Necessary and sufficient conditions for existence of maximal solutions for inf-\(\alpha\) composite fuzzy relational equations

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Abstract

On complete Brouwerian lattices, an inf-\(\alpha\) composite fuzzy relational equation and its equation system are investigated. In finite domains, a necessary and sufficient solvability condition is proposed for the equation, then all its maximal solutions and the whole solution set are determined. Subsequently, the whole solution set for the equation system is determined. In infinite domains, sufficient conditions for existence of a maximal solution for the equation and the equation system are shown, respectively. Afterwards, a necessary and sufficient condition, that there exists a maximal solution which is more than or equal to any solution, is presented for the equation.

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1. Introduction

In 1976, sup-inf composite fuzzy relational equations were introduced and their extremal solutions on complete Brouwerian lattices were first investigated by Sanchez [1]. A large amount of research about these equations had been done in later years (cf. [2–9]). In 1985, \(\oplus\)-fuzzy relational equations on complete Brouwerian lattices were introduced and their corresponding smallest solutions were given at the same time by Di Nola, Pedrycz and Sessa [10]. Further, in 1989, a sufficient and necessary condition, that the solution set of an \(\ominus\)-fuzzy relational equation is nonempty if and only if the equation has the smallest solution, was proved by Di Nola, Sessa, Pedrycz and Sanchez. Moreover, all maximal solutions for \(\ominus\)-fuzzy relational equations on linear lattices in finite domains were also constructed by them (cf. [4]). In 2000, a more general version of the equation on a complete lattice (in particular, on a distributive, complete lattice with meet-irreducible or finite meet-decomposable elements) were discussed by De Baets [3], and an analytical way was presented for obtaining the whole solution set in the terminology of crowns when the universe of discourse is finite. In this paper, an inf-\(\alpha\) composite fuzzy relational equation and its equation system are investigated. Some
sufficient conditions and some necessary and sufficient conditions for existence of maximal solutions are showed for the equation and the equation system.

The remainder of this paper is organized as follows: In Section 2, some definitions and basic lemmas are presented, and an $\mathcal{O}$-fuzzy relational equation and its equation system are introduced. In Section 3, the relationships among the solution sets of the equation and the equation system are analyzed. In Section 4, in finite domains, a necessary and sufficient solvability condition is proposed for the equation, and the solution sets of the equation and the equation system are determined. In Section 5, in infinite domains, a sufficient condition, that there exists a maximal solution which is more than or equal to any solution, is showed for the equation. And, the same condition is also obtained for the equation system. Afterwards, a necessary and sufficient condition, that there exists a maximal solution which is more than or equal to any solution, is showed for the equation. And, the same condition is also obtained for the equation system. In particular, on the real unit interval, a necessary and sufficient condition that there exist maximal solutions, and a necessary and sufficient condition that there exists a maximal solution which is more than or equal to any solution, are presented for the equation. In Section 6, some conclusions are drawn.

2. Preliminaries

Some notions, definitions and basic lemmas are introduced in this section. We assume that the reader is familiar with the basic order-theoretic notions such as ordered set, (complete) lattice, distributive lattice etc., see [11,12]. For the sake of convenience, we recall some definitions and basic lemmas in the following. Notice that the meet (join, respectively) operation of a lattice $L$ is denoted by $\land$ ($\lor$, respectively).

Definition 2.1 ([11]). A Brouwerian lattice is a lattice $L$ in which, for any element $a$ and $b$, the set of all $x \in L$ such that $a \land x \leq b$ contains a greatest element, denoted by $aab$, the relative pseudo-complement of $a$ in $b$.

In the following, we restrict our discussion to complete Brouwerian lattices $L$ with universal bounds 0 and 1.

Lemma 2.1 ([11]). For all $a, b \in L$, we have

1. $aab \geq b$.
2. If $L$ is linear, then $a > b$ implies $aab = b$.

Lemma 2.2 ([9]). For all $a, b, c \in L$, we have

1. $aab = aa(a \land b)$;
2. $a \leq b$ if and only if $aab = 1$;
3. $aa(b \lor c) = (aab) \lor (aac)$;
4. If $b \geq c$, then $aab \geq aac$.

Lemma 2.3 ([13]). For all $a, b, c \in L$, we have $(a \land b)ac = aa(bac)$.

Lemma 2.4 ([9]). If $a \in L$ and $\{x_i : i \in I\} \subseteq L$, then $aa(\bigwedge_{i \in I} x_i) = \bigwedge_{i \in I}(aax_i)$.

Definition 2.2 ([11]). We say that an element $q$ in a lattice $L$ is meet-irreducible if, for all $x, y \in L$, $q = x \land y$ implies $q = x$ or $q = y$.

Lemma 2.5 ([9]). If $q \in L$ is meet-irreducible, $a \in L$ and $a \not\leq q$, then $aaq = q$.

Definition 2.3 ([11]). If $a$ is an element of the lattice $L$, then a representation

$$a = q_1 \land \cdots \land q_n$$

of $a$ as a meet of finite many meet-irreducible elements $q_1, q_2, \ldots, q_n$ is called a finite meet-decomposition of $a$. This finite decomposition is irredundant if, for each $i = 1, 2, \ldots, n$,

$$a \neq q_1 \land \cdots \land q_{i-1} \land q_{i+1} \land \cdots \land q_n,$$

and we say $a$ has an irredundant finite meet-decomposition.
Remark 2.1. Obviously, a meet-irreducible element has an irredundant finite meet-decomposition.

Lemma 2.6 ([11]). An element in a distributive lattice has at most one irredundant finite meet-decomposition.

Lemma 2.7 ([11]). A Brouwerian lattice is distributive.

Lemma 2.8 ([11]). If \( p \) is meet-irreducible in a distributive lattice \( L \), then \( p \geq \bigwedge_{i=1}^{k} x_i \) implies that there exists an \( i \in K \) such that \( p \geq x_i \).

