx^2 - 16an^2x^2 + x^3 + 8ax^3 + 8nx^3 \\ &+ 16anx^3 + 8n^2x^3 - 2x^4 - 4nx^4)\},\end{aligned}$$

$$\begin{aligned}G_1(x;\nu;0;n) = 2\{&6n(1 + a + n - x)\nu^2 \\ &+ (4 + 6a + 2a^2 + 8n + 12an + 10a^2n + 12an^2 - 4x - 4ax \\ &- 3nx - 20anx - 8n^2x + 2x^2 + 10nx^2)\nu \\ &+ (4a + 6a^2 + 2a^3 + 8an + 6a^2n + 4a^3n + 6a^2n^2 + x - 5ax \\ &- 6a^2x + 2nx - 8anx - 12a^2nx - 8an^2x - x^2 + 6ax^2 + 2nx^2 \\ &+ 12anx^2 + 2n^2x^2 - 2x^3 - 4nx^3)\},\end{aligned}$$

$$G_0(x;\nu;0;n) = n(1 + n)\{4(n - 1)\nu^2 + 2(6 - a + 4an + x - 4nx)\nu + (12a + 2a^2 + 4a^2n + 3x - 4ax - 8anx + 2x^2 + 4nx^2)\}.$$

This differential equation has been already given in [32, Eq. (32)] (where there are some misprints) and in [18, Section 5.1] in a factorized $(2 + 2)$ form and it exactly coincides with the one obtained here. In performing this cross-check one should take into account that the co-recursive parameter (μ in [18]) is related with the one used here by $\mu = -\nu$.

Bessel polynomials: $\sigma(x) = x^2$, $\tau(x) = (ax + 2)$, $a = 2$
Co-recursive at level $k = 0$

$$G_4(x;\nu;0;n) = x^4,$$

$$G_3(x;\nu;0;n) = 10x^3,$$
$$G_2(x;\nu;0;n) = -2(2-12x^2+nx^2(n+1)),$$
$$G_1(x;\nu;0;n) = -6(n-1)(n+2)x,$$
$$G_0(x;\nu;0;n) = n(n+2)(n^2-1).$$

As stated in [32], this differential equation (independent of ν) and the one satisfied by the associated Bessel polynomials already obtained in [38] exactly coincide. So, two linearly independent polynomial solutions of this fourth-order differential equation are the co-recursive Bessel of degree n and the associated Bessel of degree $n-1$ (cf. (9) for $k=0$).

Legendre polynomials: $\sigma(x) = 1-x^2$, $\tau(x) = -2x$
Co-recursive at level $k=0$

$$G_4(x;\nu;0;n) = (1-x^2)^2,$$
$$G_3(x;\nu;0;n) = -10x(1-x^2),$$
$$G_2(x;\nu;0;n) = 2(-4+n(n+1)+12x^2-n(n+1)x^2),$$
$$G_1(x;\nu;0;n) = -6(n-1)(n+2)x,$$
$$G_0(x;\nu;0;n) = n(n^2-1)(n+2).$$

This differential equation has also two linear independent polynomial solutions: the co-recursive ($k=0$) Legendre of degree n and the associated Legendre of degree $n-1$. As pointed out in [32], these two orthogonal polynomial sequences (Bessel and Legendre) are the only classical families for which the above property holds (because in these cases $\sigma''=\tau'$, cf. [32, Eq. (27)]).

3. Distribution of zeros

In this section we will study the distribution of zeros (6) of the generalized co-recursive polynomials of the classical families via its moments (around the origin) (7). Though these moments completely characterize the distribution [33], to get it from them is not an easy task. However, the moments themselves give us valuable information about the distribution. Thus, statistical parameters as variance, skewness and kurtosis will provide useful information.

In [21] the same problem was considered for an arbitrary level of co-recursivity and for any orthogonal polynomial family belonging to the Laguerre–Hahn class. In that work a method [6,7] to obtain the moments (7) from the TTRR (4) was used. In particular, the centroid ($\mu_1^*(n,\nu,k)$) and the second moment ($\mu_2^*(n,\nu,k)$) were explicitly calculated in terms of the moments of the unmodified family, which, in what follows, will be denoted by $\mu_j(n)$, $j=0,1,2,\dots$. Their expressions are [21]:

$$\mu_0^*(n,\nu,k) = \mu_0(n) = 1, \qquad \mu_1^*(n,\nu,k) = \mu_1(n) + \frac{\nu}{n},$$
$$\mu_2^*(n,\nu,k) = \mu_2(n) + \frac{\nu}{n}[\nu + 2\beta_k], \tag{19}$$

where β_k is the coefficient appearing in the TTRR (1) satisfied by the unperturbed family. Notice that there is a shift (independent of the level of co-recursivity) in the first moment whose direction only depends on the sign of ν. For the usual co-recursive polynomials ($k = 0$) this is a consequence of the interlacing property of the zeros (cf. [4, Theorem 1]) of the perturbed and unperturbed family. Moreover, in [21], the way of calculating moments of higher order was also indicated, although the calculations involved are cumbersome and heavy.

Here we adopt a completely different approach [37,38] which allows to obtain the moments (7) of the distribution of zeros (6) in terms of the coefficients of the fourth-order differential equation satisfied by the generalized co-recursive of the classical class. The way of doing this is such that it can be programmed in any computer algebra system.

3.1. Algorithm

Although the method is already described in [37,38] we give here a brief survey of it. Let

$$P_n(x) = \sum_{k=0}^{n} (-1)^k C_{n,k} x^{n-k}, \quad C_{n,0} = 1,$$

be the monic explicit expression of a polynomial solution of an Nth-order differential equation of the form

$$\sum_{i=0}^{N} g_i(x) y^{(i)}(x) = 0, \quad g_i(x) = \sum_{j=0}^{c_i} a_j^{(i)} x^j. \tag{20}$$

Then, the $C_{n,k}$-coefficients of the polynomial can be expressed in terms of the differential equation ones by means of the recurrence relation [2]

$$C_{k,j} = -\frac{\displaystyle\sum_{m=1}^{j} (-1)^m C_{k,j-m} \sum_{i=0}^{N} \frac{(k-j+m)!}{(k-j+m-i)!} a_{i+q-m}^{(i)}}{\displaystyle\sum_{i=0}^{N} \frac{(k-j)!}{(k-j-i)!} a_{i+q}^{(i)}}, \quad j \geqslant 1, \tag{21}$$

where the initial condition is $C_{k,0} = 1$ and $q = \max\{c_i - i, i = 1, 2, \ldots, N\}$.

Once the $C_{k,n}$-coefficients of $P_n(x)$ are known, the moments (7) of the distribution of zeros (6) of such a polynomial can be generated in an exact and recurrent way by means of the following formula [11,13,26,37]:

$$\mu_k^{(n)} = (-1)^{k+1} \left\{ \frac{k}{n} C_{n,k} + \sum_{j=1}^{k-1} (-1)^j C_{n,k-j} \mu_j^{(n)} \right\}, \quad k > 1, \tag{22}$$

where the initial conditions are $\mu_0^{(n)} = 1$ and $\mu_1^{(n)} = C_{n,1}/n$.

