Pergamon

Computers Math. Applic. Vol. 36, No. 10–12, pp. 391–400, 1998

Printed in Great Britain
0898-1221/98 \$19.00 + 0.00

PII: S0898-1221(98)00230-2

Weighted Convergent and Bounded Solutions of Difference Systems

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Abstract—Using a "summable" dichotomy and fixed-point theorems, we prove the existence of convergent and bounded solutions of nonlinear difference systems. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords—Summable dichotomy, Weighted convergent solutions, Weighted bounded solutions.

1. INTRODUCTION

In several problems, to prove the existence of convergent or bounded solutions of a differential or difference equation is very important (see, [1–11]). In this paper, we study how a "summable" dichotomy insures the existence of both convergent and bounded solutions of semilinear difference systems. Assume that the linear system

$$y(n+1) = A(n)y(n) \tag{1}$$

has a dichotomy, i.e., there exists a projection P for which the behavior of the Green function

$$\Gamma(n,i) = \begin{cases} \Phi(n)P\Phi^{-1}(i+1), & n \ge i, \\ -\Phi(n)(I-P)\Phi^{-1}(i+1), & i > n \end{cases} \tag{2}$$

is known (Φ is a fundamental matrix of (1)).

In [12–15], we have studied dichotomies of "uniform type", that is dichotomies having an estimate

$$\begin{aligned} \left|\Phi(n)P\Phi^{-1}(i)\right| &\le Kh(n)h(i)^{-1}, \qquad n \ge i, \\ \left|\Phi(n)(I-P)\Phi^{-1}(i)\right| &\le Kk(n)k(i)^{-1}, \qquad i \ge n, \end{aligned} \tag{3}$$

where P is a projection matrix and h and k are two given positive sequences. These types of dichotomies, called the (h,k) dichotomies and studied in [10,16], have interesting properties as to be stable under summable perturbations, see [12–15,17–19]. In this paper, we consider "summable" dichotomies, that is satisfying

$$\sum_{i=n_0}^{\infty} |\Gamma(n,i)| h_1(i) \le K h_2(n-1), \qquad n \ge n_0, \tag{4}$$

where Γ is the Green function and h_i $(i = 1,2)$ are given two positive sequences.

We think that the dichotomies in general are decomposed in two big groups: the "uniform" dichotomies and the "summable" dichotomies. The uniform dichotomies are the natural extension

Research supported by Fondecyt 1980835.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX of the ordinary dichotomy and the summable dichotomies are an extension of the exponential dichotomy. Notice, that an exponential dichotomy satisfies (4) with $h_1 \equiv 1 \equiv h_2$. This case was studied in [20,21].

When $h_1 \equiv 1 \equiv h_2$, $\Phi(n)P$ converges exponentially to zero as $n \to \infty$ (see [1,21]), under (4) we have

$$|\Phi(n)P| \le ch_2(n-1) \prod_{i=n_0+1}^{n-1} \left[\frac{Kh_2(i)}{h_1(i)+Kh_2(i)}\right], \qquad n \ge n_0, \tag{5}$$

where c is a constant.

Under condition (4), we investigate the manifold of h_2-convergent solutions of systems

$$x(n+1) = A(n)x(n) + f(n, x(n)) \tag{6}$$

in terms of the manifold of solutions h_2-convergent to zero of the linear system (1), basically assuming that the perturbation term $f(n,x)$ is, in a h_1, h_2 weighted manner, Lipschitzian with respect to x and convergent as $n \to \infty$ (see Theorem 1). Moreover, removing the convergence condition in Theorem 1 below, it is possible to establish a boundedness result for equation (1) (see, Theorem 4).

In equations (1) and (6), x and y are m-vectors, $n \in N(n_0) = \{n_0, n_0+1, \dots\}$, ($n_0$ is a fixed nonnegative integer), $A(n)$ is a $m \times m$ matrix for $n \in N(n_0)$, $f = f(n,x)$ is defined on the product space $N(n_0) \times \mathbb{C}^m$, and $\mathbb{C}^m$ denotes the m-dimensional complex Euclidean m spaces.