Remark 2.2. Let \( b = \bigwedge_{j=1}^{m} (a_j \alpha x_j) \). If \( b \) has an irredundant finite meet-decomposition \( \bigwedge_{i=1}^{u} p_i \), then for any \( i \in U \) there exists a \( j \in m \) such that \( a_j \alpha x_j \leq p_i \) according to Lemma 2.8.

Definition 2.4 ([12]). If \( S \) is a subset of a nonempty partially ordered set \( P \), then an element \( a \in S \) is called a maximal element of \( S \) if and only if the relation \( x \geq a \) implies \( x = a \) for all \( x \in S \).

Definition 2.5. An element \( c \) in a complete lattice \( L \) is called dually compact if whenever \( c \geq \bigwedge S \) there exists a finite subset \( T \subseteq S \) with \( c \geq \bigwedge T \).

Let \( I \) and \( J \) be two index sets, both \( U = \{u_i : i \in I\} \) and \( V = \{v_j : j \in J\} \) be nonempty sets. A fuzzy set \( A \) of \( U \) is a function \( A : U \rightarrow L \), a fuzzy relation \( R \) of \( U \times V \) is a function \( R : U \times V \rightarrow L \) (see [1]). For the sake of brevity, in what follows a row vector \( A = (a_i)_{i \in I} \) (or its transposed column vector \( A^T = (a_i)_{i \in I}^T \), where the sign “\( T \)” denotes the “transposition”) is used to represent a fuzzy set of \( U \), a matrix \( A = (a_{ij})_{I \times J} \) is used to represent a fuzzy relation of \( U \times V \).

Definition 2.6 ([1]). Let \( A = (a_{ij})_{I \times J} \) and \( B = (b_{ij})_{I \times J} \) be two fuzzy relations. We say that \( B \) contains or includes \( A \), in symbols \( B \geq A \), if and only if \( b_{ij} \geq a_{ij} \) for all \( i \in I, j \in J \) and we say that \( A \) is equal to \( B \), in symbols \( A = B \), if and only if \( a_{ij} = b_{ij} \) for all \( i \in I, j \in J \). A fuzzy union (an intersection, respectively) of \( A \) and \( B \) is defined by \( A \vee B = (a_{ij} \vee b_{ij})_{I \times J} \) (\( A \wedge B = (a_{ij} \wedge b_{ij})_{I \times J} \), respectively).

Definition 2.7 ([12]). Let \( A = (a_{ij})_{I \times J} \) and \( B = (b_{jk})_{J \times K} \) be two fuzzy relations. We define the \( \odot \) composition of \( A \) and \( B \) to be the fuzzy relation \( D \), in symbols \( A \odot B = D \), given by

\[
A \odot B = D = (d_{ij})_{I \times K},
\]

where \( d_{ij} = \bigvee_{r \in J} (a_{ir} \wedge b_{rj}) \) for any \( i \in I, j \in J \).

In particular, for \( A = (a_i)_{i \in I} \) and \( b \in L \), it holds \( A \odot b = (a_i \wedge b)_{i \in I} \).

In the following, we introduce an \( \inf \)-\( \alpha \)-composite fuzzy relational equation and its equation system.

Given an element \( b \) and a fuzzy row vector \( A = (a_j)_{j \in J} \), then

\[
A \oplus X = b,
\]

or

\[
\bigwedge_{j \in J} (a_j \alpha x_j) = b
\]

is called an \( \oplus \)-fuzzy relational equation in the unknown \( X = (x_j)_{j \in J}^T \). Its solution set is denoted by \( \mathcal{X}_1 = \{X : A \oplus X = b\} \).

A fuzzy relational equation system composed of equations of the form (1), with a fuzzy relation \( A = (a_{ij})_{I \times J} \) and a fuzzy column vector \( B = (b_i)_{i \in I}^T \) given, is

\[
A \oplus X = B,
\]

or

\[
\bigwedge_{j \in J} (a_{ij} \alpha x_j) = b_i, \quad i \in I
\]
in the unknown $X = (x_j)_{j \in J}$. Its solution set is denoted by $\mathcal{X}_2 = \{X : A \circ \circ X = B\}$.

Additionally, an equation set consisting of equations of the form (1), with two row vectors $A = (a_j)_{j \in J}$ and $B = (b_i)_{i \in I}$ given, is

$$A \circ \circ X = B,$$

(3)

or

$$\bigwedge_{j \in J} (a_j \circ x_j) = b_i, \quad i \in I$$

in the unknown $X = (x_{ji})_{j \times I}$. Its solution set is denoted by $\mathcal{X}_3 = \{X : A \circ \circ X = B\}$.

**Definition 2.8** ([4]). A maximal element (if it exists) of the solution set of an $\circ \circ$-fuzzy relational equation (an equation system, an equation set, respectively) is called a maximal solution of the $\circ \circ$-fuzzy relational equation (the equation system, the equation set, respectively).

3. The relationships among the solution sets

In this section, the relationships among the solution sets of Eq. (1), equation system (2) and equation set (3) are discussed.

**Proposition 3.1** ([4]). $\mathcal{X}_1 \neq \emptyset$ if and only if $X_{s1} = (A \circ b)^T \in \mathcal{X}_1$. Further, $X \geq (A \circ b)^T$ for any $X \in \mathcal{X}_1$.

Let $A_i = (a_{ij})_{j \in J}$ be the $i$th row vector of the fuzzy relation $A = (a_{ij})_{I \times J}$, we have:

**Proposition 3.2** ([3]). For any $i \in I$, let $\mathcal{X}_{i1}$ be the solution set of $b_i = A_i \circ \circ X$. Then:

1. $\mathcal{X}_2 \neq \emptyset$ if and only if $\bigcap_{i \in I} \mathcal{X}_{i1} \neq \emptyset$. Moreover, $\mathcal{X}_2 = \bigcap_{i \in I} \mathcal{X}_{i1}$;
2. $\mathcal{X}_2 \neq \emptyset$ if and only if $X_{s2} = A^T \circ \circ B \in \mathcal{X}_2$. Further, $X \geq A^T \circ \circ B$ for any $X \in \mathcal{X}_2$.