Notice that the algorithm we have just described is general in the sense that it remains valid even in the case that the solution $P_n(x)$ does not belong to an orthogonal polynomial sequence. It should be mentioned here that an alternative way for calculating the moments (7) of a polynomial solution of an Nth-order differential equation (20) was given in [3] for $N = 2$ and in [2] for N arbitrary. However, both approaches are only valid when the zeros of the polynomial are simple.

Although the computations needed in order to obtain the moments are heavy (notice the high nonlinearity of (21)), expressions (21) and (22) allow us to use again *Mathematica* symbolic language [36] in order to calculate them. For the cases $N = 2, 4$, this has been done by means of two programs called MomentsEEx.m and MomentsDE.m described in [13,37] and [38], respectively, where they were applied to the generalized associated of the classical class. The *Mathematica* function which gives the moments (see [38] for the details) is

$$\text{MomentsDE[n\_,j\_,o\_,x\_,ST\_List]} \equiv \mu_j(n),$$

where the argument o_ is the order of the differential equation (two or four), x_ is the independent variable and the last argument is

$$\text{ST\_List} \equiv \{g_0(x), g_1(x), g_2(x), g_3(x), g_4(x)\},$$

$g_i(x)$, $i = 0, \ldots, 4$, being the polynomial coefficients of the differential equation (20).

In the next section we describe how this function can be used to compute the moments (7) of the distribution of zeros (6) of the generalized co-recursive of the classical orthogonal polynomials.

3.2. Results

Once a *Mathematica* session is started, we have to type in the following instructions:

In[1]:= << DiffEqCorecursive.m,

In[2]:= << MomentsDE.m,

for making available the functions contained in both programs. Then, the searched moments

$$\mu_j^*(n, \nu, k), \quad j = 0, 1, 2, \ldots,$$

(defined in (7)) corresponding to the co-recursive at a fixed level k of the classical class can be calculated by typing in the following *Mathematica* instruction:

$$\begin{aligned}\text{MomentsDE[n,j,4,x,\{CoeffEq[0,x,k,}\nu, \{\tfrac{1}{2}\sigma''x^2 + \sigma'(0)x + \sigma(0), \tau'x + \tau(0)\}],\\ \text{CoeffEq[1,x,k,}\nu, \{\tfrac{1}{2}\sigma''x^2 + \sigma'(0)x + \sigma(0), \tau'x + \tau(0)\}],\\ \text{CoeffEq[2,x,k,}\nu, \{\tfrac{1}{2}\sigma''x^2 + \sigma'(0)x + \sigma(0), \tau'x + \tau(0)\}],\\ \text{CoeffEq[3,x,k,}\nu, \{\tfrac{1}{2}\sigma''x^2 + \sigma'(0)x + \sigma(0), \tau'x + \tau(0)\}],\\ \text{CoeffEq[4,x,k,}\nu, \{\tfrac{1}{2}\sigma''x^2 + \sigma'(0)x + \sigma(0), \tau'x + \tau(0)\}]\}],\end{aligned} \tag{23}$$

where the order j of the moment has to be specified and CoeffEq is the function described in Section 2.1. In this case, in its last argument, σ and τ have to be replaced by

$$\sigma(x) = \tfrac{1}{2}\sigma''x^2 + \sigma'(0)x + \sigma(0), \qquad \tau(x) = \tau'x + \tau(0),$$

because the function MomentsDE needs as input data the differential equation coefficients explicitly expressed in terms of the power of the independent variable x.

For example, putting $k=0$ and $j=0,1,\ldots,4$, the above *Mathematica* instruction gives the first five moments of the distribution of zeros of the usual [4] co-recursive of the classical class:

$$\mu_0^*(n,\nu,0)=1,$$

$$\mu_1^*(n,\nu,0)=\mu_1(n)+\frac{\nu}{n},$$

$$\mu_2^*(n,\nu,0)=\mu_2(n)+\frac{\nu}{n\tau'}[\tau'\nu-2\tau(0)],$$

$$\mu_3^*(n,\nu,0)=\mu_3(n)+\frac{\nu}{n\tau'}\left[\tau'\nu^2-3\tau(0)\nu+\frac{6[\tau(0)^2+\sigma'(0)\tau(0)-\sigma(0)\tau']}{2\tau'+\sigma''}\right],$$

$$\begin{aligned}\mu_4^*(n,\nu,0)&\\ &=\mu_4(n)+\frac{\nu}{n\tau'}\\ &\times\left\{\tau'\nu^3-4\tau(0)\nu^2+\left[2\frac{[\sigma''+6\tau']\tau(0)^2+4\sigma'(0)\tau'\tau(0)-4\sigma(0)\tau'^2}{\tau'(2\tau'+\sigma'')}\right]\nu\right.\\ &\left.\quad-8\frac{[\tau(0)+3\sigma'(0)]\tau(0)^2+[2\sigma'(0)^2-\sigma(0)(\sigma''-3\tau')]\tau(0)-2\tau'\sigma(0)\sigma'(0)}{(\sigma''+\tau')(2\tau'+\sigma'')}\right\},\end{aligned}$$

where $\mu_j(n)$, $j=0,1,\ldots,4$, are the moments corresponding to the unperturbed classical family (which could be also computed in terms of σ and τ by using a similar *Mathematica* instruction). As it could be expected, when the co-recursive parameter ν is zero, the above moments become the corresponding to the unperturbed classical family. This is so due to the above stated fact that when $\nu=0$, the resulting fourth-order differential equation has as polynomial solutions the classical ones. Moreover, it is clear that asymptotically (i.e., when $n\to\infty$) the perturbation of co-recursive type has no effect in the moments.

On the other hand, it should be noticed that the values of $\mu_0^*(n,\nu,0)$, $\mu_1^*(n,\nu,0)$ and $\mu_2^*(n,\nu,0)$ exactly coincide with the ones given in (19), although they were computed [21] by using a completely different approach. For comparison, one should take into account that, for the classical class, the coefficient β_n of the recurrence relation (1) can be expressed in terms of σ and τ in the following way [30]:

$$\beta_n=\frac{n[(n-1)\sigma'(0)+\tau(0)]}{(n-1)\sigma''+\tau'}-\frac{(n+1)[n\sigma'(0)+\tau(0)]}{n\sigma''+\tau'},$$

so $\beta_0=-\tau(0)/\tau'$.

Of course, moments of higher order could be computed and also they could be obtained for higher levels of co-recursivity. These possibilities only depend on the computer memory capacity.

As a final and concrete illustration, we are going to consider here the Hermite polynomials.

