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ORIGINAL PAPER

Usage of the homotopy analysis method for solving the nonlinear and linear integral equations of the second kind

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Abstract The paper presents an application of the homotopy analysis method for solving the nonlinear and linear integral equations of the second kind. In this method a series is created, sum of which (if the series is convergent) gives the solution of discussed equation. Conditions ensuring convergence of this series are presented in the paper. Error of approximate solution, obtained by considering only partial sum of the series, is also estimated. Examples illustrating usage of the investigated method are presented as well, including the example having practical application for calculating the charge in supply circuit of flash lamps used in cameras.

Keywords Homotopy analysis method · Nonlinear integral equation · Linear integral equation · Convergence · Error estimations

Mathematics Subject Classifications (2010) 65R20 · 45G10 · 45A05

1 Introduction

Homotopy analysis method was developed by Shijun Liao [28–31, 33]. It enables to solve the operator equations of different kind. In particular, the method has found a number of applications in heat conduction problems [1, 16, 18, 26, 56]. It is also used, among others, for solving the nonlocal initial boundary value problem [35], nonlinear

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reaction-diffusion-convection problems [41] and fractional differential equations [4, 54, 58]. In several papers the method was used for solving the integro-differential equations [9, 14, 20, 43, 57]. Theoretical results concerning, among others, convergence of the method in case of differential equations are included in papers [29, 31, 38, 39, 44–46, 52]. Various modifications of homotopy analysis method have been also elaborated, for example, the spectral homotopy analysis method [37], the optimal homotopy analysis method [19] and the optimal homotopy asymptotic method [22, 34] (see also [33]).

In recent time there have appeared some papers describing application of the homotopy analysis method for solving integral equations. In papers [40, 42] the examples of using the method for solving the system of integral equations are presented. Whereas the examples of applying the method for solving the two-dimensional integral equations can be found in works [5, 12]. Next, in papers [13, 36] the possibility of using the method for solving the fuzzy integral equations is shown. Paper [48] includes the examples of applying the discussed method for solving the Fredholm and Volterra integral equations of the first and second kind. Abbasbandy with the co-authors described the usage of homotopy analysis method for solving the nonlinear Fredholm and Volterra integral equations of the second kind [2, 53]. In case of the Fredholm equation [2] it was proven that if the series obtained in result of homotopy analysis method is convergent then its sum satisfies the discussed equation. The same was proven for the Volterra equation as well [53]. Moreover, in case of this equation the uniqueness of solution was proven and the sufficient condition for convergence of created series was given. Additionally, it was proven in both papers that in the case of investigated integral equations the Adomian decomposition method represents a special case of the homotopy analysis method. Particular case of nonlinear equation (sought function is in power and inside of integral) is considered also in paper [10], in which the uniqueness of solution is proven together with the fact that sum of the series gives the sought solution.

Some modifications of homotopy analysis method have been also used for solving the integral equations. In particular, in paper [8] the discrete homotopy analysis method is applied which has been obtained by combining the homotopy analysis method with quadrature rules. Whereas, in paper [27] the multistage homotopy analysis method is used. In this method the interval, in which the considered equation is defined, is divided into several subintervals and next, in each one of them, the homotopy analysis method is applied.

In the current paper we intend to use the homotopy analysis method for solving the nonlinear and linear integral equations of the second kind. Discussed equations are in the more general form than the ones considered in [2, 10, 48, 53]. In particular, we prove in this paper that under appropriate assumptions the investigated equations possess unique solutions. We prove also that if the series, obtained in the course of using the method, is convergent then its sum is a solution of considered equation. Condition ensuring the series convergence is presented and the error of approximate solution, obtained by taking the partial sum of the series, is estimated. The paper includes also the example of using the homotopy analysis method for determining the approximate solution of the equation having some practical application for calculating the charge in supply circuit of flash lamps used in cameras.

2 Homotopy analysis method

Homotopy analysis method serves for solving the operator equations

$$N(u(x)) = 0, \qquad x \in \Omega, \tag{1}$$

where N denotes the operator (in particular, it can be the nonlinear operator), whereas u is the unknown function. In the first step of method we define the homotopy operator $\mathcal{H}$ in the following way

$$\mathcal{H}(\Phi, p) \equiv (1-p)\, L\big(\Phi(x;p) - u_0(x)\big) - p\, h\, N\big(\Phi(x;p)\big), \tag{2}$$

where $p \in [0,1]$ is the embedding parameter, $h \neq 0$ denotes the convergence control parameter [31, 33, 39, 52], u_0 describes the initial approximation of the solution of problem (1) and L is the auxiliary linear operator with property $L(0) = 0$.

Considering equation $\mathcal{H}(\Phi, p) = 0$ we get the so-called zero-order deformation equation

$$(1-p)\, L\big(\Phi(x;p) - u_0(x)\big) = p\, h\, N\big(\Phi(x;p)\big). \tag{3}$$

For $p = 0$ we have $L(\Phi(x;0) - u_0(x)) = 0$ which implies that $\Phi(x;0) = u_0(x)$. Whereas for $p = 1$ we have $N(\Phi(x;1)) = 0$ which means that $\Phi(x;1) = u(x)$, where u is the sought solution of (1). In this way, the change of parameter p from zero to one corresponds with the change of problem from the trivial problem to the original one (and with the change of solution from u_0 to u).

By expanding function $\Phi(x;p)$ into the Maclaurin series with respect to parameter p we receive

$$\Phi(x;p) = \Phi(x;0) + \sum_{m=1}^{\infty} \frac{1}{m!} \frac{\partial^m \Phi(x;p)}{\partial p^m}\Big|_{p=0} p^m. \tag{4}$$

By designating

$$u_m(x) = \frac{1}{m!} \frac{\partial^m \Phi(x;p)}{\partial p^m}\Big|_{p=0}, \qquad m = 1, 2, 3, \ldots, \tag{5}$$

the previous relation can be written in the following form

$$\Phi(x;p) = u_0(x) + \sum_{m=1}^{\infty} u_m(x)\, p^m. \tag{6}$$

If the above series is convergent for $p = 1$ then we get the sought solution

$$u(x) = \sum_{m=0}^{\infty} u_m(x). \tag{7}$$

In order to determine function u_m we differentiate the left and the right side of relation (3) m times with respect to parameter p, next we divide the received result by $m!$ and we substitute $p = 0$. In this way we obtain the so-called mth-order deformation equation ($m > 0$):

$$L\big(u_m(x) - \chi_m\, u_{m-1}(x)\big) = h\, R_m\big(\overline{u}_{m-1}, x\big), \tag{8}$$

where $\overline{u}_{m-1} = \{u_0(x), u_1(x), \ldots, u_{m-1}(x)\}$ and

$$\chi_m = \begin{cases} 0 & m \leq 1, \\ 1 & m > 1 \end{cases} \tag{9}$$

and

$$R_m\left(\overline{u}_{m-1}, x\right) = \frac{1}{(m-1)!} \left(\frac{\partial^{m-1}}{\partial p^{m-1}} N\Big(\sum_{i=0}^{\infty} u_i(x)\, p^i \Big) \right) \bigg|_{p=0}. \tag{10}$$

If we are not able to determine the sum of series in (7) then we can accept the partial sum of this series

$$\widehat{u}_n(x) = \sum_{m=0}^{n} u_m(x) \tag{11}$$

as the approximate solution of considered equation.