**Lemma 3.1** ([3,4,14,15]). If $X_1, X_2 \in \mathcal{X}_1$ ( $\mathcal{X}_2$, $\mathcal{X}_3$, respectively) and $X_1 \leq X \leq X_2$, then $X \in \mathcal{X}_1$ ( $\mathcal{X}_2$, $\mathcal{X}_3$, respectively).

For the rest of this section, let $I = n$ (where $n =\{1, 2, \ldots, n\}$), $J = m$ be two finite index sets.

According to Proposition 3.2, if $\mathcal{X}_2 \neq \emptyset$, then $X_{s2} = A^T \circ \circ B$ is the smallest solution of (2) and $X_{s2} \in \mathcal{X}_{i1}$ for any $i \in n$. If for any $i \in n$, there exist maximal elements in the solution set $\mathcal{X}_{i1}$ of $b_i = A_i \circ \circ X$, then we define

$M_i = \{X_i : X_i \text{ is a maximal element in } \mathcal{X}_{i1} \text{ and } X_i \geq X_{s2}\}$

for $i \in n$.

**Proposition 3.3** ([3]). Let $\mathcal{X}_2 \neq \emptyset$. If for any $i \in n$, the number of maximal elements of $\mathcal{X}_{i1}$ is finite and for any $X \in \mathcal{X}_{i1}$ there exists a maximal element $F_X \in \mathcal{X}_{i1}$ such that $F_X \geq X$, then $M = \{X : X = \bigwedge_{i \in n} X_i, X_i \in M_i, i \in n\}$ is a finite subset of $\mathcal{X}_2$. Moreover, $X$ is a maximal element in $M$ if and only if $X$ is a maximal element in $\mathcal{X}_2$.

As for equation set (3), we have the next two propositions, which are similar to those of Wang [5].

**Proposition 3.4.** Let $X_i = (x_{i1}, x_{i2}, \ldots, x_{im_i})^T$ and $\mathcal{X}_{i1}$ be the solution set of $b_i = A_i \circ \circ X_i$ with $i \in n$, then

1. $\mathcal{X}_3 \neq \emptyset$ if and only if $\mathcal{X}_{i1} \neq \emptyset$ for any $i \in n$. Further,
   $$\mathcal{X}_3 = \{X = (X_1, X_2, \ldots, X_n) : X_i \in \mathcal{X}_{i1}, i \in n\};$$
2. if $\mathcal{X}_3 \neq \emptyset$, then
   $$X_{s3} = ((A \circ b_1)^T, (A \circ b_2)^T, \ldots, (A \circ b_n)^T)$$

is the smallest solution of (3).

**Proposition 3.5.** $X^* = (x^*_1, x^*_2, \ldots, x^*_n)$ is a maximal element in $\mathcal{X}_3$ if and only if $X^*_i$ is a maximal element in $\mathcal{X}_{i1}$ for any $i \in n$.

**Remark 3.1.** According to Propositions 3.4 and 3.5, it is easy to verify that Eq. (1) and equation set (3) are essentially equivalent.
4. A necessary and sufficient solvability condition

In this section, a necessary and sufficient solvability condition for Eq. (1) is showed.

In the rest of this section, let \( I = n \) and \( J = m \) be two finite index sets, and suppose that \( b \) has an irredundant finite meet-decomposition. Due to Lemmas 2.6 and 2.7, we may always let the irredundant finite meet-decomposition of \( b \) be \( \bigwedge_{i=1}^{u} p_{i} \), where \( p_{i} \) is meet-irreducible for \( i = 1, 2, \ldots, u \). Denote \( P = \{ p_{1}, p_{2}, \ldots, p_{u} \} \).

Let \( A \) and \( B \) be two sets, define \( A \setminus B = \{ x \in A : x \notin B \} \).

**Proposition 4.1** ([16]). *If \( b \) has an irredundant finite meet-decomposition and for \( i \in u \), let \( G(p_{i}) = \{ j \in m : a_{j} \not\leq p_{i} \} \), then \( \mathcal{X}_{1} \neq \emptyset \) if and only if \( G(p_{i}) \neq \emptyset \) for any \( i \in u \).*

**Proof.** Let us first assume that \( \mathcal{X}_{1} \neq \emptyset \). According to Lemmas 2.2 and 2.5, it follows that \( a_{j} \alpha p_{i} = 1 \) if \( a_{j} \leq p_{i} \), and \( a_{j} \alpha p_{i} = p_{i} \) if \( a_{j} \not\leq p_{i} \) for all \( j \in m, i \in u \). Now if \( \mathcal{X}_{1} \neq \emptyset \), then Proposition 3.1 implies \( (A \odot b)^{T} \in \mathcal{X}_{1} \).

Suppose that there exists a \( k \in u \) such that \( G(p_{k}) = \emptyset \). Then \( a_{j} \leq p_{k} \) for any \( j \in m \), hence \( a_{j} \wedge p_{k} = a_{j} \) for any \( j \in m \). Therefore, the proof above implies that

\[
A \odot b = \left( a_{1} \wedge \left( \bigwedge_{i \in u} p_{i} \right) \right) \wedge \left( a_{2} \wedge \left( \bigwedge_{i \in u} p_{i} \right) \right) \wedge \cdots \wedge \left( a_{m} \wedge \left( \bigwedge_{i \in u} p_{i} \right) \right)
\]

Thus from Lemmas 2.2, 2.4 and 2.5, we have

\[
(A \odot b)^{T} = \left( a_{1} \alpha \left( \bigwedge_{i \in u, i \neq k} (a_{1} \wedge p_{i}) \right) \right) \wedge \left( a_{2} \alpha \left( \bigwedge_{i \in u, i \neq k} (a_{2} \wedge p_{i}) \right) \right) \wedge \cdots \wedge \left( a_{m} \alpha \left( \bigwedge_{i \in u, i \neq k} (a_{m} \wedge p_{i}) \right) \right)
\]

\[
\geq \bigwedge_{i \in u, i \neq k} p_{i}
\]

\[
> b,
\]

which contradicts the fact that \( (A \odot b)^{T} \in \mathcal{X}_{1} \). Therefore, \( G(p_{i}) \neq \emptyset \) for any \( i \in u \).