Hermite polynomials

The moments of the zero distribution of the classical Hermite polynomials were given for the first time in [3]. Since the zeros of these polynomials are symmetric with respect to the origin, every odd moment is zero and the first few even ones are [3]

$$\mu_0 = 1, \qquad \mu_2(n) = \tfrac{1}{2}(n-1), \qquad \mu_4(n) = \tfrac{1}{4}(n-1)(2n-3),$$
$$\mu_6(n) = \tfrac{1}{8}(n-1)(15-17n+5n^2).$$

(See also [37] where the MomentsDE *Mathematica* function was used to compute them.)

Taking into account that in this case $\sigma(x) = 1$ and $\tau(x) = -2x$, putting $k = 0$ and $j = 0, 1, \ldots, 6$ in the *Mathematica* instruction (23), the following moments for the co-recursive of Hermite at level $k = 0$ are obtained:

$$\mu_0^*(n,\nu,0) = 1, \qquad \mu_1^*(n,\nu,0) = \frac{\nu}{n},$$
$$\mu_2^*(n,\nu,0) = \mu_2(n) + \frac{\nu^2}{n}, \qquad \mu_3^*(n,\nu,0) = \frac{\nu(3+2\nu^2)}{2n},$$
$$\mu_4^*(n,\nu,0) = \mu_4(n) + \frac{\nu^2(2+\nu^2)}{n}, \qquad \mu_5^*(n,\nu,0) = \frac{\nu(15+10\nu^2+4\nu^4)}{4n},$$
$$\mu_6^*(n,\nu,0) = \mu_6(n) + \frac{\nu^2(21+12\nu^2+4\nu^6)}{4n}.$$

The same procedure with $k = 1$ gives the moments for the co-recursive of Hermite at first level. Their values are

$$\mu_0^*(n,\nu,1) = 1, \qquad \mu_1^*(n,\nu,1) = \frac{\nu}{n},$$
$$\mu_2^*(n,\nu,1) = \mu_2 + \frac{\nu^2}{n}, \qquad \mu_3^*(n,\nu,1) = \frac{\nu(9+2\nu^2)}{2n},$$
$$\mu_4^*(n,\nu,1) = \mu_4 + \frac{\nu^2(6+\nu^2)}{n}, \qquad \mu_5^*(n,\nu,1) = \frac{\nu(75+30\nu^2+4\nu^4)}{4n},$$
$$\mu_6^*(n,\nu,1) = \mu_6 + \frac{\nu^2(117+36\nu^2+4\nu^4)}{4n}.$$

From these moments one can obtain information about the effect of a perturbation of recursive type on the zeros of classical Hermite polynomials. In the first place, as pointed out in [21], the centroid undergoes a shift of ν/n which does not depend on the level of recursivity. It also indicates that the zeros of the perturbed polynomials are not symmetric with respect to the origin.

More details of the qualitative effect of the perturbation can be obtained by considering, e.g., the variance ($\delta_1(n,\nu,k)$), skewness ($\delta_2(n,\nu,k)$) and kurtosis ($\delta_3(n,\nu,k)$) parameters. They are defined in terms of the moments around the centroid (to be denoted by $\mu_j^{(c)}(n,\nu,k)$) as [17]

$$\delta_1(n,\nu,k) = [\mu_2^{(c)}(n,\nu,k)]^{1/2}, \qquad \delta_2(n,\nu,k) = \frac{\mu_3^{(c)}(n,\nu,k)}{[\mu_2^{(c)}(n,\nu,k)]^{3/2}},$$
$$\delta_3(n,\nu,k) = \frac{\mu_4^{(c)}(n,\nu,k)}{[\mu_2^{(c)}(n,\nu,k)]^2} - 3.$$

Here the first few moments around the centroid are related to the already obtained moments around the origin (defined in (7)) by [17]

$$\mu_2^{(c)}(n,\nu,k) = \mu_2(n,\nu,k) - [\mu_1(n,\nu,k)]^2,$$

$$\mu_3^{(c)}(n,\nu,k) = \mu_3(n,\nu,k) - 3\mu_1(n,\nu,k)\mu_2(n,\nu,k) + 2[\mu_1(n,\nu,k)]^3,$$

$$\mu_4^{(c)}(n,\nu,k) = \mu_4(n,\nu,k) - 4\mu_1(n,\nu,k)\mu_3(n,\nu,k) + 6[\mu_1(n,\nu,k)]^2\mu_2(n,\nu,k) - 3[\mu_1(n,\nu,k)]^4.$$

Thus the above-mentioned parameters for the distribution of zeros of the co-recursive Hermite polynomials at levels $k=0$ and $k=1$ are

$$\delta_1(n,\nu,0) = \delta_1(n,\nu,1) = \frac{1}{n}\sqrt{\tfrac{1}{2}(n-1)(n^2+2\nu^2)},$$

$$\delta_2(n,\nu,0) = \frac{\sqrt{2}\nu[2(n^2-3n+2)\nu^2 - 3n^2(n-2)]}{[(n-1)(n^2+2\nu^2)]^{3/2}},$$

$$\delta_2(n,\nu,1) = \frac{\sqrt{2}\nu[2(n^2-3n+2)\nu^2 - 3n^2(n-4)]}{[(n-1)(n^2+2\nu^2)]^{3/2}},$$

$$\delta_3(n,\nu,0) = \frac{4(n^3-7n^2+12n-6)\nu^4 - 4n^2(3n^2-11n+12)\nu^2 - (n^6-n^5)}{[(n-1)(n^2+2\nu^2)]^2},$$

$$\delta_3(n,\nu,1) = \frac{4(n^3-7n^2+12n-6)\nu^4 - 12n^2(n^2-5n+8)\nu^2 - (n^6-n^5)}{[(n-1)(n^2+2\nu^2)]^2}$$

and those corresponding to the zero distribution of the classical Hermite polynomials are obtained from these ones by taking $\nu = 0$.

From these expressions the following qualitative effects of the co-recursive perturbation in the distribution of zeros can be deduced.

(a) For fixed ν and n large enough ($\nu \ll n$), the co-recursive perturbation (at both levels $k=0,1$) has a negligible effect in the zero distribution. Moreover, as pointed out above, in the asymptotic limit ($n \to \infty$) the distribution of zeros of the perturbed and unperturbed polynomials coincide. When contracted to the interval $[-1,1]$, this zero distribution follows the semicircular law (cf., e.g., [12,22]).

(b) For fixed n and ν large enough ($\nu \gg n$), the perturbations at both levels considered here ($k=0,1$) produce almost the same effect on the distribution of zeros of the starting Hermite family. Besides the afore-mentioned shift on the centroid, since $\delta_1(n,\nu,0)$ ($=\delta_1(n,\nu,1)$) is an increasing function of ν, the scattering of the zeros on the real line of the nth-degree perturbed polynomial increases with $|\nu|$ (notice that it has its minimum for the unperturbed family (i.e., at $\nu=0$)). Moreover, in this case, for $k=0,1$, $\operatorname{sign}(\nu)\delta_2(n,\nu,k) > 0$, so the extent of departure from symmetry of the zero distribution with respect to the centroid also increases with $|\nu|$. Notice that the skewness is zero for three different values of ν, in spite of which the corresponding zero distribution is symmetric only for the unperturbed family (this is the only case where every odd moment is zero). Concerning the kurtosis parameter, one has $\delta_3(n,\nu,k) > 0$, $k=0,1$, which allows to conclude that for $\nu \gg n$, the zero distribution is leptokurtic [17], i.e., it is more sharply peaked around its centroid than the Gaussian distribution, and this effect also increases with $|\nu|$.