Appropriate selection of the convergence control parameter h has a big influence on the convergence region of series (7) and on the convergence rate as well [33, 38, 45]. One of the methods for selecting the value of convergence control parameter is the so-called h-curve. To obtain this curve we need to investigate the behavior of a certain quantity of the exact solution as a function of parameter h [29, 46]. This method enables to determine the effective region of the convergence control parameter, however it does not give the possibility to determine the value ensuring the fastest convergence [33]. Another method is the so-called "optimization method" proposed in paper [55] (see also [6, 33]). In this method we define the squared residual of governing equation

$$E_n(h) = \int_{\Omega} \Big(N\big[\widehat{u}_n(x)\big] \Big)^2 dx. \tag{12}$$

Optimal value of the convergence control parameter is obtained by finding minimum of this squared residual. Whereas the effective region of the convergence control parameter is defined as

$$\mathbf{R}_h = \big\{ h : \lim_{n\to\infty} E_n(h) = 0 \big\}. \tag{13}$$

For speed up the calculations Liao [33] suggested to replace the integral in formula (12) by its approximate value obtained by applying the quadrature rules. In examples presented by Liao the received optimal values of the convergence control parameter differ not much from the values obtained by applying formula (12). The residual error method for obtaining the convergence control parameter (h), shown in (12), has been applied recently to a number of problems for nonlinear ODEs and PDEs [3, 7, 11, 19, 21, 32, 39, 49–51].

3 Nonlinear integral equation

We consider equation of the form

$$u(x) - \int_{f(x)}^{g(x)} K(x,t)\, R(u(t))\, dt = F(x), \tag{14}$$

where $x \in [a, b]$, $R : C[a, b] \to C[a, b]$ is the nonlinear operator, $f, g \in C[a, b]$, $a \leq f(x) \leq g(x) \leq b$, $K \in C([a, b] \times [a, b])$ and $F \in C[a, b]$, whereas function u is sought. We assume that R is the operator satisfying the Lipschitz condition

$$\|R(v_1) - R(v_2)\| \leq s\, \|v_1 - v_2\| \qquad \text{for every} \quad v_1, v_2 \in C[a, b]$$

and some $s > 0$. As norm of the function we take the supremum norm

$$\|v\| = \sup_{x \in \Omega} |v(x)|,$$

in particular

$$\|K\| = \sup_{(x,t) \in [a,b] \times [a,b]} |K(x, t)| \qquad \text{and} \qquad \|F\| = \sup_{x \in [a,b]} |F(x)|.$$

Special cases of the above equation are the Fredholm and Volterra integral equations of the second kind.

Operators L and N can be define in the following way

$$L(v) = v, \qquad N(v) = v(x) - F(x) - \int_{f(x)}^{g(x)} K(x, t)\, R(v(t))\, dt. \tag{15}$$

Let $u_0 \in C[a, b]$. In this case, by applying the homotopy analysis method we get the following formula for functions u_m:

$$u_m(x) = \chi_m\, u_{m-1}(x) + h\, R_m\big(\overline{u}_{m-1}, x\big), \tag{16}$$

where χ_m and R_m are defined by relations (9) and (10), respectively. By using definitions of the respective operators we obtain

$$u_1(x) = h \left(u_0(x) - F(x) - \int_{f(x)}^{g(x)} K(x, t)\, R(u_0(t))\, dt \right), \tag{17}$$

and for $m \geq 2$:

$$\begin{aligned} u_m(x) &= (1 + h)\, u_{m-1}(x) \\ &\quad - \frac{h}{(m-1)!} \int_{f(x)}^{g(x)} K(x, t) \left(\frac{\partial^{m-1}}{\partial p^{m-1}} R \left(\sum_{i=0}^{\infty} u_i(t)\, p^i \right) \right)_{p=0} dt. \end{aligned} \tag{18}$$

In literature one can find the expression $\frac{\partial^{m-1}}{\partial p^{m-1}} R \left(\sum_{i=0}^{\infty} u_i(x)\, p^i \right)_{p=0}$ calculated for various nonlinear operators R. Most of these results are collected in monograph [33].

Now we proceed to prove that under appropriate assumptions (14) possesses a unique solution.

Theorem 1 *If the following condition is fulfilled*

$$s\, \|K\|\, (b - a) < 1, \tag{19}$$

then (14) *possesses at most one solution.*

Proof Let us suppose that there exist two solutions u_1 and u_2. Thus we have

$$\begin{aligned}\|u_1 - u_2\| &= \| \int_{f(x)}^{g(x)} K(x,t) \big(R(u_1(t)) - R(u_2(t))\big)\, dt\| \\ &\leq \|K\| \int_{f(x)}^{g(x)} \|R(u_1) - R(u_2)\|\, dt \leq s\, \|K\|\, (b-a)\, \|u_1 - u_2\|.\end{aligned}$$

Hence we get

$$\big(1 - s\, \|K\|\, (b-a)\big)\, \|u_1 - u_2\| \leq 0.$$

So, if condition (19) is satisfied then equality $u_1 = u_2$ must hold true. □

We proceed now to prove the theorem ensuring that the sum of determined series is the solution of discussed equation.

Theorem 2 *Let functions u_m, $m \geq 1$, be defined by relations* (17) *and* (18). *Then, if $s < 1$ and series in* (7) *is convergent, the sum of this series is the solution of* (14).

Proof Let series (7) be convergent. From the necessary condition for the series convergence we get that for any $x \in [a, b]$:

$$\lim_{m\to\infty} u_m(x) = 0.$$

Let us designate

$$H_m(x) = \frac{1}{m!} \left(\frac{\partial^m}{\partial p^m} R\left(\sum_{i=0}^{+\infty} u_i(x)\, p^i \right) \right) \Bigg|_{p=0}.$$

If R is the contraction mapping ($s < 1$) and series (7) converges to $u(x)$ then series $\sum_{m=0}^{\infty} H_m(x)$ converges to $R(u(x))$ (see [17]).