Conversely, if \( b \) has an irredundant finite meet-decomposition and \( G(p_{i}) \neq \emptyset \) for any \( i \in u \), set

\[
A_{1} = \{ p_{i} \in P : a_{1} \not\leq p_{i} \},
\]

\[
A_{2} = \{ p_{i} \in P \setminus A_{1} : a_{2} \not\leq p_{i} \},
\]

\[
\ldots
\]

\[
A_{j} = \{ p_{i} \in P \setminus A_{1} \cup A_{2} \cup \cdots \cup A_{j-1} : a_{j} \not\leq p_{i} \},
\]

\[
\ldots
\]

\[
A_{m} = \{ p_{i} \in P \setminus A_{1} \cup A_{2} \cup \cdots \cup A_{m-1} : a_{m} \not\leq p_{i} \},
\]

where \( \bigwedge_{i=1}^{u} p_{i} \) is meet-irreducible for \( i \in u \).
then \( A_1 \cup A_2 \cup \cdots \cup A_m = P = \{ p_1, p_2, \ldots, p_n \} \), and \( A_i \cap A_j = \emptyset \) if \( i \neq j \). Therefore, \( A_1, A_2, \ldots, A_m \) is a partition of \( P \) (where, there may be some \( A_j = \emptyset \)). Let \( X^* = (x_1^*, x_2^*, \ldots, x_m^*)^T \) be defined by

\[
x_j^* = \begin{cases} \bigwedge_{p_i \in A_j} p_i & A_j \neq \emptyset, \\ 1 & \text{otherwise} \end{cases}
\]

for any \( j \in m \). Then from Lemmas 2.2, 2.4 and 2.5,

\[
A \ominus X^* = \bigwedge_{j=1}^m (a_j \alpha x_j^*)
= \bigwedge_{k \in m, A_k \neq \emptyset} (a_k \alpha x_k^*)
= \bigwedge_{k \in m, A_k \neq \emptyset} \left( a_k \alpha \left( \bigwedge_{p_i \in A_k} p_i \right) \right)
= \bigwedge_{k \in m, A_k \neq \emptyset} \left( \bigwedge_{p_i \in A_k} (a_k \alpha p_i) \right)
= \bigwedge_{p_i \in P} p_i
= b.
\]

Hence, \( X^* \in \mathcal{X}_1 \), i.e. \( \mathcal{X}_1 \neq \emptyset \). \( \square \)

**Remark 4.1 (\cite{16}).** If \( \mathcal{X}_1 \neq \emptyset \) and \( b \) has an irredundant finite meet-decomposition, then \( X^* \) defined by the formula (4) is a maximal element of \( \mathcal{X}_1 \).

The next four conclusions are included in our other papers, we omit their proofs.

**Lemma 4.1 (\cite{16}).** If \( X = (x_j)_{j \in m}^T \in \mathcal{X}_1 \) and \( b \) has an irredundant finite meet-decomposition, then for any \( i \in u \) there exists a \( j \in m \) such that \( a_j \alpha x_j \leq p_i \), and \( a_j \alpha x_j \leq p_i \) implies \( a_j \not\leq p_i \).

**Proposition 4.2 (\cite{16}).** If \( \mathcal{X}_1 \neq \emptyset \) and \( b \) has an irredundant finite meet-decomposition, then there exists a maximal element \( F_X \in \mathcal{X}_1 \) such that \( F_X \geq X \) for any \( X \in \mathcal{X}_1 \).

**Proposition 4.3 (\cite{16}).** If \( \mathcal{X}_1 \neq \emptyset \), \( b \) has an irredundant finite meet-decomposition and for \( i \in u \), let \( G(p_i) = \{ j \in m : a_j \not\leq p_i \} \). Then the number of maximal elements in \( \mathcal{X}_1 \) is \( \prod_{i \in u} | G(p_i) | \) exactly and all maximal elements of \( \mathcal{X}_1 \) are of the form (4).

**Proposition 4.4 (\cite{16}).** If \( \mathcal{X}_2 \neq \emptyset \) and each component \( b_i \) (\( i \in n \)) of \( B \) in (2) has an irredundant finite meet-decomposition, then for any \( X \in \mathcal{X}_2 \) there exists a maximal element \( F_X \in \mathcal{X}_2 \) such that \( F_X \geq X \).

**Remark 4.2.** If \( \mathcal{X}_2 \neq \emptyset \) and each component \( b_i \) (\( i \in n \)) of \( B \) in (2) has an irredundant finite meet-decomposition, then according to Proposition 3.2 we can give the smallest solution \( A^T \ominus B \) firstly, then construct all maximal solutions for each equation \( b_i = A_i \ominus X \) with \( i \in n \), due to Proposition 4.3. Finally, according to Proposition 3.3 we can obtain all maximal elements of \( \mathcal{X}_2 \), and therefore determine the whole solution set \( \mathcal{X}_2 \).