(c) To compare the effects of the perturbation at level $k=0$ with those corresponding to the level $k=1$, an adequate range of the parameters n and ν has to be chosen because, as pointed out in (a)

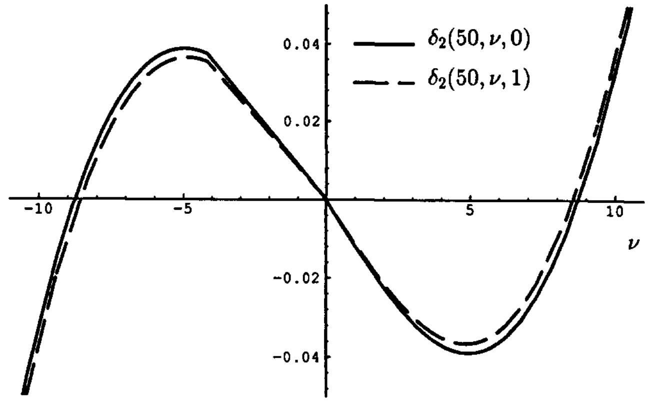

Fig. 1. Skewness ($\delta_2(50,\nu,k)$) of the zero distribution of the 50th-degree co-recursive Hermite polynomial at levels $k=0$ (solid line) and $k=1$ (dashed line). The value of this parameter for the 50th-degree classical Hermite polynomial corresponds to $\nu=0$.

and (b), when $\nu \gg n$ or $n \gg \nu$, there is no appreciable difference between the two modifications. Moreover, notice that for the classical Hermite polynomials, the shift on the centroid and the variance do not depend on the level of recursivity [21], because in this case one has $\forall n \in \mathbb{N}, \beta_n = 0$. Thus, to make this comparison, only the skewness and kurtosis parameters should be considered.

For the sake of completeness, in Figs. 1 and 2 these two statistical parameters have been plotted respectively, for $n=50$ and $\nu \in [-10,10]$. Moreover, Fig. 3 shows the behaviour of the differences

$$\Delta(\delta_2) = \delta_2(n,\nu,1) - \delta_2(n,\nu,0), \qquad \Delta(\delta_3) = \delta_3(n,\nu,1) - \delta_3(n,\nu,0)$$

for the 50th-degree polynomial and $\nu \in [-100,100]$. In particular, one can conclude that the differences between the two perturbations become more important when $|\nu|$ goes from 25 to 36 approximately, while for $\nu \gg n$ this difference goes to zero, as it has been pointed out in (b). On the other hand, for the range of the parameters here considered, the kurtosis is negative, which means that the zero distribution is platykurtic [17], i.e., more flat-topped than the Gaussian one. Moreover, for

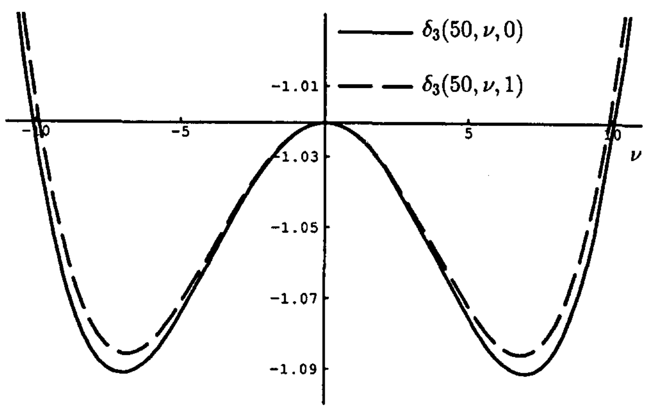

Fig. 2. Kurtosis ($\delta_3(50,\nu,k)$) of the zero distribution of the 50th-degree co-recursive Hermite polynomial at levels $k=0$ (solid line) and $k=1$ (dashed line). The value of this parameter for the 50th-degree classical Hermite polynomial corresponds to $\nu=0$.

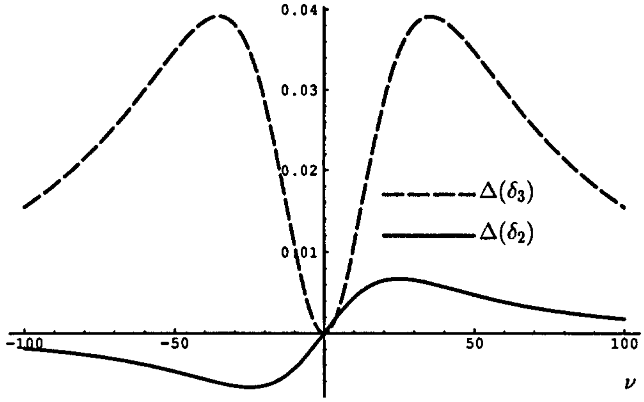

Fig. 3. Differences between the skewness: $\Delta(\delta_2) = \delta_2(50,\nu,1) - \delta_2(50,\nu,0)$ (solid line), and the kurtosis: $\Delta(\delta_3) = \delta_3(50,\nu,1) - \delta_3(50,\nu,0)$ (dashed line) of the zero distribution of the 50th-degree co-recursive Hermite polynomial at levels $k=0$ and $k=1$.

both perturbations ($k = 0, 1$), it becomes mesokurtic for $|\nu| \approx 16.8$ and leptokurtic when $|\nu| > 16.9$. Of course more information could be obtained from these figures and also by taking into account more moments. This is left to the reader.

Appendix A. Coefficients $C_{pq}^{(t)}$ appearing in the fourth-order differential equation satisfied by the co-recursive at level $k = 1$ of the classical class

The indexes p, q and t are related to the $G_p(x;\nu;1;n)$-coefficient, the power of ν and the power of σ, respectively.