The convergence problem of ordinary differential equations has been widely investigated by many researchers most notably by Avramescu [2], Kartsatos and Michaelides [7], Hallam [5,6]. Some results concerning convergent solutions of difference equations were established by Cheng, Li and Patula [3], Drosdowicz and Popenda [4], Agarwal [1], Kelly and Peterson [22], Szafranski and Szmanda [11], Lakshmikantham and Trigiante [23], Aulbach [24]. However, many of them are related with special classes of second-order difference equations, and particularly with solutions convergent to zero (see [1,3]).

2. PRELIMINARIES

Let h_1, h_2 be two positive sequences. First, we prove estimate (5).

LEMMA 1. *Let $\Phi(n)$ be a nonsingular $m \times m$ matrix for $n \ge n_0$, let P be a projection. The relation*

$$\sum_{i=n_0}^{n-1} \left|\Phi(n)P\Phi^{-1}(i+1)\right| h_1(i) \le Kh_2(n-1)$$

for a positive constant K, implies that

$$|\Phi(n)P| \le ch_2(n-1) \prod_{i=n_0+1}^{n-1} \left[\frac{Kh_2(i)}{h_1(i)+Kh_2(i)}\right], \qquad n \ge n_0,$$

where

$$c = \frac{K|\Phi(n_0+1)P|}{h_1(n_0)}.$$

PROOF. Putting

$$\varphi(n) = |\Phi(n)P|^{-1}, \qquad \psi(n) = \sum_{i=n_0}^{n-1} \varphi(i+1)h_1(i),$$

we have

$$\sum_{i=n_0}^{n-1} \varphi(i+1)\Phi(n)Ph_1(i) = \sum_{i=n_0}^{n-1} \Phi(n)P\Phi^{-1}(i+1)\Phi(i+1)P\varphi(i+1)h_1(i),$$

and hence, $\varphi(n)^{-1}\psi(n) \le Kh_2(n-1)$. Moreover, $\psi(n+1)-\psi(n) = \varphi(n+1)h_1(n) \ge K^{-1}\psi(n+1)$ $h_1(n)/h_2(n) \ge K^{-1}\psi(n)h_1(n)/h_2(n)$. Then,

$$\psi(n+1) \ge \left(1 + \frac{h_1(n)}{Kh_2(n)}\right)\psi(n), \qquad n \ge n_0,$$

and hence,

$$Kh_2(n-1)\varphi(n) \ge \prod_{i=n_0+1}^{n-1}\left(1+\frac{h_1(i)}{Kh_2(i)}\right)\varphi(n_0+1)h_1(n_0), \qquad \text{for} \quad n \ge n_0.$$

From where,

$$|\Phi(n)P| \le \frac{|\Phi(n_0+1)P|}{h_1(n_0)}h_2(n-1)\prod_{i=n_0+1}^{n-1}\left(\frac{Kh_2(i)}{h_1(i)+Kh_2(i)}\right), \qquad n \ge n_0.$$

The particular case $h_1 = h_2 = h$ in Lemma 1 gives the following corollary.

COROLLARY 1. *The relation*

$$\sum_{i=n_0}^{n-1}\left|\Phi(n)P\Phi^{-1}(i+1)\right| h(i) \le Kh(n-1), \qquad n \ge n_0$$

implies

$$|\Phi(n)P| \le ch(n-1)\beta^{n-n_0-1}\prod_{i=n_0+1}^{n-1} h(i), \qquad n \ge n_0,$$

where $c = K|\Phi(n_0+1)P|h(n_0)^{-1}$ *and* $\beta = K(1+K)^{-1}$.

REMARK. Results as Lemma 1 and Corollary 1 are not known in the literature. In the recent Agarwal's book [1], the case $h_1 = h_2 = 1$ is proven. Moreover, we do not impose $K > 1$ as in [1, Lemma 5.6.5].