**Example 4.1.** Let \( L = [0, 1]^2 \), for \( \langle c_1, c_2 \rangle, \langle d_1, d_2 \rangle \in L \), define \( \langle c_1, c_2 \rangle \leq \langle d_1, d_2 \rangle \) if and only if \( c_1 \leq d_1 \) and \( c_2 \leq d_2 \). \( \langle c_1, c_2 \rangle \setminus \langle d_1, d_2 \rangle = \langle c_1 \setminus d_1, c_2 \setminus d_2 \rangle \), \( \langle c_1, c_2 \rangle \cap \langle d_1, d_2 \rangle = \langle c_1 \cap d_1, c_2 \cap d_2 \rangle \), \( \langle c_1, c_2 \rangle \vee \langle d_1, d_2 \rangle = \langle c_1 \vee d_1, c_2 \vee d_2 \rangle \). For \( e, f \in L \), denote \( [e, f] = \{ x \in L : e \leq x \leq f \} \) (see \cite{9}). Let

\[
A = \begin{pmatrix} 0.2 & 0.5 \\ 0.3 & 0.2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1.0 & 0.4 \\ 0.4 & 1.0 \end{pmatrix},
\]

consider the equation \( A \ominus X = B \).
Solution. $A^T \odot B = \left( \begin{pmatrix} 0.3 & 0.4 \\ 0.4 & 0.4 \end{pmatrix} \right)$ and $A \odot (A^T \odot B) = B$, thus the solution set of $A \odot X = B$ is nonempty. Obviously both $b_1 = (1.0, 0.4)$ and $b_2 = (0.4, 1.0)$ are meet-irreducible, and

$$G(b_1) = \{ i : a_{1i} \not\leq b_1 \} = \{ 1, 2 \}, \quad G(b_2) = \{ i : a_{2i} \not\leq b_2 \} = \{ 2 \}.$$ 

So, two maximal solutions of $A \odot X = b_1$ are

$$X_{11}^* = \left( \begin{pmatrix} 1.0, 0.4 \\ 1.0, 1.0 \end{pmatrix} \right) \quad \text{and} \quad X_{12}^* = \left( \begin{pmatrix} 1.0, 1.0 \\ 1.0, 0.4 \end{pmatrix} \right),$$

respectively by Proposition 4.3. A maximal solution of $A \odot X = b_2$ is

$$X_{21}^* = \left( \begin{pmatrix} 1.0, 1.0 \\ 0.4, 1.0 \end{pmatrix} \right).$$

From Proposition 3.3, it follows that two maximal solutions of $A \odot X = B$ are

$$X_1^* = X_{11}^* \land X_{21}^* = \left( \begin{pmatrix} 1.0, 0.4 \\ 0.4, 1.0 \end{pmatrix} \right) \quad \text{and} \quad X_2^* = X_{12}^* \land X_{21}^* = \left( \begin{pmatrix} 1.0, 1.0 \\ 0.4, 0.4 \end{pmatrix} \right),$$

respectively. Therefore, the solution set of $A \odot X = B$ is

$$\mathcal{X}_2 = \left( \left[ \left( \begin{pmatrix} 0.3, 0.4 \\ 0.4, 0.4 \end{pmatrix} \right) \right] \cup \left[ \left( \begin{pmatrix} 0.3, 0.4 \\ 1.0, 0.4 \end{pmatrix} \right) \right] \cup \left[ \left( \begin{pmatrix} 0.3, 0.4 \\ 0.4, 1.0 \end{pmatrix} \right) \right] \cup \left[ \left( \begin{pmatrix} 0.4, 0.4 \\ 0.4, 0.4 \end{pmatrix} \right) \right] \right).$$

5. Necessary and sufficient conditions for existence of maximal solutions

In this section, some sufficient conditions and some necessary and sufficient conditions about maximal solutions are presented for Eq. (1) and equation system (2) in infinite domains. Let $J$ be an infinite set and $I$ a finite set.

5.1. Sufficient conditions

In this subsection, we present a sufficient condition in the case that $b$ is dually compact and has an irredundant finite meet-decomposition, and show some necessary and sufficient conditions for existence of a maximal solution $F_X$ such that $F_X \geq X$ for any solution $X$ of Eq. (1). Moreover, some properties of maximal solutions of Eq. (1) are analyzed, and a sufficient condition for existence of a maximal solution $F_X$ such that $F_X \geq X$ for any solution $X$ of equation system (2) is given in the case that each $b_i (i \in I)$ of $B$ is dually compact and has an irredundant finite meet-decomposition.

In the rest of this subsection, we always suppose that $b$ has an irredundant finite meet-decomposition. Due to Lemmas 2.6 and 2.7, we may always let the irredundant finite meet-decomposition of $b$ be $\bigwedge_{i=1}^u p_i$, where $p_i$ is meet-irreducible for $i = 1, 2, \ldots, u$. Denote $P = \{ p_1, p_2, \ldots, p_u \}$.

Let $X = (x_j)_{j \in J}^T$ and $H \subseteq J$ with $X|_H = (\overline{x}_j)_{j \in H}^T$ defined by: $\overline{x}_j = x_j$ for any $j \in H$.

**Proposition 5.1.1.** If $\mathcal{X}_1 \neq \emptyset$, and $b$ is dually compact and has an irredundant finite meet-decomposition, then for any $X \in \mathcal{X}_1$ there exists a maximal element $F_X$ in $\mathcal{X}_1$ such that $F_X \geq X$.

**Proof.** Suppose $M = (m_j)_{j \in J}^T \in \mathcal{X}_1$, then $b = \bigwedge_{j \in J} (a_j \alpha m_j)$. From Definition 2.5, it follows that there exists a finite subset $H \subseteq J$ such that $b = \bigwedge_{j \in H} (a_j \alpha x_j)$. So, $M|_H$ is a solution of $b = \bigwedge_{j \in H} (a_j \alpha x_j)$. Then Proposition 4.2 implies that there exists a maximal solution $N = (n_j)_{j \in H}^T$ of $b = \bigwedge_{j \in H} (a_j \alpha x_j)$ such that $N \geq M|_H$. Let $X^* = (x_j^*)_{j \in J}^T$ be defined by

$$x_j^* = \begin{cases} n_j & j \in H, \\ 1 & j \in J \setminus H. \end{cases}$$

It is easy to see that $X^* \in \mathcal{X}_1$. In order to prove that $X^*$ is a maximal element in $\mathcal{X}_1$, suppose $X = (x_j)_{j \in J}^T \in \mathcal{X}_1$ and $X \geq X^*$, we need show $X = X^*$ by Definition 2.4. Since $x_j = x_j^* = 1$ for any $j \in J \setminus H$, it is sufficient to show
Under the condition of $X^{*}$ and $M$, there exists a finite subset $(1)$ Since $G$ implies that there exists a maximal solution $G = (g_j)_{j \in H}$ of $b = \bigwedge_{j \in H} (a_j \alpha x^*_j)$ such that $G \geq X^*|H$. Let $G = (\overline{g}_j)_{j \in J}$ be defined by:

$$\overline{g}_j = \begin{cases} g_j & j \in H, \\ 1 & j \notin H. \end{cases}$$

Then $\overline{G} \geq X^*$ and $\bigwedge_{j \in J} (a_j \alpha \overline{g}_j) = \bigwedge_{j \in H} (a_j \alpha g_j) = b$, hence $\overline{G} \in \mathcal{X}$. Therefore, $\overline{G} = X^*$ since $X^*$ is a maximal element in $\mathcal{X}$, and it follows that $x^*_j = 1$ for any $j \in J \setminus H$.