By using definition of operator L we can write

$$\begin{aligned}\sum_{m=1}^{n} L\big(u_m(x) - \chi_m\, u_{m-1}(x)\big) &= \sum_{m=1}^{n} \big(u_m(x) - \chi_m\, u_{m-1}(x)\big) \\ &= u_1(x) + (u_2(x) - u_1(x)) + (u_3(x) - u_2(x)) + \ldots \\ &\quad + (u_n(x) - u_{n-1}(x)) = u_n(x).\end{aligned}$$

Hence

$$\sum_{m=1}^{\infty} L\big(u_m(x) - \chi_m\, u_{m-1}(x)\big) = \lim_{n\to\infty} u_n(x) = 0.$$

From (8) we receive

$$h \sum_{m=1}^{\infty} R_m\big(\overline{u}_{m-1}, x\big) = \sum_{m=1}^{\infty} L\big(u_m(x) - \chi_m\, u_{m-1}(x)\big),$$

that is, since $h \neq 0$ thus we have

$$\sum_{m=1}^{\infty} R_m\left(\overline{u}_{m-1}, x\right) = 0.$$

In result of some transformations we get successively

$$\begin{aligned}
0 &= \sum_{m=1}^{\infty} R_m\left(\overline{u}_{m-1}, x\right) \\
&= \sum_{m=1}^{\infty} \left(\frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial p^{m-1}} \left[\sum_{i=1}^{\infty} u_i(x)\, p^i - F(x) \right.\right. \\
&\qquad \left.\left. - \int_{f(x)}^{g(x)} K(x,t)\, R\left(\sum_{i=1}^{\infty} u_i(t)\, p^i \right) dt \right]_{p=0} \right) \\
&= \sum_{m=1}^{\infty} \left(u_{m-1}(x) - \frac{1-\chi_m}{(m-1)!} F(x) \right. \\
&\qquad \left. - \int_{f(x)}^{g(x)} K(x,t) \left[\frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial p^{m-1}} R\left(\sum_{i=1}^{\infty} u_i(t)\, p^i \right) \right]_{p=0} dt \right) \\
&= \sum_{m=1}^{\infty} \left(u_{m-1}(x) - \frac{1-\chi_m}{(m-1)!} F(x) - \int_{f(x)}^{g(x)} K(x,t)\, H_{m-1}(t)\, dt \right) \\
&= \sum_{m=1}^{\infty} u_{m-1}(x) - F(x) - \int_{f(x)}^{g(x)} K(x,t) \sum_{m=1}^{\infty} H_{m-1}(t)\, dt \\
&= u(x) - F(x) - \int_{f(x)}^{g(x)} K(x,t)\, R(u(t))\, dt.
\end{aligned}$$

□

Remark 1 In the above proof we used the fact that R is the contraction mapping in order to ensure the convergence of series $\sum_{m=0}^{\infty} H_m(x)$ to $R(u(x))$, in case when series (7) converges to $u(x)$. The same convergence can be obtained under another assumptions, for example, when R belongs to class C^{∞}.

Now we present the sufficient condition for convergence of considered series.

Theorem 3 *If parameter h is selected in such a way that the constants $\beta_h \in (0,1)$ and $k_0 \in \mathbb{N}$ exist such that for each $k \geq k_0$ the following inequality*

$$\|u_{k+1}\| \leq \beta_h \|u_k\|, \tag{20}$$

is satisfied, then the series appearing in (7) is uniformly convergent in interval [a,b].

Proof Let S_n denote the partial sum of considered series

$$S_n = \sum_{m=0}^{n} u_m(x).$$

We intend to show that sequence $\{S_n\}$ is the Cauchy sequence. For this purpose we begin by estimating the following norm

$$\|S_n - S_{n-1}\| = \|u_n\| \leq \beta \|u_{n-1}\| \leq \ldots \leq \beta^{n-k_0} \|u_{k_0}\|.$$

Now for any $n, k \in \mathbb{N}$, $n \geq k \geq k_0$, we have

$$\begin{aligned} \|S_n - S_k\| &\leq \|S_n - S_{n-1}\| + \ldots + \|S_{k+1} - S_k\| \\ &\leq \beta^{n-k_0} \|u_{k_0}\| + \ldots + \beta^{k+1-k_0} \|u_{k_0}\| \\ &= \beta^{k+1-k_0} \left(\beta^{n-k-1} + \ldots + \beta + 1\right) \|u_{k_0}\| = \beta^{k+1-k_0} \frac{1-\beta^{n-k}}{1-\beta} \|u_{k_0}\|. \end{aligned}$$

Since $\beta \in (0, 1)$, therefore it implies that sequence $\{S_n\}$ is the Cauchy sequence. By using the completeness of space $\mathbb{R}$ we can deduce that this sequence is convergent, which implies convergence of the discussed series. □

Remark 2 The above theorem can be generalized in the following way.

If parameter h is selected in such a way that constant $k_0 \in \mathbb{N}$ exists such that for each $k \geq k_0$ there exists $\beta_{h,k} \in (0, 1)$ satisfying conditions

$$\prod_{k=k_0}^{\infty} \beta_{h,k} = 0 \qquad \text{and} \qquad \|u_{k+1}\| \leq \beta_{h,k} \|u_k\|,$$

then the series appearing in (7) is uniformly convergent in interval $[a, b]$.

Next theorem concerns the estimation of error of the approximate solution $\widehat{u}_n$.

Theorem 4 *If assumptions of Theorem 3 are satisfied and additionally if $n \in \mathbb{N}$ and $n \geq k_0$, then we get the following estimation of error of the approximate solution*

$$\|u - \widehat{u}_n\| \leq \frac{\beta_h^{n+1-k_0}}{1-\beta_h} \|u_{k_0}\|. \tag{21}$$

Proof Let $n \in \mathbb{N}$ and $n \geq k_0$. Thus we obtain

$$\begin{aligned} \|u - \widehat{u}_n\| &= \sup_{x\in[a,b]} \left| u(x) - \sum_{m=0}^{n} u_m(x) \right| = \sup_{x\in[a,b]} \left| \sum_{m=n+1}^{\infty} u_m(x) \right| \\ &\leq \sup_{x\in[a,b]} \left(\sum_{m=n+1}^{\infty} |u_m(x)| \right) \leq \sum_{m=n+1}^{\infty} \left(\sup_{x\in[a,b]} |u_m(x)| \right) = \sum_{m=n+1}^{\infty} \|u_m\| \\ &\leq \sum_{m=n+1}^{\infty} \beta_h^{m-k_0} \|u_{k_0}\| = \frac{\beta_h^{n+1-k_0}}{1-\beta_h} \|u_{k_0}\|. \end{aligned}$$

□

Remark 3 In particular case when $k_0 = 0$ the formula for estimation of error of the approximate solution takes the form

$$\|u - \widehat{u}_n\| \leq \frac{\beta_h^{n+1}}{1 - \beta_h} \|u_0\|. \tag{22}$$

Generalization from Remark 2 can be also applied for Theorem 4.