We will use the space $\mathcal{B}_{\infty,h_2}$ of all m-vector functions x defined on $N(n_0)$ and h_2-convergent, i.e., for which $\lim_{n\to\infty} h_2(n-1)^{-1}x(n)$ exists. $\mathcal{B}_{\infty,h_2}$ is a Banach space with the norm

$$\|x\| = \operatorname*{Sup}_{n\in N(n_0)} h_2(n-1)^{-1}|x(n)|.$$

We denote by $z_\infty = \lim_{n\to\infty} h_2(n-1)^{-1}x(n)$, and for Φ a fundamental matrix of equation (1)

$$\Gamma_k(n,i) = \Phi(n)P_k\Phi^{-1}(i+1); \qquad k = 1,2, \quad P_1 = P, \quad P_2 = I - P.$$

3. h_2-CONVERGENT SOLUTIONS

We begin our study of equation (6) by showing that there is a unique h_2-convergent solution if $f(n,x)$ is Lipschitzian with respect to the second variable and is convergent as $n \to \infty$ in a weighted manner.

THEOREM 1. *Let the following conditions be satisfied.*

(i) *There exist supplementary projections* P_i, $i = 1,2$ *and a constant* $K > 0$ *such that*

$$\sum_{i=n_0}^{\infty}|\Gamma(n,i)|h_1(i) \le Kh_2(n-1), \qquad n \in N(n_0).$$

(ii) *For all* $x_1, x_2 \in \mathbb{C}^m$ *and* $n \geq n_0$

$$|f(n, h_2(n-1)x_1) - f(n, h_2(n-1)x_2)| \leq \gamma h_1(n)|x_1 - x_2|,$$

where $\gamma K < 1$.

(iii) *There exists a function* $g : \mathbb{C}^m \to \mathbb{C}^m$ *such that*

$$\lim_{n\to\infty} \frac{f(n, h_2(n-1)z) - g(h_2(n-1)z)}{h_1(n)} = 0,$$

uniformly in z.

(iv) *The limit* $\lim_{n\to\infty} G(n,z) = G(z)$ *exists, where*

$$G(n,z) = \sum_{i=n_0}^{\infty} \Gamma(n,i) \frac{g(h_2(i-1)z)}{h_2(n-1)}.$$

(v) *The sequences* h_1 *and* h_2 *satisfy*

$$\lim_{n\to\infty} \prod_{i=n_0+1}^{n-1} \left(\frac{K h_2(i)}{h_1(i) + k h_2(i)} \right) = 0.$$

Then, for any $\xi \in P_1\mathbb{C}^m$, *there is a unique solution* $x = x(n, n_0, x_0)$ *with* $P_1 x_0 = \xi$ *of equation* (6) *such that* $\lim_{n\to\infty} h_2(n-1)^{-1}x(n) = z_\infty$ *exists.*

Furthermore, this h_2*-limit of* $x(n)$, z_∞ *must satisfy the equation*

$$z_\infty = G(z_\infty).$$

Moreover, if $z_\xi = \Phi\xi$, *then we have the asymptotic formula*

$$x_\xi = z_\xi + G(n, z_\infty) + o(h_2(n-1)), \qquad \text{as } n \to \infty. \tag{7}$$

The correspondence $x_\xi \leftrightarrow z_\xi$ *is bicontinuous and the application* $\xi \to x_\xi$ *is continuous.*

PROOF. For any sequence $x \in \mathcal{B}_{\infty,h_2}$ let T be the operator

$$Tx(n) = \sum_{i=n_0}^{\infty} \Gamma(n,i) f(i, x(i)), \qquad \text{for } n \geq n_0. \tag{8}$$

It must be shown that $\lim_{n\to\infty} h_2(n-1)^{-1} Tx(n)$ exists for $x \in \mathcal{B}_{\infty,h_2}$. We write (8) as

$$\begin{aligned} Tx(n) = &\sum_{i=n_0}^{n-1} \Gamma_1(n,i)[f(i,x(i)) - g(h_2(i-1)z_\infty)] \\ &- \sum_{i=n}^{\infty} \Gamma_2(n,i)[f(i,x(i)) - g(h_2(i-1)z_\infty)] + \sum_{i=n_0}^{\infty} \Gamma(n,i) g(h_2(i-1)z_\infty), \end{aligned} \tag{9}$$

where $z_\infty = \lim_{n\to\infty} h_2(n-1)^{-1}x(n)$.