(2) Since $p_i \geq b = \bigwedge_{j \in J} (a_j \alpha x^*_j)$, for any $p_i \in P$, Remark 2.2 and Lemma 2.1 imply that there exists a $j_0 \in H$ such that $p_i \geq a_{j_0} \alpha x^*_{j_0}$ for any $p_i \in P$. Then we can construct a family of subsets of $P$ as follows:

- $A_{j_1} = \{ p_i \in P : p_i \geq a_{j_1} \alpha x^*_{j_1} \}$,
- $A_{j_2} = \{ p_i \in P : p_i \geq a_{j_2} \alpha x^*_{j_2} \}$ but $p_i \not\geq a_{j_1} \alpha x^*_{j_1}$,
- $\ldots$
- $A_{j_k} = \{ p_i \in P : p_i \geq a_{j_k} \alpha x^*_{j_k} \}$ but $p_i \not\geq a_{j_{k-1}} \alpha x^*_{j_{k-1}}$, $l = 1, 2, \ldots, k-1$,
- $\ldots$
- $A_{j_m} = \{ p_i \in P : p_i \geq a_{j_m} \alpha x^*_{j_m} \}$ but $p_i \not\geq a_{j_{m-1}} \alpha x^*_{j_{m-1}}$, $l = 1, 2, \ldots, m-1$,
- $A_j = \emptyset$ for any $j \notin J \setminus H$.

Then $\bigcup_{j \in J} A_j = \bigcup_{j \in H} A_j = P$ and there exists at least a $j \in J$ such that $A_j \neq \emptyset$. Consider $R = (r_j)_{j \in J}$ with

$$r_j = \begin{cases} \bigwedge_{p_i \in A_j} p_i & A_j \neq \emptyset, \\ 1 & A_j = \emptyset. \end{cases}$$

Then $R|_H \geq X^*|H$ and $R \geq X^*$. Again, if $A_j \neq \emptyset$, then $j \in H$. From Lemmas 2.4, 2.2, 4.1 and 2.5, it follows that $A \otimes R = \bigwedge_{j \in J} (a_j \alpha r_j) = \bigwedge_{j \in H} [a_j \alpha (\bigwedge_{p_i \in A_j} p_i)] = \bigwedge_{j \in H, p_i \in A_j} (a_j \alpha p_i) = \bigwedge_{p_i \in P} p_i = b$, i.e. $R \in \mathcal{X}$. Hence $R = X^*$ since $X^*$ is a maximal element in $\mathcal{X}$. Therefore, if there exists a $j \in J$ such that $x^*_j \neq 1$, then there exist $p_{i_1}, p_{i_2}, \ldots, p_{i_l} \in P$ such that $x^*_j = p_{i_1} \wedge p_{i_2} \wedge \cdots \wedge p_{i_l}$. \hfill $\square$

In the rest of this subsection, let each component $b_i$ ($i \in I$) of $B$ in (2) has an irredundant finite meet-decomposition. Then, by Lemmas 2.6 and 2.7, we assume that

$$b_i = \bigwedge_{r=1}^{n_i} p_{i_r}, \quad p_i = \{ p_{i_1}, p_{i_2}, \ldots, p_{i_{n_i}} \},$$

and

$$X|_H = X^*|H.$$
where \( p_{ii} \) is meet-irreducible for all \( i \in I, t \in \{1, 2, \ldots, n_i\} \). That is,

\[
b_i = \bigwedge_{p_{ii} \in P_i} p_{ii},
\]

for any \( i \in I \).

**Proposition 5.1.2.** If \( \mathcal{X}_2 \neq \emptyset \), and each component \( b_i \) \((i \in I)\) of \( B \) is dually compact and has an irredundant finite meet-decomposition, then there exists a maximal element \( F_X \in \mathcal{X}_2 \) such that \( F_X \geq X \) for any \( X \in \mathcal{X}_2 \).

**Proof.** Suppose \( \overline{X} = (x_j)_{j \in J}^T \in \mathcal{X}_2 \), then according to Proposition 3.2, \( \overline{X} \in \mathcal{X}_{i1} \) for any \( i \in I \). Also from Proposition 5.1.1, there exists a maximal element \( X_i^{1*} \in \mathcal{X}_{i1} \) such that \( X_i^{1*} \geq \overline{X} \). Let

\[
X^1 = \bigwedge_{i \in I} X_i^{1*},
\]

thus \( X^1 \geq \overline{X} \) and \( X_i^{1*} \geq X^1 \geq \overline{X} \) for any \( i \in I \). Therefore, \( X^1 \in \mathcal{X}_{i1} \) for any \( i \in I \). From Proposition 3.2, it follows that \( X^1 \in \mathcal{X}_2 \) and \( X^1 \geq \overline{X} \). Now, let \( X^1 = (x_j)_{j \in J}^T \). Then since \( I \) is finite, Remark 5.1.1 implies that \( X^1 \) satisfies:

(5.1.1) there exists a finite subset \( H \) of \( J \) such that \( x_j^1 = 1 \) for any \( j \in J \setminus H \);