4 Linear integral equation

Let us proceed to consideration of the case when operator $R : C[a, b] \to C[a, b]$ is a bounded linear operator. Thus we assume $\|R\| < \infty$. In this case operator R_m (see (10)) has the form

$$\begin{aligned}
R_m(\overline{u}_{m-1}, x) &= \frac{1}{(m-1)!} \left(\frac{\partial^{m-1}}{\partial p^{m-1}} N \left(\sum_{i=0}^{\infty} u_i(x)\, p^i \right) \right)_{p=0} \\
&= \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial p^{m-1}} \left[\sum_{i=1}^{\infty} u_i(x)\, p^i - F(x) - \int_{f(x)}^{g(x)} K(x,t)\, R \left(\sum_{i=1}^{\infty} u_i(t)\, p^i \right) dt \right]_{p=0} \\
&= \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial p^{m-1}} \left[\sum_{i=1}^{\infty} u_i(x)\, p^i - F(x) - \sum_{i=1}^{\infty} \int_{f(x)}^{g(x)} K(x,t)\, R(u_i(t))\, p^i\, dt \right]_{p=0} \\
&= \frac{1}{(m-1)!} \Bigg((m-1)!\, u_{m-1}(x) - (1 - \chi_m)\, F(x) \\
&\qquad - \int_{f(x)}^{g(x)} K(x,t)\, (m-1)!\, R(u_{m-1}(t))\, dt \Bigg) \\
&= u_{m-1}(x) - \frac{1 - \chi_m}{(m-1)!} F(x) - \int_{f(x)}^{g(x)} K(x,t)\, R(u_{m-1}(t))\, dt.
\end{aligned}$$

By using the above relation and formula (16) we receive the following formulas for functions u_m:

$$u_1(x) = h \left(u_0(x) - F(x) - \int_{f(x)}^{g(x)} K(x,t)\, R(u_0(t))\, dt \right), \tag{23}$$

and for $m \geq 2$:

$$u_m(x) = (1 + h)\, u_{m-1}(x) - h \int_{f(x)}^{g(x)} K(x,t)\, R(u_{m-1}(t))\, dt. \tag{24}$$

In case of the linear integral equation Theorem 1 remains true if we take $s = \|R\|$. Whereas in Theorem 2 we do not have to assume that R is the contraction mapping. Respective theorems can be then formulated in the following way.

Theorem 5 *If R is the bounded linear operator and the condition given below is fulfilled*

$$\|R\|\, \|K\|\, (b - a) < 1, \tag{25}$$

then (14) *possesses at most one solution.*

Theorem 6 *Let R be the bounded linear operator and let functions u_m, $m \geq 1$, be defined by relations* (23) *and* (24)*. Then, if series in* (7) *is convergent, the sum of this series is the solution of* (14).

Proofs of these two above theorems run similarly as the proofs of Theorems 1 and 2.

Theorem corresponding to Theorem 3 can be, in case of linear equation, formulated in the following way.

Theorem 7 *If inequality* (25) *is satisfied then the series appearing in* (7) *is uniformly convergent in interval* $[a, b]$.

Proof Let u_0 be the function of class $C[a, b]$. We intend to find the boundaries for function u_m in interval $[a, b]$:

$$\begin{aligned} |u_1(x)| &= \left| h \left(u_0(x) - F(x) - \int_{f(x)}^{g(x)} K(x, t)\, R(u_0(t))\, dt \right) \right| \\ &\leqslant |h| \Big(|u_0(x)| + |F(x)| + \int_{f(x)}^{g(x)} |K(x, t)|\, |R(u_0(t))|\, dt \Big) \\ &\leqslant |h| \big(\|F\| + \|u_0\| + \|K\|\, \|R\|\, (b - a)\, \|u_0\| \big), \\ |u_2(x)| &= \left| (1 + h) u_1(x) - h \int_{f(x)}^{g(x)} K(x, t)\, R(u_1(t))\, dt \right| \\ &\leqslant \big(|1 + h| + |h|\, \|K\|\, \|R\|\, (b - a) \big)\, \|u_1\| = \beta_h\, \|u_1\|, \end{aligned}$$

where

$$\beta_h := |1 + h| + |h|\, \|K\|\, \|R\|\, (b - a).$$

By using the above result one can easily prove by induction that for $m \geqslant 1$ we have

$$\|u_m\| \leqslant \beta_h^{m-1}\, \|u_1\|.$$

In this way, for the considered series (7) we get

$$\sum_{m=0}^{\infty} u_m(x) \leqslant \sum_{m=0}^{\infty} |u_m(x)| \leqslant \|u_0\| + \|u_1\| \sum_{m=1}^{\infty} \beta_h^{m-1},$$

Last series in the above estimation is the geometrical series with quotient β_h. Therefore, if $\beta_h < 1$ (we have certainly $\beta_h > 0$) then, by virtue of the comparison test, the discussed series is uniformly convergent in interval $[a, b]$.

Let us answer the question whether parameter h can be selected such that $\beta_h < 1$, it means such that

$$|1 + h| + |h|\, \|K\|\, \|R\|\, (b - a) < 1.$$

Last inequality is equivalent to condition (since $h \neq 0$):

$$\|K\|\, \|R\|\, (b - a) < \frac{1 - |1 + h|}{|h|}. \tag{26}$$

One can easily notice that

$$\frac{1-|1+h|}{|h|} = \begin{cases} -1-\frac{2}{h} & \text{for } h < -1, \\ 1 & \text{for } h \in [-1,0), \\ -1 & \text{for } h > 0. \end{cases}$$

It implies that if condition (25) is fulfilled then we are able to choose the value of parameter h such that inequality (26) will be satisfied (for this aim it is enough to take any $h \in [-1, 0)$), which means that $\beta_h < 1$. □

By using the estimation evaluated in the last proof we can prove the following theorem.

Theorem 8 *If inequality* (25) *is fulfilled and* $n \in \mathbb{N}$ *then we get the following estimation of error of the approximate solution*

$$\|u - \widehat{u}_n\| \leqslant \frac{\beta_h^n}{1-\beta_h}\,\|u_1\|, \tag{27}$$

where $\beta_h = |1+h| + |h|\,\|K\|\,\|R\|\,(b-a)$.

Remark 4 Let us notice that if we take $h = -1$ and $u_0(x) = 0$ or $u_0(x) = F(x)$ then the investigated homotopy analysis method is equivalent to the method of successive approximation as well as to the Adomian decomposition method (in case $u_0(x) = 0$ after removing the first term which is identically equal to zero – see also [2, 53]). It concerns the case of nonlinear equations as well. Whereas for $h = -1$ the method is identical with the homotopy perturbation method (see [23–25]).