$$C_{41}^{(0)} = \tau[(4n-3)\tau\tau' + 2(3-2n)\sigma'\tau' + (2n^2-2n-1)\tau\sigma'' + 2n(1-n)\sigma'\sigma''],$$

$$C_{41}^{(1)} = 4(-1+n)\tau'(2\tau' + n\sigma''),$$

$$C_{40}^{(0)} = 4\tau^4(\tau-\sigma')[(2n-1)\tau\tau' + 2(2-n)\sigma'\tau' + (n^2-n-1)\tau\sigma'' + n(1-n)\sigma'\sigma''],$$

$$C_{40}^{(1)} = 2\tau^2[(16n-15)\tau^2\tau'^2 + 4(3-4n)\tau\sigma'\tau'^2 - 12\sigma'^2\tau'^2 + 2(1-4n+4n^2)\tau^2\tau'\sigma'' + 4(3+2n-2n^2)\tau\sigma'\tau'\sigma'' - 3\tau^2\sigma''^2],$$

$$C_{40}^{(2)} = 8\tau\tau'^2[(4n-1)\tau\tau' + 6\sigma'\tau' + (2n^2-2n-3)\tau\sigma''],$$

$$C_{40}^{(3)} = -24\tau'^4;$$

$$C_{31}^{(0)} = 5\tau^2\sigma'[(4n-3)\tau\tau' + 2(3-2n)\sigma'\tau' + (2n^2-2n-1)\tau\sigma'' + 2n(1-n)\sigma'\sigma''],$$

$$C_{31}^{(1)} = -\tau[5(4n-3)\tau\tau'^2 + 8(7-6n)\sigma'\tau'^2 + (10n^2-14n+1)\tau\tau'\sigma'' + 24n(1-n)\sigma'\tau'\sigma'' + 2n(1-n)\tau\sigma''^2],$$

$$C_{31}^{(2)} = -12(-1+n)\tau'^2(2\tau' + n\sigma''),$$

$$C_{30}^{(0)} = 20\tau^4(\tau - \sigma')\sigma'[(2n-1)\tau\tau' + 2(2-n)\sigma'\tau' + (n^2-n-1)\tau\sigma'' + n(1-n)\sigma'\sigma''],$$

$$\begin{aligned} C_{30}^{(1)} = &-4\tau^2[6(2n-1)\tau^3\tau'^2 + (55-52n)\tau^2\sigma'\tau'^2 + 40(n-1)\tau\sigma'^2\tau'^2 + 24\sigma'^3\tau'^2 \\ &+ (6n^2-10n-1)\tau^3\tau'\sigma'' + (30n-26n^2-7)\tau^2\sigma'\tau'\sigma'' \\ &+ 4(5n^2-5n-6)\tau\sigma'^2\tau'\sigma'' + (1+2n-2n^2)\tau^3\sigma''^2 + 2(n^2-n+3)\tau^2\sigma'\sigma''^2], \end{aligned}$$

$$\begin{aligned} C_{30}^{(2)} = &-8\tau\tau'[(16n-15)\tau^2\tau'^2 + 12(1-2n)\tau\sigma'\tau'^2 - 24\sigma'^2\tau'^2 + (5-12n+8n^2)\tau^2\tau'\sigma'' \\ &+ 12(1+n-n^2)\tau\sigma'\tau'\sigma'' + 2n(1-n)\tau^2\sigma''^2], \end{aligned}$$

$$C_{30}^{(3)} = -16\tau'^3[(4n-1)\tau\tau' + 6\sigma'\tau' + 2n(n-1)\tau\sigma''];$$

$$C_{22}^{(0)} = \tau^2(\tau - 3\sigma')(\tau + \sigma'),$$

$$C_{22}^{(1)} = \tau[2(2+n)\tau\tau' + 15\sigma'\tau' + (n^2-n-6)\tau\sigma''],$$

$$C_{22}^{(2)} = -12\tau'^2,$$

$$\begin{aligned} C_{21}^{(0)} = &\,2\tau^3(\tau - 3\sigma')(-\tau - \sigma') \\ &\times [(3-4n)\tau\tau' + 2(2n-3)\sigma'\tau' + (2n+1-2n^2)\tau\sigma'' + 2n(n-1)\sigma'\sigma''], \end{aligned}$$

$$\begin{aligned} C_{21}^{(1)} = &\,\tau^2[4(4n^2+8n-9)\tau^2\tau'^2 + 2(51n-13-8n^2)\tau\sigma'\tau'^2 + 4(42-27n)\sigma'^2\tau'^2 \\ &+ 8(3-8n+2n^3)\tau^2\tau'\sigma'' + (67n^2-33n-66-16n^3)\tau\sigma'\tau'\sigma'' \\ &+ 54n(1-n)\sigma'^2\tau'\sigma'' + 4(n^4-2n^3-5n^2+6n+3)\tau^2\sigma''^2 \\ &+ n(5n+8n^2-4n^3-9)\tau\sigma'\sigma''^2], \end{aligned}$$

$$\begin{aligned} C_{21}^{(2)} = &\,2\tau\tau'[(16n^2-44n+13)\tau\tau'^2 + 2(45n-57)\sigma'\tau'^2 + (33-2n-38n^2+16n^3)\tau\tau'\sigma'' \\ &+ 45n(n-1)\sigma'\tau'\sigma'' + 4n(n^3-2n^2-2n+3)\tau\sigma''^2], \end{aligned}$$

$$C_{21}^{(3)} = -48(-1+n)\tau'^3(2\tau' + n\sigma''),$$

$$\begin{aligned} C_{20}^{(0)} = &\,4\tau^4(\tau - 3\sigma')(-\tau - \sigma')(\tau - \sigma') \\ &\times [(1-2n)\tau\tau' + 2(n-2)\sigma'\tau' + (1+n-n^2)\tau\sigma'' + n(n-1)\sigma'\sigma''], \end{aligned}$$

$$
\begin{aligned}
C_{20}^{(1)} = 2\tau^3[& (8n^2+24n-21)\tau^3\tau'^2 + (933+26n-16n^2)\tau^2\sigma'\tau'^2 \\
& + 2(4n^2-49n+51)\tau\sigma'^2\tau'^2 \\
& + 24(2n+3)\sigma'^3\tau'^2 + 2(3-18n+2n^2+4n^3)\tau^3\tau'\sigma'' \\
& + (5n+29n^2-44-16n^3)\tau^2\sigma'\tau'\sigma'' + (8n^3-57n^2+55n-6)\tau\sigma'^2\tau'\sigma'' \\
& + 24n(n-1)\sigma'^3\tau'\sigma'' + (7+12n-10n^2-4n^3+2n^4)\tau^3\sigma''^2 \\
& + (3-9n+5n^2+8n^3-4n^4)\tau^2\sigma'\sigma''^2 + n(2n^3-4n^2+5n-3)\tau\sigma'^2\sigma''^2],
\end{aligned}
$$

$$
\begin{aligned}
C_{20}^{(2)} = 2\tau^2[& (32n^2-42n-11)\tau^2\tau'^3 + 2(64n-16n^2-87)\tau\sigma'\tau'^3 - 132n\sigma'^2\tau'^3 \\
& + (48+9n-53n^2+32n^3)\tau^2\tau'^2\sigma'' + 2(57-2n+48n^2-16n^3)\tau\sigma'\tau'^2\sigma'' \\
& + 2(36+33n-33n^2)\sigma'^2\tau'^2\sigma'' + (8n^4-16n^3+2n^2-12n-27)\tau^2\tau'\sigma''^2 \\
& + 2(8n^3-4n^4+11n^2-15n-36)\tau\sigma'\tau'\sigma''^2 + 9(2+n-n^2)\tau^2\sigma''^3],
\end{aligned}
$$