(5.1.2) if \( x_j^1 \neq 1 \) for any \( j \in J \), then there exists \( p_{ii1}, p_{ii2}, \ldots, p_{iiw(j)} \in P_i \) for \( i \in I \) such that

\[
x_j^1 = \bigwedge_{i \in I} (p_{ii1} \wedge p_{ii2} \wedge \cdots \wedge p_{iiw(j)}).
\]

Suppose \( X^{1*} \geq X^1 \) for any \( X^{1*} \in \mathcal{X}_2 \). If \( X^{1*} = X^1 \), then Definition 2.4 implies that \( X^1 \) is a maximal element in \( \mathcal{X}_2 \) and \( X^1 \geq \overline{X} \). Otherwise, repeat the construction steps of \( X^1 \), we can obtain an element \( X^2 = (x_j^2)_{j \in J}^T \) of \( \mathcal{X}_2 \) such that

\[
X^2 = \bigwedge_{i \in I} X_i^{2*},
\]

where \( X_i^{2*} \) is a maximal element in \( \mathcal{X}_{i1} \) for any \( i \in I \) and \( X_i^{2*} \geq X^{1*} \). Moreover, both (5.1.1) and (5.1.2) are satisfied by \( X^2 \) (just replace \( x_j^1 \) with \( x_j^2 \) in (5.1.1) and (5.1.2), respectively). Suppose \( X^{2*} \geq X^2 \) for any \( X^{2*} \in \mathcal{X}_2 \). If \( X^{2*} = X^2 \), then by Definition 2.4, \( X^2 \) is a maximal element in \( \mathcal{X}_2 \) and \( X^2 \geq X^{1*} \geq X^1 \geq \overline{X} \). Otherwise, repeat the steps above, and so on, we can construct a series of \( X^k = (x_j^k)_{j \in J}^T \) in \( \mathcal{X}_2 \) with \( k = 1, 2, \ldots \), such that:

\[
\cdots \geq X^k \geq \cdots \geq X^2 \geq X^1
\]

(5)

and both (5.1.1) and (5.1.2) are satisfied by all \( X^k \) (just replace \( x_j^1 \) with \( x_j^2 \) in (5.1.1) and (5.1.2), respectively). Notice that for any \( i \in I \), if \( v \neq w \) with \( v, w \in \{1, 2, \ldots, n_i\} \), then \( p_{iv} \leq p_{iw} \) and \( p_{iv} \leq p_{bw} \). On the other hand, the property (5.1.1) implies that there exists a finite subset \( H \) of \( J \) such that \( x_j^k = 1 \) for any \( j \in J \setminus H \) for each \( X^k \) in (5). Also, since \( I \) is finite and \( b_j \in I \) has an irredundant finite meet-decomposition, i.e. \( P_i \) is also finite for any \( i \in I \), then there must exist an \( X^m \) in (5) such that \( X^m = X^n \) if \( m \geq n \). We claim that \( X^n \) is a maximal element in \( \mathcal{X}_2 \). Indeed, for any \( X \in \mathcal{X}_2 \) if \( X \geq X^n \), then we can construct an \( X^{n+1} \) such that \( X^{n+1} \geq X \) and both (5.1.1) and (5.1.2) are satisfied by \( X^{n+1} \). Since \( n + 1 \geq n \), then \( X^{n+1} = X^n \) and \( X = X^n \). From Definition 2.4, it then follows that \( X^n \) is a maximal element in \( \mathcal{X}_2 \). Obviously, \( X^n \geq \overline{X} \). □

It is easy to see that the proof of Proposition 5.1.2 gives a method to construct a maximal solution \( F_X \) such that \( F_X \geq X \) for any solution \( X \) of equation system (2).

5.2. Necessary and sufficient conditions

In this subsection, we give a necessary and sufficient condition that there exists a maximal solution \( F_X \) such that \( F_X \geq X \) for any solution \( X \) of Eq. (1). Particularly, on the real unit interval, we give a necessary and sufficient condition that there exist maximal solutions, and a necessary and sufficient condition that there exists a maximal solution \( F_X \) such that \( F_X \geq X \) for any solution \( X \) of Eq. (1).
Proposition 5.2.1. If $\mathcal{X}_1 \neq \emptyset$ and $X^* = (x^*_j)_{j \in J}$ is a maximal element in $\mathcal{X}_1$, then $b = \bigwedge_{j \in J} x^*_j$.

Proof. Since $X^* \in \mathcal{X}_1$, i.e. $b = \bigwedge_{j \in J} (a_j x^*_j)$, Lemma 2.1 implies that $a_j x^*_j \geq x^*_j$ for any $j \in J$. If $a_j x^*_j > x^*_j$ for some $j_0 \in J$, then consider $R = (r_j)_{j \in J}$ with

$$r_j = \begin{cases} a_j x^*_j, & j = j_0, \\ x^*_j, & \text{otherwise.} \end{cases}$$

It is easy to verify that $R \geq X^*$, but $R \neq X^*$. From Lemma 2.3, it follows that

$$\bigwedge_{j \in J} (a_j x^*_j) = \left[ \bigwedge_{j \in J, j \neq j_0} (a_j x^*_j) \right] \vee \left[ (a_{j_0} \wedge a_{j_0}) x^*_j \right]$$

which holds (i), (ii)

Corollary 5.2.1. Let $\mathcal{X}_1 \neq \emptyset$ and $X^* = (x^*_j)_{j \in J}$ be a maximal element of $\mathcal{X}_1$, then:

1. $b < \bigwedge_{j \in J, j \neq i} x^*_j$ whenever $x^*_i \neq 1$ for any $i \in J$;
2. $a_j x^*_j = x^*_j$ for any $j \in J$.

Proposition 5.2.2. Let $\mathcal{X}_1 \neq \emptyset$. Then a necessary and sufficient condition for existence of a maximal element $F_X$ in $\mathcal{X}_1$ such that $F_X \geq X$ for any $X \in \mathcal{X}_1$ is that there exists a subset $B$ of $L$ which holds that:

(i) $\bigwedge B = b$;
(ii) for any $p \in B$, if $p \neq 1$, then $b \neq \bigwedge B \setminus \{p\}$;
(iii) if $X = (x^*_j)_{j \in J}$ holds that $b = \bigwedge_{j \in J} (a_j x^*_j)$, then for any $p \in B$ there exists a $j \in J$ such that $p \geq a_j x^*_j$, and $a_j p = p$.