The h_2-limit of the last term in (9) exists by virtue of Hypothesis (iv). The h_2-limits of the remaining terms on the right side of (9) tend to zero as $n \to \infty$. The h_2-limit for the term with Γ_1

$$\lim_{n\to\infty} h_2(n-1)^{-1} \sum_{i=n_0}^{n-1} \Gamma_1(n,i)[f(i,x(i)) - g(h_2(i-1)z_\infty)] = 0 \tag{10}$$

will be established. The proof that the h_2-term in (9) with Γ_2 approaches to zero follows similarly.

Using (iii), for $\varepsilon > 0$, we can choose $n_1 \in N(n_0)$ such that

$$|f(n,x(n)) - g(h_2(n-1)z_\infty)| < \frac{\varepsilon}{2K} h_1(n), \qquad n \geq n_1.$$

By Lemma 1, and Hypotheses (i) and (v), $\lim_{n\to\infty} h_2(n-1)^{-1}|\Phi(n)P_1| = 0$, therefore, $n_2 \geq n_1$ can be chosen so that

$$|\Phi(n)P_1| \left| \sum_{i=n_0}^{n-1} P_1\Phi^{-1}(i+1)[f(i,x(i)) - g(h_2(i-1)z_\infty)] \right| < \frac{\varepsilon}{2} h_2(n-1), \qquad n \geq n_2.$$

From this inequality, we obtain

$$\begin{aligned} &\left| \sum_{i=n_0}^{n-1} \Gamma_1(n,i)[f(i,x(i)) - g(h_2(i-1)z_\infty)] \right| \\ &\quad \leq |\Phi(n)P_1| \left| \sum_{i=n_0}^{n-1} P_1\Phi^{-1}(i+1)[f(i,x(i)) - g(h_2(i-1)z_\infty)] \right| \\ &\quad\quad + \sum_{i=n_1}^{n-1} |\Gamma_1(n,i)|\,|f(i,x(i)) - g(h_2(i-1)z_\infty)| < \varepsilon h_2(n-1), \qquad n \geq n_2. \end{aligned}$$

This verifies (10). It follows from (9) that

$$Tx(n) = G(n,z_\infty) + o(h_2(n-1)), \qquad \text{as } n \to \infty. \tag{11}$$

On the other hand, for any two sequences $x_1(n), x_2(n)$ in $\mathcal{B}_{\infty,h_2}$, we obtain from (i) and (ii)

$$\begin{aligned} |Tx_1(n) - Tx_2(n)| &\leq \sum_{i=n_0}^{\infty} |\Gamma(n,i)|\,|f(i,x_1(i)) - f(i,x_2(i))| \\ &\leq \sum_{i=n_0}^{\infty} |\Gamma(n,i)|h_1(i)\gamma h_2(i-1)^{-1}|x_1(i) - x_2(i)| \\ &\leq K\gamma h_2(n-1)\|x_1 - x_2\|. \end{aligned}$$

Then,

$$\|Tx_1 - Tx_2\| \leq \gamma K \|x_1 - x_2\|.$$

Therefore, by the Contraction Principle, the equation $x = \Phi\xi + Tx$ has a unique solution $x \in \mathcal{B}_{\infty,h_2}$. It can easily be checked that x is a solution of equation (6). As a consequence of (11), we note that the fixed point x has h_2-limit z_∞, which satisfies formula (7).

Finally, if $x_{\xi_i} = z_{\xi_i} + Tx_{\xi_i}$, then

$$(1-\gamma K)\|x_{\xi_i} - x_{\xi_2}\| \leq \|z_{\xi_1} - z_{\xi_2}\| \leq (1+\gamma K)\|x_{\xi_1} - x_{\xi_2}\|.$$

Moreover,

$$\|x_{\xi_1} - x_{\xi_2}\| \leq M|\xi_1 - \xi_2| + \gamma K \|x_{\xi_1} - x_{\xi_2}\|,$$

where $M = \|\Phi P_1\|_\infty$. This establishes the continuity of $\xi \to x_\xi$ and the bicontinuity of $z_\xi \leftrightarrow x_\xi$. This completes the proof of the Theorem 1.

Theorem 1 admits a factorized form.