$$
\begin{aligned}
C_{20}^{(3)} = 4\tau\tau'^2[& (16n^2-20n+43)\tau\tau'^2 + 4(15+21n)\sigma'\tau'^2 + (16n^3-26n^2-26n-51)\tau\tau'\sigma'' \\
& + 6(7n^2-7n-12)\sigma'\tau'\sigma'' + 2(18+9n-7n^2-4n^3+2n^4)\tau\sigma''^2],
\end{aligned}
$$

$$
C_{20}^{(4)} = -72(1+n)\tau'^4(2\tau'-2\sigma''+n\sigma'');
$$

$$
C_{12}^{(0)} = \tau^2[2\tau^2\tau' + 2(7+3n)\tau\sigma'\tau' + 12\sigma'^2\tau' - 6\tau^2\sigma'' + 3(n^2-n-2)\tau\sigma'\sigma''],
$$

$$
C_{12}^{(1)} = -4\tau\tau'[2(2+n)\tau\tau' + 15\sigma'\tau' + (6-n+n^2)\tau\sigma''],
$$

$$
C_{12}^{(2)} = 48\tau'^3,
$$

$$
\begin{aligned}
C_{11}^{(0)} = 2\tau^3[& 2(n-1)\tau^3\tau'^2 + 2(6n^2+6n-5)\tau^2\sigma'\tau'^2 + 2(8+11n-6n^2)\tau\sigma'^2\tau'^2 \\
& + 12(2-n)\sigma'^3\tau'^2 + (5-11n+n^2)\tau^3\tau'\sigma'' + (12n^3-6n^2-10n-13)\tau^2\sigma'\tau'\sigma'' \\
& + (23n^2-5n-12n^3-18)\tau\sigma'^2\tau'\sigma'' + 6n(1-n)\sigma'^3\tau'\sigma'' + (3+5n-5n^2)\tau^3\sigma''^2 \\
& + (3+2n+n^2-6n^3+3n^4)\tau^2\sigma'\sigma''^2 + 3n(2n^2-n^3-1)\tau\sigma'^2\sigma''^2],
\end{aligned}
$$

$$
\begin{aligned}
C_{11}^{(1)} = -\tau^2[& 2(20n^2+13n-15)\tau^2\tau'^3 + (236n-64n^2-52)\tau\sigma'\tau'^3 + (192-72n)\sigma'^2\tau'^3 \\
& + (36-83n-35n^2+40n^3)\tau^2\tau'^2\sigma'' + (182n^2-72-142n-64n^3)\tau\sigma'\tau'^2\sigma'' \\
& + 36n(1-n)\sigma'^2\tau'^2\sigma'' + (10n^4-28n^3-17n^2+59n-6)\tau^2\tau'\sigma''^2 \\
& + (12n-28n^2+32n^3-16n^4)\tau\sigma'\tau'\sigma''^2 + (10n^2-12n+4n^3-2n^4)\tau^2\sigma''^3],
\end{aligned}
$$

$$C_{11}^{(2)} = -12\tau\tau'^2[(7-16n+4n^2)\tau\tau'^2 + 4(3n-5)\sigma'\tau'^2 + (3+8n-12n^2+4n^3)\tau\tau'\sigma'' + 6n(n-1)\sigma'\tau'\sigma'' + n^2(1-2n+n^2)\tau\sigma''^2],$$

$$C_{11}^{(3)} = 48(-1+n)\tau'^4(2\tau' + n\sigma''),$$

$$C_{10}^{(0)} = 4\tau^3(\tau-\sigma')[-4(1+3n+3n^2)\tau^2\sigma'\tau'^2 + (12n^2-36n-28)\tau\sigma'^2\tau'^2 - 24\sigma'^3\tau'^2 + 2(1+4n)\tau^3\tau'\sigma'' + (26+14n+6n^2-12n^3)\tau^2\sigma'\tau'\sigma'' + (24+18n-30n^2+12n^3)\tau\sigma'^2\tau'\sigma'' + (4n^2-4n-6)\tau^3\sigma''^2 + (n^2+6n^3-3n^4-4n-6)\tau^2\sigma'\sigma''^2 + 3n^2(1-2n+n^2)\tau\sigma'^2\sigma''^2],$$

$$C_{10}^{(1)} = 4\tau^2[4(1+2n+6n^2)\tau^3\tau'^3 + (58+112n-72n^2)\tau^2\sigma'\tau'^3 + (108-60n+48n^2)\tau\sigma'^2\tau'^3 + 24(n-3)\sigma'^3\tau'^3 + (24n^3-28n^2-14n-29)\tau^3\tau'^2\sigma'' + (136n^2-120-110n-72n^3)\tau^2\sigma'\tau'^2\sigma'' + (6n-12-78n^2+48n^3)\tau\sigma'^2\tau'^2\sigma'' + 12n(n-1)\sigma'^3\tau'^2\sigma'' + (32+19n+9n^2-20n^3+6n^4)\tau^3\tau'\sigma''^2 + (54+33n-53n^2+44n^3-18n^4)\tau^2\sigma'\tau'\sigma''^2 + 12n(1-2n^2+n^3)\tau\sigma'^2\tau'\sigma''^2 + (4n^3-2n^4-15-7n+5n^2)\tau^3\sigma''^3 + (2n^4-4n^3+5n^2-3n)\tau^2\sigma'\sigma''^3],$$

$$C_{10}^{(2)} = 8\tau\tau'[(32n^2-54n-17)\tau^2\tau'^3 + (8n-32n^2-114)\tau\sigma'\tau'^3 + (36-24n)\sigma'^2\tau'^3 + (61+61n-67n^2+32n^3)\tau^2\tau'^2\sigma'' + (48+8n+36n^2-32n^3)\tau\sigma'\tau'^2\sigma'' + 12n(1-n)\sigma'^2\tau'^2\sigma'' + (-30-17n+33n^2-24n^3+8n^4)\tau^2\tau'\sigma''^2 + (-6n-2n^2+16n^3-8n^4)\tau\sigma'\tau'\sigma''^2 + (-2n^2+4n^3-2n^4)\tau^2\sigma''^3],$$

$$C_{10}^{(3)} = 16\tau'^3[(29+2n+8n^2)\tau\tau'^2 + 6(n-1)\sigma'\tau'^2 + (-15-n-7+8n^3)\tau\tau'\sigma'' + 3n(n-1)\sigma'\tau'\sigma'' + (2n^2-4n^3+2n^4)\tau\sigma''^2];$$

$$C_{02}^{(0)} = \tau^2[4(1-2n+n^2)\tau^2\tau'^2 + 4(3n+7)\tau\sigma'\tau'^2 + 24\sigma'^2\tau'^2 + 4(n^3-2n^2-3)\tau^2\tau'\sigma'' + 6(n^2-n-2)\tau\sigma'\tau'\sigma'' + (2n-n^2-2n^3+n^4)\tau^2\sigma''^2],$$