Remark 5 In literature (see for example [29, 33]) some other formulation for homotopy operator can be found, which is

$$\mathcal{H}(\Phi, p) \equiv (1-p)\,L\big(\Phi(x;p) - u_0(x)\big) - p\,h\,H(x)\,N\big(\Phi(x;p)\big),$$

where H is the auxiliary function. In this case as well, all the results evaluated in this paper (for nonlinear and linear equations too) hold true. Only the formula for constant β_h will change.

5 Examples

Example 1 Let us use the investigated method for solving equation of the form

$$u(x) - \frac{1}{3}\int_0^1 (x-t)\,(u(t))^2\,dt = \frac{8}{9}x + \frac{1}{12}, \tag{28}$$

for $x \in [0, 1]$. Solution of the above equation is given by function $u_e(x) = x$. In considered equation we can take $K(x, t) = x - t$ and $R(u) = u^2/3$. Then we have

$$\|K\| = 1, \qquad s = \frac{2}{3}.$$

Hence

$$s\,\|K\|\,(b-a) = \frac{2}{3},$$

which means that the discussed equation possesses at most one solution and if the constructed series is convergent, then its sum is the solution of this equation.

By taking the initial approximation as zero: $u_0(x) = 0$, we get successively

$$\begin{aligned}
u_1(x) &= -\frac{1}{36}\,h\,(3+32x),\\
u_2(x) &= -\frac{1}{36}\,h\,(1+h)\,(3+32x),\\
u_3(x) &= -\frac{1}{23328}\,h\,\big(648(3+32x)+1296h(3+32x)+h^2(-3+23414x)\big),\\
u_4(x) &= -\frac{1}{7776}\,h\,(1+h)\,\big(216(3+32x)\\
&\quad +432\,h(3+32x)+h^2(-1299+9590x)\big),\\
u_5(x) &= -\frac{1}{7558272}\,h\,\big(209952(3+32x)+839808\,h(3+32x)\\
&\quad +3888\,h^3(-1299+9590x)+1944\,h^2(-3+23414x)\\
&\quad +h^4(-3157593+11898032x)\big),\\
&\;\;\vdots
\end{aligned}$$

In Fig. 1 the plots of logarithm of squared residual E_n for $n = 3, 5, 7$ are presented. In this case the numerically determined, by minimizing the squared residual of governing equations, optimal value of the convergence control parameter was equal to -1 ($h = -1$). Figure 2 presents the h-curve of $u'(0)$.

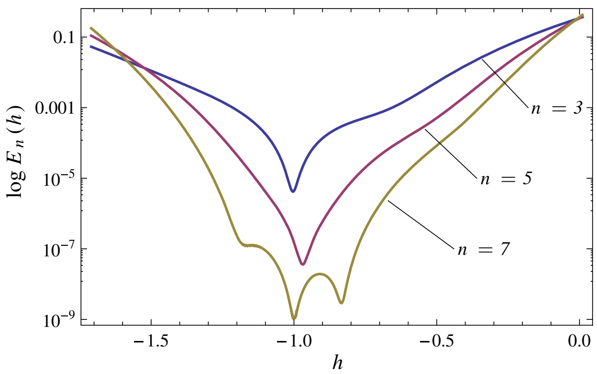

Fig. 1 Logarithm of squared residual E_n for $n = 3, 5, 7$

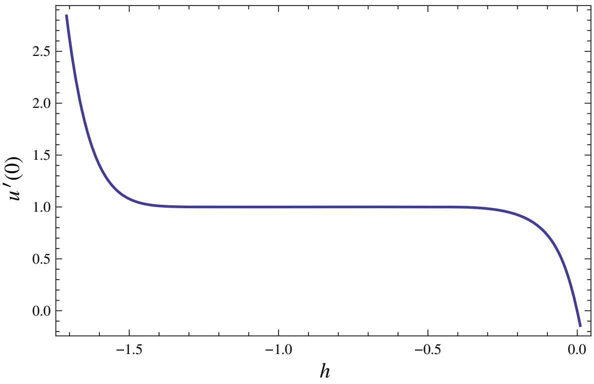

Fig. 2 The h-curve of $u'(0)$

By calculating the partial sums of series in (7) for $h = -1$, which means by determining the successive approximate solutions $\widehat{u}_n$, we obtain

$$\begin{aligned}
\widehat{u}_1(x) &= 0.0833333 + 0.888889\,x,\\
\widehat{u}_3(x) &= -1.28601 \cdot 10^{-4} + 1.00369\,x,\\
\widehat{u}_5(x) &= -4.5685 \cdot 10^{-4} + 1.00019\,x,\\
\widehat{u}_7(x) &= 7.06342 \cdot 10^{-5} + 0.999873\,x,\\
\widehat{u}_9(x) &= 5.05985 \cdot 10^{-6} + x,\\
\widehat{u}_{11}(x) &= -1.57935 \cdot 10^{-6} + x,\\
\widehat{u}_{13}(x) &= 6.01773 \cdot 10^{-8} + x,\\
\widehat{u}_{15}(x) &= 3.35277 \cdot 10^{-8} + x,
\end{aligned}$$

where $x \in [0, 1]$. In considered equation for $h = -1$ the equality $\widehat{u}_{2n-1}(x) \equiv \widehat{u}_{2n}(x)$ holds.

Table 1 compiles the percentage relative errors of the exact solution reconstruction for various values of the convergence control parameter h. As revealed by the above results, together with increase of the components number in sum (11) the errors quickly decrease. The fastest error decrease can be observed for optimal value $h = -1$. For this value the approximate solution $\widehat{u}_5$ provides the approximation of the sought function with the error not higher than $6.3764 \cdot 10^{-2}$ %, while the approximate solution $\widehat{u}_{15}$ gives the error not higher than $4.0685 \cdot 10^{-6}$ %. Whereas, by moving further away from this value, the errors decrease slower.

Differences $|u_e(x) - \widehat{u}_n(x)|$ for $n = 5$ and $n = 15$ are displayed in Fig. 3. Obtained results indicate that the method is very rapidly convergent and calculation of only few first terms of the series ensures very good approximation of the exact solution.