THEOREM 2. *Assume that Conditions (i), (ii), and (v) of Theorem 1 hold. Suppose also that:*

(iii)′ *the limit* $\lim_{n\to\infty} L(n) = L(\infty)$ *exists, where*

$$L(n) = h_2(n-1)^{-1} \sum_{i=n_0}^{\infty} \Gamma(n,i)h_1(i),$$

(iv)′ *the limit*

$$\lim_{n\to\infty} h_1(n)^{-1} f(n, h_2(n-1)z) = \tilde{g}(z),$$

exists uniformly in $z \in \mathbb{C}^m$.

Then, the conclusions of Theorem 1 follows with $G(n,z) = L(n)\tilde{g}(z)$ *and* $G(z) = L(\infty)\cdot\tilde{g}(z)$, *i.e.,* z_∞ *satisfies the functional equation*

$$z_\infty = L(\infty)\tilde{g}(z_\infty),$$

and the asymptotic formula (7) takes the form

$$x_\xi = z_\xi + L(n)\tilde{g}(z_\infty) + o(h_2(n-1)), \qquad \text{as } n \to \infty.$$

The case $h_1 = h_2 = h$ gives the statement.

THEOREM 3. *Assume the following hypotheses.*

(i)* *There exists a constant* $K > 0$ *such that*

$$\sum_{i=n_0}^{\infty} |\Gamma(n,i)| h(i) \le K h(n-1), \qquad n \ge n_0.$$

(ii)* *The function* f *satisfies the Lipschitz condition*

$$|f(n,x_1) - f(n,x_2)| \le \gamma |x_1 - x_2|$$

with $\gamma K < 1$.

(iii)* *There is a function* $g : \mathbb{C}^m \to \mathbb{C}^m$ *such that*

$$\lim_{i\to\infty} \frac{f(i,h(i-1)z) - g(h(i-1)z)}{h(i)} = 0$$

exists uniformly in z.

(iv)* *For any* $z \in \mathbb{C}^m$, *there exists the limit* $\lim_{n\to\infty} G(n,z) \equiv G(z)$, *where*

$$G(n,z) = h(n-1)^{-1} \sum_{i=n_0}^{\infty} \Gamma(n,i) g(h(i-1)z).$$

(v)* $\lim_{n\to\infty} \beta^{n-n_0} \prod_{i=n_0+1}^{n-1} h(i) = 0$, $\beta = K/(1+K)$.

Then, for any $\xi \in P_1\mathbb{C}^n$, *there exists a unique solution* $x = x(n; n_0, x_0)$ *with* $P_1 x_0 = \xi$ *of equation (6) such that* $\lim_{n\to\infty} h(n-1)^{-1}x(n) = z_\infty$. *This limit satisfies the functional equation*

$$z_\infty = G(z_\infty).$$

Moreover, if $z_\xi = \Phi\xi$, *then we have the asymptotic formula*

$$x_\xi = z_\xi + G(n,z_\infty) + o(h(n-1)), \qquad n \to \infty.$$

The correspondence $x_\xi \leftrightarrow z_\xi$ *is bicontinuous and* $\xi \to x_\xi$ *is continuous.*

4. h_2-BOUNDED SOLUTIONS

Assuming Condition (i) and the Lipschitz Condition (ii) of our Theorem 1, the existence of a h_2-bounded solution of equation (6) can be proved. However, without the convergence Conditions (iii),(iv), in general, it is not possible to have h_2-convergent solutions. If in Theorem 1, we remove Hypotheses (iii) and (iv), then we can establish an h_2-boundedness result.

THEOREM 4. *Let the following conditions be satisfied.*

(a) *Condition (i) of Theorem 1 holds.*

(b) *For all* $(n,u),(n,v)$, *where* $n \in N(n_0)$ *and* $|u| \le \delta_1$, $|v| \le \delta_1$, *and*

$$|f(n,h_2(n-1)u) - f(n,h_2(n-1)v)| \le \lambda h_1(n)|u-v|$$

with $\lambda K < 1$.

(c) *The function* $f = f(n,x)$ *satisfies*

$$\rho = \operatorname*{Sup}_{(n,x)\in N(n_0)\times B_{\delta_1}} h_1(n)^{-1}|f(n,x)| < \infty \quad \text{and} \quad \delta_1 - \rho K > 0,$$

where $B_{\delta_1} = \{x \in \mathbb{C}^m \mid \|x\| \le \delta_1\}$.