$$C_{02}^{(1)} = -8\tau\tau'^2[2(2+n)\tau\tau' + 15\sigma'\tau' + (n^2-n-6)\tau\sigma''],$$

$$C_{02}^{(2)} = 96\tau'^4,$$

$$C_{01}^{(0)} = -\tau^2[(16n^3-36n^2+28n-8)\tau^3\tau'^3+(132n^2-48-12n-16n^3)\tau^2\sigma'\tau'^3$$
$$+(8+120n-24n^2)\tau\sigma'^2\tau'^3+48\sigma'^3\tau'^3$$
$$+(24-30n+22n^2-60n^3+24n^4)\tau^3\tau'^2\sigma''$$
$$+(156n^3-24n^4-66n-134n^2)\tau^2\sigma'\tau'^2\sigma''+(84n^2-24n^3-24-84n)\tau\sigma'^2\tau'^2\sigma''$$
$$+(20n+11n^2+2n^3-33n^4+12n^5)\tau^3\tau'\sigma''^2$$
$$+(48n-7n^2-74n^3+57n^4-12n^5)\tau^2\sigma'\tau'\sigma''^2$$
$$+(12n-18n^2+12n^3-6n^4)\tau\sigma'^2\tau'\sigma''^2$$
$$+(2n^6-6n^5-n^4+12n^3-n^2-6n)\tau^3\sigma''^3$$
$$+(-6n+7n^2-5n^4+6n^5-2n^6)\tau^2\sigma'\sigma''^3],$$

$$C_{01}^{(1)} = -2\tau\tau'[(34+2n-88n^2+16n^3)\tau^2\tau'^3+(112-268n+36n^2)\tau\sigma'\tau'^3$$
$$+(-48-72n)\sigma'^2\tau'^3+(-64+105n+89n^2-112n^3+24n^4)\tau^2\tau'^2\sigma''$$
$$+(-48+278n-170n^2+36n^3)\tau\sigma'\tau'^2\sigma''+36n(1-n)\sigma'^2\tau'^2\sigma''$$
$$+(30-101n+31n^2+56n^3-46n^4+12n^5)\tau^2\tau'\sigma''^2$$
$$+(-72n+81n^2-18n^3+9n^4)\tau\sigma'\tau'\sigma''^2$$
$$+(24n-24n^2-2n^3+6n^4-6n^5+2n^6)\tau^2\sigma''^3],$$

$$C_{01}^{(2)} = 8\tau'^3[(29-52n+8n^2)\tau\tau'^2+(-6-8n)\sigma'\tau'^2+(-15+50n-34n^2+8n^3)\tau\tau'\sigma''$$
$$+9n(1-n)\sigma'\tau'\sigma''+(-12n+14n^2-4n^3+2n^4)\tau\sigma''^2],$$

$$C_{00}^{(0)} = 2\tau^2(\tau-\sigma')[(2-6n+4n^2)\tau^3\tau'^2+(-10+42n-4n^2)\tau^2\sigma'\tau'^2+52\tau\sigma'^2\tau'^2+24\sigma'^3\tau'^2$$
$$+(-7n-7n^2+4n^3)\tau^3\tau'\sigma''+(-32-23n+25n^2-4n^3)\tau^2\sigma'\tau'\sigma''$$
$$-24\tau\sigma'^2\tau'\sigma''+(6+5n-4n^2-2n^3+n^4)\tau^3\sigma''^2$$
$$+(6+n-2n^2+2n^3-n^4)\tau^2\sigma'\sigma''^2],$$

$$C_{00}^{(1)} = \tau[(46-150n+32n^2)\tau^3\tau'^3+(-300+136n-32n^2)\tau^2\sigma'\tau'^3+(-80-112n)\tau\sigma'^2\tau'^3$$
$$+96\sigma'^3\tau'^3+(58+111n-107n^2+32n^3)\tau^3\tau'^2\sigma''$$
$$+(236+28n+100n^2-32n^3)\tau^2\sigma'\tau'^2\sigma''$$
$$+(72+56n-56n^2)\tau\sigma'^2\tau'^2\sigma''+(-62-48n+26n^2-16n^3+8n^4)\tau^3\tau'\sigma''^2$$
$$+(-120-48n+40n^2+16n^3-8n^4)\tau^2\sigma'\tau'\sigma''^2+(30+15n-15n^2)\tau^3\sigma''^3],$$

$$C_{00}^{(2)} = 4\tau'^2[(49-18n+8n^2)\tau^2\tau'^2 + (82+56n)\tau\sigma'\tau'^2 - 12\sigma'^2\tau'^2$$
$$+ (-65-13n-17n^2+8n^3)\tau^2\tau'\sigma'' + (-60-28n+28n^2)\tau\sigma'\tau'\sigma''$$
$$+ (30+11n-9n^2-4n^3+2n^4)\tau^2\sigma''^2],$$
$$C_{00}^{(3)} = -12\tau'^4[(16+10n)\tau' + (-10-5n+5n^2)\sigma''].$$

Appendix B. Polynomial form (in terms of powers of x) of the coefficients $J_{pq}^{(t)}$ appearing in the fourth-order differential equation satisfied by the co-recursive Jacobi at level $k = 1$

The indexes p, q and t are related to the $G_p(x;\nu;1;n)$-coefficient, the power of ν and the power of σ, respectively. Moreover, the coefficients $\alpha_{pq}^{(t)}$, $\beta_{pq}^{(t)}$, $\gamma_{pq}^{(t)}$ and $\delta_{pq}^{(t)}$ are polynomials in the Jacobi parameters and in the degree n. Their explicit expressions have been omitted because it would take several pages to write them down explicitly. Anyway, they can be obtained from the general expressions given in Appendix A.

$$J_{41}^{(0)} = (b-a+b-(a+b+2)x)[\alpha_{41}^{(0)} + \beta_{41}^{(0)}x],$$
$$J_{41}^{(1)} = 8(2+a+b)(n-1)(2+a+b+n),$$
$$J_{40}^{(0)} = 4(b-a-(a+b)x)(a-b+(2+a+b)x)^4[\alpha_{40}^{(0)} + \beta_{40}^{(0)}x],$$
$$J_{40}^{(1)} = 2(a-b+(2+a+b)x)^2[\alpha_{40}^{(1)} + \beta_{40}^{(1)}x + \gamma_{40}^{(1)}x^2],$$
$$J_{40}^{(2)} = 8(2+a+b)^2(b-a-(2+a+b)x)[\alpha_{40}^{(2)} + \beta_{40}^{(2)}x],$$
$$J_{40}^{(3)} = -24(2+a+b)^4;$$