Table 1 Values of the percentage relative errors in reconstruction of the exact solution

n	$h=-1.25$	$h=-1.1$	$h=-1$	$h=-0.9$	$h=-0.75$	$h=-0.5$
1	28.217	13.997	7.349	10.897	24.562	49.437
2	8.382	7.155	7.349	7.155	8.382	24.562
3	6.853	2.399	0.350	2.011	4.104	12.632
4	3.691	0.633	0.350	0.538	1.866	6.937
5	1.960	0.336	$6.376 \cdot 10^{-2}$	0.126	0.770	4.092
6	0.948	$3.643 \cdot 10^{-2}$	$6.376 \cdot 10^{-2}$	$2.733 \cdot 10^{-2}$	0.286	2.531
7	0.293	$5.240 \cdot 10^{-2}$	$6.458 \cdot 10^{-3}$	$2.289 \cdot 10^{-2}$	$8.986 \cdot 10^{-2}$	1.592
8	0.115	$2.194 \cdot 10^{-2}$	$6.458 \cdot 10^{-3}$	$9.243 \cdot 10^{-3}$	$1.926 \cdot 10^{-2}$	0.997
9	0.140	$1.082 \cdot 10^{-2}$	$1.097 \cdot 10^{-3}$	$3.426 \cdot 10^{-3}$	$1.544 \cdot 10^{-3}$	0.616
10	0.113	$4.319 \cdot 10^{-3}$	$1.097 \cdot 10^{-3}$	$1.406 \cdot 10^{-3}$	$4.313 \cdot 10^{-3}$	0.373
11	$7.162 \cdot 10^{-2}$	$2.661 \cdot 10^{-3}$	$1.472 \cdot 10^{-4}$	$6.982 \cdot 10^{-4}$	$3.141 \cdot 10^{-3}$	0.221
12	$4.285 \cdot 10^{-2}$	$1.424 \cdot 10^{-3}$	$1.472 \cdot 10^{-4}$	$2.721 \cdot 10^{-4}$	$1.719 \cdot 10^{-3}$	0.128
13	$3.094 \cdot 10^{-2}$	$4.876 \cdot 10^{-4}$	$1.858 \cdot 10^{-5}$	$7.296 \cdot 10^{-5}$	$8.493 \cdot 10^{-4}$	$7.184 \cdot 10^{-2}$
14	$2.319 \cdot 10^{-2}$	$1.745 \cdot 10^{-4}$	$1.858 \cdot 10^{-5}$	$2.246 \cdot 10^{-5}$	$4.242 \cdot 10^{-4}$	$3.914 \cdot 10^{-2}$
15	$1.471 \cdot 10^{-2}$	$1.150 \cdot 10^{-4}$	$4.069 \cdot 10^{-6}$	$1.474 \cdot 10^{-5}$	$2.208 \cdot 10^{-4}$	$2.055 \cdot 10^{-2}$

Example 2 In the next example we deal with equation which may be practically applied for calculating the charge in supply circuit of flash lamps used in cameras [15, 24]. Supply circuit of flash lamps may be pictured as a simple electrical circuit consisting of source and series connected ideal switch, resistor and capacitor. For modeling the charging or discharging process the first convolution integral may be used. In this example we consider the equation representing the charge referred to some value which gives the input signal for main controller to stop, for example, the charging process of the capacitor or discharging the capacitor by connected flash lamp. Part of this equation, connected with second integral, may represent the

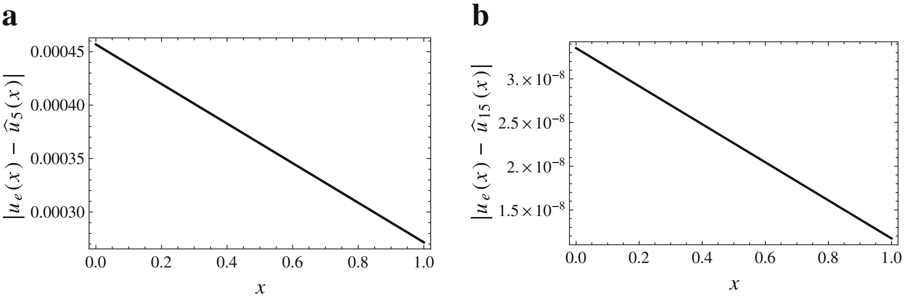

Fig. 3 Distribution of error of the exact solution approximation for $n = 5$ (a) and $n = 15$ (b)

simplified measurement circuit (with characteristic described by integral function) calculating the charge collected on capacitor in some time interval.

Thus we seek for the solution of integral equation of the form

$$u(x) - \int_0^x (x-t)\left(\int_0^t v(\tau)\, u(t-\tau)\, d\tau\right) dt = F(x), \tag{29}$$

for $x \in [0, 1]$ where (see Fig. 4):

$$v(t) = \begin{cases} \frac{1}{2}, & t \in \left[0, \frac{1}{4}\right) \cup \left[\frac{3}{4}, 1\right], \\ 1, & t \in \left[\frac{1}{4}, \frac{3}{4}\right), \end{cases}$$

and

$$F(x) = \begin{cases} \frac{1}{24}\left(48x - x^4\right), & x \in \left[0, \frac{1}{4}\right], \\ \frac{1}{6144}\left(-1 + 12304x - 96x^2 + 256x^3 - 512x^4\right), & x \in \left(\frac{1}{4}, \frac{1}{2}\right], \\ \frac{1}{6144}\left(12319 - 12528x + 672x^2 - 768x^3\right), & x \in \left(\frac{1}{2}, \frac{3}{4}\right], \\ \frac{1}{3072}\left(6281 - 6912x + 1632x^2 - 1536x^3 + 384x^4\right), & x \in \left(\frac{3}{4}, 1\right]. \end{cases}$$

Plot of function F looks illusively like some polyline, but certainly it is not (see formula above).

Exact solution of the above equation has the form (see Fig. 5):

$$u_e(x) = \begin{cases} 2\,x, & x \in \left[0, \frac{1}{2}\right), \\ 2\,(1-x), & x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

In Fig. 6 the plots of logarithm of squared residual E_n for $n = 3, 4, 5$ are shown. Numerically determined, by minimizing the squared residual of governing equations, optimal value of the convergence control parameter was $h = -1$. Figure 7 presents the h-curve of $u'(0)$.

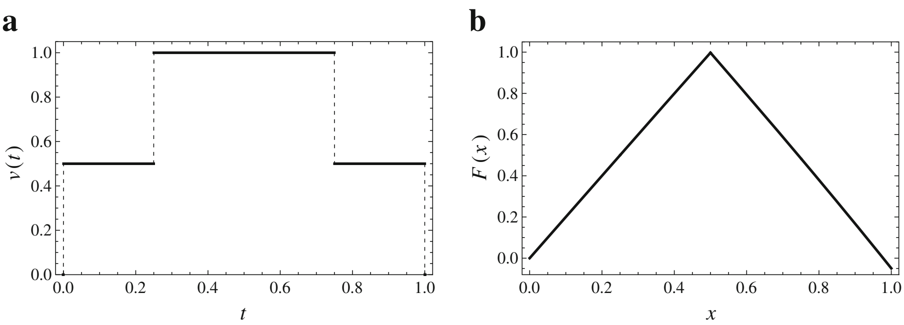

Fig. 4 Functions v and F

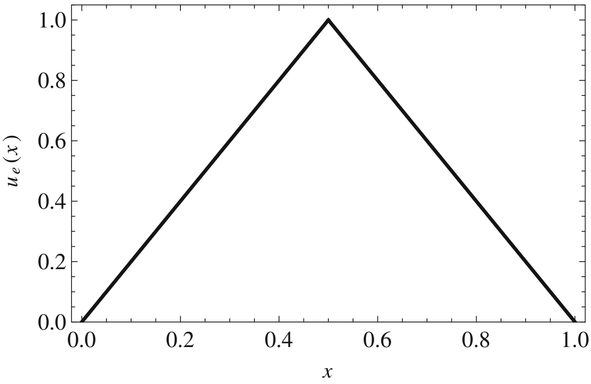

Fig. 5 Exact solution of (29)