(d) *The sequence*

$$\prod_{i=n_0+1}^{n-1} \left(\frac{Kh_2(i)}{h_1(i)+Kh_2(i)}\right),$$

is bounded.

Then, for any $\xi \in P_1\mathbb{C}^m$ *such that* $|\xi|\,\|\Phi P_1\|_{\infty,h_2} \le \delta_1 - \rho K$, *there is a unique* h_2*-bounded solution* $x = x(n,n_0,x_0)$, *with* $P_1x_0 = \xi$, *of equation (6) such that* $\|x\|_{\infty,h_2} \le \delta_1$. *Furthermore, we have the asymptotic formula*

$$x_\xi = z_\xi + |\Phi(n)P_1|O(1) + o(h_2(n-1)), \qquad \text{as } n \to \infty,$$

where $z_\xi = \Phi(n)\xi$. *The correspondence* $x_\xi \leftrightarrow z_\xi$ *is bicontinuous and the application* $\xi \to x_\xi$ *is continuous.*

PROOF. Let $\mathcal{B}_\delta = \mathcal{B}_{\delta,h_2} = \{x \in \mathcal{B}(N(n_0)) \mid h_2(n-1)^{-1}|x(n)| \le \delta\}$, where $\mathcal{B}(N(n_0))$ is the space of all real m-vector functions defined and h_2-bounded on $N(n_0)$. On $\mathcal{B}_{\delta_1}$ we define an operator $\mathcal{N}$ as follows:

$$\mathcal{N}x(n) = \Phi(n)\xi + \sum_{i=n_0}^{\infty} \Gamma(n,i)f(i,x(i)).$$

By Lemma 1, we find that there exists a positive constant M such that for all $n \in N(n_0)$, $|\Phi(n)P_1| \le Mh_2(n-1)$. Choose ξ so that $M|\xi| \le \delta_1 - \rho K$. Then, if $x \in \mathcal{B}_{\delta_1}$,

$$\begin{aligned} |\mathcal{N}x(n)|h_2(n-1)^{-1} &\le M|\xi| + \sum_{i=n_0}^{\infty} |\Gamma(n,i)|h_1(i)|f(i,x(i))|h_1(i)^{-1} \\ &\le (\delta_1 - \rho K) + \rho K = \delta_1. \end{aligned}$$

Thus, $\mathcal{N}$ maps $\mathcal{B}_{\delta_1}$ into itself. Further, for $x_1, x_2 \in \mathcal{B}_{\delta_1}$, as in Theorem 1, we obtain

$$\|\mathcal{N}x_1 - \mathcal{N}x_2\| \le \lambda K\|x_1 - x_2\|.$$

Therefore, by Contraction Principle, $\mathcal{N}$ has a unique fixed point $x \in \mathcal{B}_{\delta_1}$; i.e., $x(n) = \mathcal{N}x(n)$. This fixed point is indeed a solution of equation (1) on $N(n_0)$. The other statements can be demonstrated as in Theorem 1.

The case $h_1 = h_2 = h$ gives the result.

THEOREM 5. *Assume that the following conditions hold.*

(a)′ *Condition (i)* of Theorem 3 holds.*

(b)′ *For all $(n,u),(n,v)$, where $n \in N(n_0)$ and $|u| \le \delta_1$, $|v| \le \delta_1$, we have*

$$|f(n,u)-f(n,v)| \le \gamma|u-v|, \qquad \gamma K < 1.$$

(c)′ *The function f satisfies*

$$\rho = \sup_{(n,x)\in N(n_0)\times B_{\delta_1}} h(n)^{-1}|f(n,x)| < \infty \quad \text{and} \quad \rho < \delta_1 K^{-1}.$$

(d)′ *The sequence h satisfies*

$$\beta^n \prod_{i=n_0+1}^{n-1} h(i), \qquad \beta = K(1+K)^{-1}$$

is bounded.

Then, the conclusions of Theorem 4 are true with $h_2 = h$.

We are able to obtain Agarwal's Theorem 5.8.5 [1, pp. 272–273] as a corollary of Theorem 5.

5. SOME EXAMPLES

Now, we present some simple examples of the results obtained.