$$J_{31}^{(0)} = 10x(a-b+(2+a+b)x)^2[\alpha_{31}^{(0)} + \beta_{31}^{(0)}x],$$
$$J_{31}^{(1)} = (b-a-(2+a+b)x)[\alpha_{31}^{(1)} + \beta_{31}^{(1)}x],$$
$$J_{31}^{(2)} = 24(2+a+b)^2(n-1)(2+a+b+n),$$
$$J_{30}^{(0)} = 40x(a-b+(a+b)x)(a-b+(2+a+b)x)^4[\alpha_{30}^{(0)} + \beta_{30}^{(0)}x],$$
$$J_{30}^{(1)} = 8(a-b+(2+a+b)x)^2[\alpha_{30}^{(1)} + \beta_{30}^{(1)}x + \gamma_{30}^{(1)}x^2 + \delta_{30}^{(1)}x^3],$$
$$J_{30}^{(2)} = 8(2+a+b)(b-a-(2+a+b)x)[\alpha_{30}^{(2)} + \beta_{30}^{(2)}x + \gamma_{30}^{(2)}x^2],$$
$$J_{30}^{(3)} = 16(2+a+b)^3[\alpha_{30}^{(3)} + \beta_{30}^{(3)}x];$$

$$J_{22}^{(0)} = (a-b+(a+b-4)x)(a-b+(2+a+b)x)^2(a-b+(4+a+b)x),$$
$$J_{22}^{(1)} = 2(b-a-(2+a+b)x)[\alpha_{22}^{(1)} + \beta_{22}^{(1)}x],$$

$$J_{22}^{(2)} = -12(2+a+b)^2,$$

$$J_{21}^{(0)} = 2(b-a-(2+a+b)x)^3(a-b+(a+b-4)x) \times (a-b+(4+a+b)x)[\alpha_{21}^{(0)} + \beta_{21}^{(0)}x],$$

$$J_{21}^{(1)} = 4(a-b+(2+a+b)x)^2[\alpha_{21}^{(1)} + \beta_{21}^{(1)}x + \gamma_{21}^{(1)}x^2],$$

$$J_{21}^{(2)} = 2(2+a+b)(b-a-(2+a+b)x)[\alpha_{21}^{(2)} + \beta_{21}^{(2)}x],$$

$$J_{21}^{(3)} = 96(2+a+b)^3(1-n)(2+a+b+n),$$

$$J_{20}^{(0)} = 4(b-a-(4+a+b)x)(a-b+(a+b)x)(a-b+(a+b-4)x) \times (a-b+(2+a+b)x)^4[\alpha_{20}^{(0)} + \beta_{20}^{(0)}x],$$

$$J_{20}^{(1)} = 2(a-b+(2+a+b)x)^3[\alpha_{20}^{(1)} + \beta_{20}^{(1)}x + \gamma_{20}^{(1)}x^2 + \delta_{20}^{(1)}x^3],$$

$$J_{20}^{(2)} = 2(a-b+(2+a+b)x)^2[\alpha_{20}^{(2)} + \beta_{20}^{(2)}x + \gamma_{20}^{(2)}x^2],$$

$$J_{20}^{(3)} = 4(2+a+b)^2(a-b+(2+a+b)x)[\alpha_{20}^{(3)} + \beta_{20}^{(3)}x],$$

$$J_{20}^{(4)} = 144(2+a+b)^4(1+n)(a+b+n);$$

$$J_{12}^{(0)} = 2(a-b+(2+a+b)x)^2[\alpha_{12}^{(0)} + \beta_{12}^{(0)}x + \gamma_{12}^{(0)}x^2],$$

$$J_{12}^{(1)} = 8(2+a+b)(b-a-(2+a+b)x)[\alpha_{12}^{(1)} + \beta_{12}^{(1)}x],$$

$$J_{12}^{(2)} = -48(2+a+b)^3,$$

$$J_{11}^{(0)} = 4(a-b+(2+a+b)x)^3[\alpha_{11}^{(0)} + \beta_{11}^{(0)}x + \gamma_{11}^{(0)}x^2 + \delta_{11}^{(0)}x^3],$$

$$J_{11}^{(1)} = 2(a-b+(2+a+b)x)^2[\alpha_{11}^{(1)} + \beta_{11}^{(1)}x + \gamma_{11}^{(1)}x^2],$$

$$J_{11}^{(2)} = 12(2+a+b)^2(b-a-(2+a+b)x)[\alpha_{11}^{(2)} + \beta_{11}^{(2)}x],$$

$$J_{11}^{(3)} = 96(2+a+b)^4(1-n)(2+a+b+n),$$

$$J_{10}^{(0)} = 16(a-b+(a+b)x)(a-b+(2+a+b)x)^3[\alpha_{10}^{(0)} + \beta_{10}^{(0)}x + \gamma_{10}^{(0)}x^2 + \delta_{10}^{(0)}x^3],$$

$$J_{10}^{(1)} = 8(a-b+(2+a+b)x)^2[\alpha_{10}^{(1)} + \beta_{10}^{(1)}x + \gamma_{10}^{(1)}x^2 + \delta_{10}^{(1)}x^3],$$

$$J_{10}^{(2)} = 8(2+a+b)(b-a-(2+a+b)x)[\alpha_{10}^{(2)} + \beta_{10}^{(2)}x + \gamma_{10}^{(2)}x^2],$$

$$J_{10}^{(3)} = 16(2+a+b)^3[\alpha_{10}^{(3)} + \beta_{10}^{(3)}x];$$

$$J_{02}^{(0)} = 4(a-b+(2+a+b)x)^2[\alpha_{02}^{(0)} + \beta_{02}^{(0)}x + \gamma_{02}^{(0)}x^2],$$

$$J_{02}^{(1)} = 16(2+a+b)^2(a-b+(2+a+b)x)[\alpha_{02}^{(1)} + \beta_{02}^{(1)}x],$$

$$J_{02}^{(2)} = 96(2+a+b)^4,$$

$$J_{01}^{(0)} = 4(a-b+2x+ax+bx)^2[\alpha_{01}^{(0)} + \beta_{01}^{(0)}x + \gamma_{01}^{(0)}x^2\delta_{01}^{(0)}x^3],$$

$$J_{01}^{(1)} = 4(2+a+b)(a-b+(2+a+b)x)[\alpha_{01}^{(1)} + \beta_{01}^{(1)}x + \gamma_{01}^{(1)}x^2],$$

$$J_{01}^{(2)} = 8(2+a+b)^3[\alpha_{01}^{(2)} + \beta_{01}^{(2)}x],$$

$$J_{00}^{(0)} = 4(a-b+(a+b)x)(a-b+(2+a+b)x)^2[\alpha_{00}^{(0)} + \beta_{00}^{(0)}x + \gamma_{00}^{(0)}x^2\delta_{00}^{(0)}x^3],$$

$$J_{00}^{(1)} = 2(b-a-(2+a+b)x)[\alpha_{00}^{(1)} + \beta_{00}^{(1)}x + \gamma_{00}^{(1)}x^2\delta_{00}^{(1)}x^3],$$

$$J_{00}^{(2)} = 4(2+a+b)^2[\alpha_{00}^{(2)} + \beta_{00}^{(2)}x + \gamma_{00}^{(2)}x^2],$$

$$J_{00}^{(3)} = 24(2+a+b)^4(6+8a+8b+5n+5an+5bn+5n^2).$$

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