By taking the zero initial approximation $u_0(x) = 0$ and the optimal value of convergence control parameter we receive successively

$$u_1(x) = F(x),$$

$$u_2(x) = \begin{cases} \frac{1}{10080}x^4\left(420 - x^3\right), & x \in \left[0, \frac{1}{4}\right], \\ \frac{1}{82575360}\left(13441 - 215068x + 1290576x^2 - 3442880x^3 \right. \\ \left. + 6890240x^4 - 21504x^5 + 28672x^6 - 24576x^7\right), & x \in \left(\frac{1}{4}, \frac{1}{2}\right], \\ \frac{1}{82575360}\left(-416703 + 3226468x - 9036720x^2 \right. \\ \left. + 10337600x^3 - 26880x^4 + 21504x^5 - 16384x^7\right), & x \in \left(\frac{1}{2}, \frac{3}{4}\right], \\ \frac{1}{10321920}\left(-461148 + 2588243x - 5514768x^2 \right. \\ + 5230960x^3 - 1384320x^4 + 75264x^5 - 32256x^6 \\ \left. + 4096x^7\right), & x \in \left(\frac{3}{4}, 1\right]. \end{cases}$$

$$\vdots$$

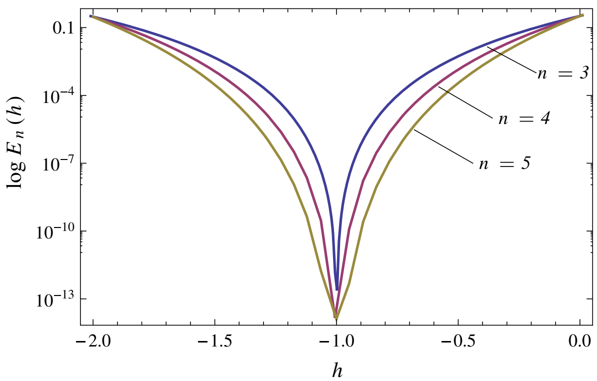

Fig. 6 Logarithm of squared residual E_n for $n = 3, 4, 5$

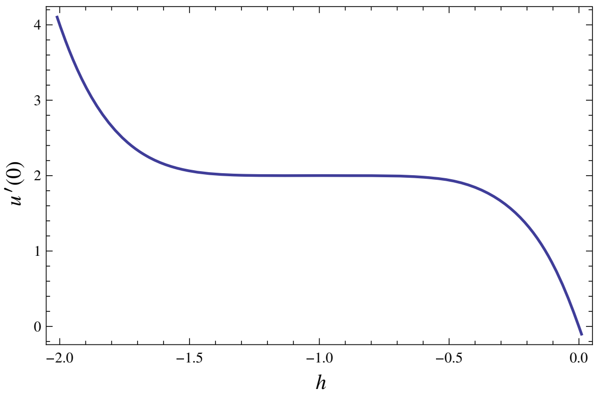

Fig. 7 The h-curve of $u'(0)$

Table 2 displays the absolute errors (Δ) and the percentage relative errors (δ) with which the approximate solution $\widehat{u}_n$ approaches the exact solution u_e. Differences $|u_e(x) - \widehat{u}_n(x)|$ for $n = 2, 4$ are displayed in Fig. 8.

There are also presented in table the estimations of error of the successive approximate solutions resulting from relation (27). In considered example we have $\|K\| = 1$, $\|F\| = 1$ and $\|R\| = 3/4$. Thereby for the optimal value of convergence control parameter $h = -1$ we obtain $\beta_{-1} = 3/4$, $\|u_1\| = \|F\| = 1$ and

$$\|u - \widehat{u}_n\| \leqslant \frac{\beta_{-1}^n}{1 - \beta_{-1}} \|u_1\| = 4\left(\frac{3}{4}\right)^n .$$

Presented data imply that the real errors of approximate solution are significantly smaller than the estimations following from relation (27).

As indicated by the example, with the properly chosen value of convergence control parameter h, if it is impossible to predict a general form of function u_m or calculate the sum of series in (7), it is sufficient to use the sum of several first functions u_m to obtain a very good approximation of sought solution.

Table 2 Errors of the exact solution approximation (Δ – absolute error, δ – percentage relative error) and estimation of the error resulting from relation (27)

n	Δ	δ [%]	(27)
1	$1.6444\,10^{-2}$	2.8481	3.0000
2	$3.1343\,10^{-5}$	$5.4288\,10^{-3}$	2.2500
3	$1.7221\,10^{-8}$	$2.9828\,10^{-6}$	1.6875
4	$4.1791\,10^{-12}$	$7.2385\,10^{-10}$	1.2656
5	$5.4237\,10^{-16}$	$9.3942\,10^{-14}$	0.9492
6	$4.2031\,10^{-20}$	$7.2800\,10^{-18}$	0.7119
7	$2.0945\,10^{-24}$	$3.6277\,10^{-22}$	0.5339
8	$7.0892\,10^{-29}$	$1.2279\,10^{-26}$	0.4005

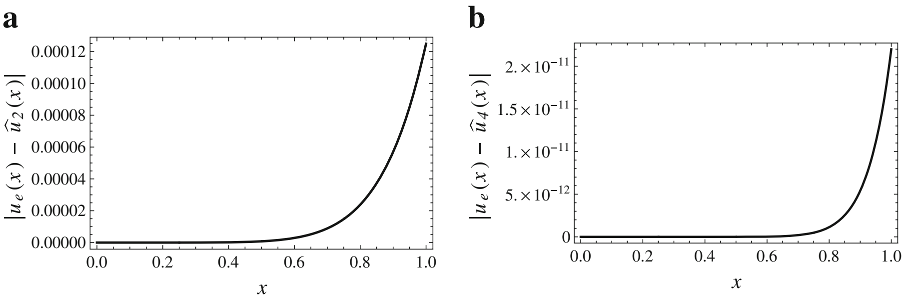

Fig. 8 Distribution of error of the exact solution approximation for $n = 2$ (a) and $n = 4$ (b)