(1) The solutions of equation (6) described by Theorem 1 are defined on all of $N(n_0)$. If (iii) is valid only for $n \ge n_1$ large enough, then the conclusions of Theorem 1 are also valid. This is the case when (ii) is satisfied with

$$|f(n,x_1)-f(n,x_2)| \le \gamma(n)|x_1-x_2|, \qquad x_i \in \mathbb{C}^m,\ i=1,2, \quad n \ge n_0,$$

and $\gamma(n) \to 0$ as $n \to \infty$.

(2) Furthermore, if in (1) above $\gamma \in \ell_1(N(n_0))$, then $h_2(n-1)^{-1}Tx(n) \to 0$ as $n \to \infty$, and the solutions $x = x_\xi$ of equation (6) in Theorem 1 h_2-converge to zero as $n \to \infty$.

(3) If $f(n,x) = f(n)$ and $h_1(n)^{-1}f(n) \to v$ as $n \to \infty$, v constant, then for any $\xi \in P_1\mathbb{C}^m$, there is a unique solution $x = x_\xi$ of equation

$$x(n+1) = A(n)x(n) + f(n),$$

such that

$$\lim_{n\to\infty} h_2(n-1)^{-1}x(n) = L(\infty)(v).$$

(4) For the linearly perturbed linear system

$$x(n+1) = (A(n)+B(n))x(n),$$

if $h_1(n)^{-1}B(n)h_2(n-1) \to B$ as $n \to \infty$, with B a constant matrix, then Theorem 2 establishes the existence of a unique h_2-convergent solution $x = x(n,n_0,x_0)$ such that for any $\xi \in P_1\mathbb{C}^m$,

$$x(n) = \Phi(n)\xi + L(n)Bz_\infty + o(h_2(n-1)), \qquad \text{as } n \to \infty.$$

(5) An interesting particular case of our results is present when $P = I$, the identity matrix.

We conclude with a simple illustration that describes a setting where the Theorem 2 with $h_1 = h_2 = 1$ is applicable. Let the coefficient matrix A be given as

$$A(k) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}, \qquad k \geq 0.$$

A fundamental matrix solution is given explicitly by

$$\Phi(k) = \begin{bmatrix} 2^{-(k-1)} & 0 \\ 0 & 2^{k-1} \end{bmatrix}.$$

The projection employed in the Theorem 2 can be taken as the matrices $P_1 = \text{diag}(1,0)$ and $P_2 = \text{diag}(0,1)$. A computation shows that

$$\Gamma_1(n,k) = \text{diag}\left(2^{-(n-k)}, 0\right), \qquad 0 \leq k \leq n-1, \qquad \text{and}$$
$$\Gamma_2(n,k) = \text{diag}\left(0, 2^{-(n-k)}\right), \qquad n \leq k.$$

For $n \in N(0)$, we have

$$L(n) = \text{diag}(1 - 2^{-n}, -2), \qquad \sum_{i=n_0}^{\infty} \mid \Gamma(n,i) = 3 - 2^{-n}, \qquad \text{and}$$
$$L(\infty) = \lim_{n\to\infty} L(n) = \text{diag}(1,-2).$$

Then, Hypotheses (i) and (iii) of Theorem 2 are satisfied.

Now, to complete the illustration, we exhibit a function which is admissible as a perturbation term: for $a(n), b(n) \leq 1/4$, and $a(n), b(n) \to 1/4$ as $n \to \infty$, let $f(n,x)$ be defined as

$$f(n,x) = (a(n)|x|, b(n)z_1), \qquad x = (z_1, z_2).$$

Thus, $|f(n,x_1) - f(n,x_2)| \leq (1/4)|x_1 - x_2|$, and $\gamma = 1/4$ and $K = 3$ satisfy $\gamma K < 1$. The limit

$$\lim_{n\to\infty} f(n,x) = \frac{1}{4}(|x|, z_1) = g(x),$$

is uniform for all constant x. Then, for any $\xi \in P_1\mathbb{R}^2$ there is a unique solution $x = x_\xi$ of the difference equation (6) described above and $\lim_{n\to\infty} x(n) = x_\infty$ exists. Furthermore, this limit x_∞ of $x(n)$ satisfies the equation

$$x_\infty = L(\infty)g(x_\infty).$$

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