Example 3 Next example shows that the discussed method may be effectively applied for solving linear equations not satisfying inequality (25). Let us consider equation of the form [47]:

$$u(x) - \int_0^\pi \cos x \ \cos t \, u(t) \, dt = \left(1 - \frac{\pi}{2}\right) \cos x + \sin x, \tag{30}$$

for $x \in [0, \pi]$. Solution of the above equation is given by function $u_e(x) = \cos x + \sin x$. In this case we have

$$\|K\| = 1, \qquad \|R\| = 1.$$

Hence

$$\|R\| \, \|K\| \, (b - a) = \pi > 1.$$

By taking the initial approximation of the form

$$u_0(x) = \left(1 - \frac{\pi}{2}\right) \cos x + \sin x,$$

we receive successively

$$\begin{aligned}
u_1(x) &= \frac{1}{4} h \,(\pi - 2)\, \pi \ \cos x, \\
u_2(x) &= -\frac{1}{8} \big(h \,(\pi - 2) - 2\big)\, h \,(\pi - 2)\, \pi \ \cos x, \\
u_3(x) &= \frac{1}{16} \big(h \,(\pi - 2) - 2\big)^2 h \,(\pi - 2)\, \pi \ \cos x, \\
u_4(x) &= -\frac{1}{32} \big(h \,(\pi - 2) - 2\big)^3 h \,(\pi - 2)\, \pi \ \cos x, \\
u_5(x) &= \frac{1}{64} \big(h \,(\pi - 2) - 2\big)^4 h \,(\pi - 2)\, \pi \ \cos x, \\
&\vdots
\end{aligned}$$

One can easily prove by induction that for any $m \in \mathbb{N}$ we get

$$u_m(x) = \left(-\frac{1}{2}\right)^{m+1} \big(h \,(\pi - 2) - 2\big)^{m-1} h \,(\pi - 2)\, \pi \ \cos x.$$

From this we have

$$\begin{aligned} u(x) &= u_0(x) + \sum_{m=1}^{\infty} u_m(x) = \left(1 - \frac{\pi}{2}\right)\cos x + \sin x \\ &\quad + h\,(\pi-2)\,\pi\ \cos x \sum_{m=1}^{\infty}\left(\left(-\frac{1}{2}\right)^{m+1}\ \big(h\,(\pi-2)-2\big)^{m-1}\right) \\ &= \left(1 - \frac{\pi}{2}\right)\cos x + \sin x + h\,(\pi-2)\,\pi\ \cos x\ \frac{\frac{1}{4}}{1-\left(-\frac{1}{2}\right)(h(\pi-2)-2)} \\ &= \left(1 - \frac{\pi}{2}\right)\cos x + \sin x + h\,(\pi-2)\,\pi\ \cos x\ \frac{1}{2h(\pi-2)} \\ &= \cos x + \sin x, \end{aligned}$$

if only the condition of convergence of the geometrical sequence is satisfied, which in this case takes the form

$$\left|\left(-\frac{1}{2}\right)\ \big(h(\pi-2)-2\big)\right| < 1.$$

The above inequality is satisfied if

$$h \in \left(0, \frac{4}{\pi-2}\right).$$

Certainly the above interval determines the effective region of the convergence control parameter $\mathbf{R}_h = (0, 4/(\pi-2))$. It means that by taking any value from the above interval we obtain the exact solution of the above equation.

In considered example we can determine in analytical way the squared residual of governing equations

$$E_n(h) = \frac{1}{512}\,4^{2-n}\,\pi^3\,(\pi-2)^2\,\big(h\,(\pi-2)-2\big)^{2n}, \qquad n \in \mathbb{N}.$$

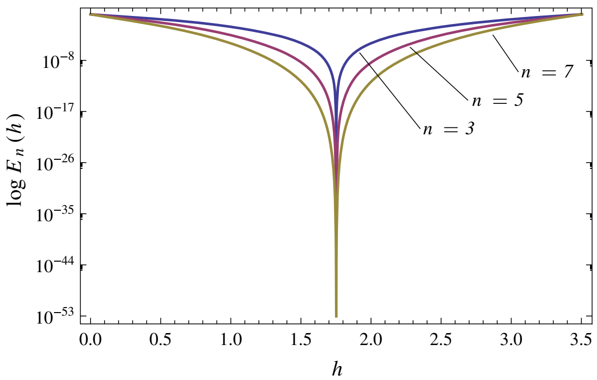

Fig. 9 Logarithm of squared residual E_n for $n = 3, 5, 7$

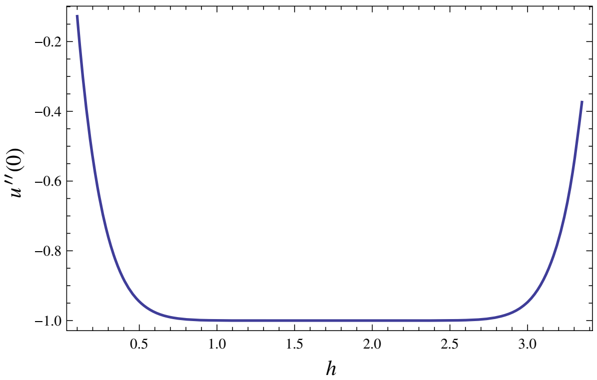

Fig. 10 The h-curve of $u''(0)$

Logarithms of squared residual E_n, for $n = 3, 5, 7$, are presented in Fig. 9. Whereas, Fig. 10 presents the h-curve of $u''(0)$. One can easily verify that for any $n \in \mathbb{N}$ function E_n takes minimum for

$$h = \frac{2}{\pi - 2}.$$

It means that the optimal value of convergence control parameter h is equal to $2/(\pi - 2)$.

6 Conclusion

In this paper the homotopy analysis method has been successfully applied for solving the nonlinear and linear integral equations of the second kind. The homotopy analysis method consists in formulating the series, elements of which are iteratively determined. We prove that if this series is convergent then its sum is a solution of considered equation. This series contains the convergence control parameter h. Appropriate selection of this parameter influence the region of convergence of the series and the convergence rate. In many case we are able to determine analytically the sum of obtained series and, in this way, to compute the exact solution of considered equation. Whereas in cases when analytical determination of sum of the series is impossible, we can use its initial terms for creating the approximate solution. In most of cases, with regard to the rapid convergence of considered series, only few initial terms ensures a very small error of the exact solution approximation. Condition ensuring the convergence of obtained series is presented in the paper and the error of approximate solution received by taking the partial sum of the series is estimated as well. Presented examples show that investigated method is effective in solving the equations of considered kind. Proved Theorems 3 and 7 define the sufficient conditions for convergence of the series constructed in the course of applying the homotopy analysis method. Additionally the third example shows that the method can be used as well for equations not satisfying the conditions given in the above theorems.

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