Proof. Necessity. (i) Let $X^* = (x^*_j)_{j \in J}$ be a maximal element of $\mathcal{X}_1$ and $B^* = \{x^*_j : j \in J\}$, then $\bigwedge B^* = b$ by Proposition 5.2.1.

(ii) For any $x^*_j \in B^*$, if $x^*_j \neq 1$, then $b \neq \bigwedge B^* \setminus \{x^*_j\}$ by Corollary 5.2.1.

(iii) If $X = (x^*_j)_{j \in J}$ holds that $b = \bigwedge_{j \in J} (a_j x^*_j)$, i.e. $X \in \mathcal{X}_1$, then by hypothesis, there exists a maximal element $S = (s^*_j)_{j \in J}$ of $\mathcal{X}_1$ such that $S \geq X$, i.e. $s^*_j \geq x^*_j$ for any $j \in J$. Let $B = \{s^*_j : j \in J\}$, then we similarly have that $\bigwedge B = b$ and $b \neq \bigwedge B \setminus \{p\}$ when $p \neq 1$ for any $p \in B$. Moreover, from Corollary 5.2.1 and Lemma 2.2 it follows that $s^*_j \geq x^*_j$ and $s^*_j = a_j s^*_j \geq a_j x^*_j$ for any $j \in J$. Hence, there exists a $j \in J$ such that $p \geq a_j x^*_j$ for any $p \in B$, and $a_j p = p$.

Sufficiency. Let $X = (x^*_j)_{j \in J} \in \mathcal{X}_1$, then $b = \bigwedge_{j \in J} (a_j x^*_j)$. Suppose that $B$ is a subset of $L$ which holds (i), (ii) and (iii). Then from (iii), there exists a $j \in J$ such that $p \geq a_j x^*_j$ for any $p \in B$. Now let $B_s = B \setminus \{1\}$, then we can construct a family of subsets $A_j$ with $j \in J$, of $B_s$, which satisfies:

(5.2.1) for any $j \in J, A_j = \{p \in B_s : p \geq a_j x^*_j\}$;
(5.2.2) if $i \neq j$ for any $i, j \in J$ then $A_i \cap A_j = \emptyset$.

It follows that $\bigcup_{j \in J} A_j = B_s$, therefore $\bigwedge (\bigcup_{j \in J} A_j) = \bigwedge B_s = \bigwedge B = b$. Let $X^* = (x^*_j)_{j \in J}$ be defined by

$$x^*_j = \begin{cases} \bigwedge_{p \in A_j} p, & A_j \neq \emptyset, \\ 1, & A_j = \emptyset. \end{cases}$$
Then
\[
\bigwedge_{j \in J} (a_j \alpha x_j^*) = \bigwedge_{j \in J, A_j \neq \emptyset} (a_j \alpha x_j^*)
\]
\[
= \bigwedge_{j \in J, A_j \neq \emptyset} \left[ a_j \alpha \left( \bigwedge_{p \in A_j} p \right) \right]
\]
\[
= \bigwedge_{j \in J, A_j \neq \emptyset} \left( \bigwedge_{p \in A_j} (a_j \alpha p) \right)
\]
\[
= \bigwedge_{j \in J, A_j \neq \emptyset} \left( \bigwedge_{p \in A_j} p \right)
\]
\[
= \bigwedge B_\ast, = b,
\]
i.e. \(X^* \in \mathcal{D}_1\) and \(X^* \geq X\). We only need to show that \(X^*\) is a maximal element in \(\mathcal{D}_1\). Suppose that there exists an \(R = (r_j)_{j \in J} \in \mathcal{D}_1\) such that \(R \geq X^*\), then we need to prove \(R = X^*\). For any \(j \in J\), it is easy to see that \(r_j = x_j^*\) if \(x_j^* = 1\). Now we assume that \(a_j \neq 0\), thus \(A_j = \emptyset\) and \(x_j^* = \bigwedge_{p \in A_j} p\). Also, by \(b = \bigwedge_{j \in J} (a_j \alpha r_j)\) there exists a \(j_0 \in J\) such that \(p \geq a_j \alpha r_j\) for any \(p \in A_j \subseteq B_\ast \subseteq B\). Due to Lemma 2.1, \(a_j \alpha r_j \geq r_{j_0}\), hence \(p \geq r_{j_0}\). Since \(p \neq 1\) and \(r_{j_0} \geq x_{j_0}^*,\) we have \(x_{j_0}^* \neq 1\) which means that \(A_{j_0} \neq \emptyset\) and \(x_{j_0}^* = \bigwedge_{q \in A_{j_0}} q\). Thus, \(p \geq \bigwedge_{q \in A_{j_0}} q\).

If \(j_0 \neq j\), then \(A_{j_0} \cup A_j = \emptyset\) and \(p \neq A_{j_0}\), hence \(\bigwedge B \setminus \{p\} = b\), which contradicts (ii). Therefore, \(j_0 = j\) and \(p \geq r_j\). Since \(p\) is arbitrary in \(A_j\), we have \(\bigwedge_{p \in A_j} p \geq r_j\), i.e. \(x_j^* \geq r_j\), thus \(x_j^* = r_j\). Also since \(j\) is arbitrary in \(\{j \in J : x_j^* \neq 1\}\), we have \(R = X^*\), i.e. \(X^*\) is a maximal element in \(\mathcal{D}_1\). \(\square\)

**Proposition 5.2.3.** Let \(L = [0, 1]\), \(\mathcal{D}_1 \neq \emptyset\) and \(b \neq 1\). Then a necessary and sufficient condition for existence of a maximal element in \(\mathcal{D}_1\) is that there exists an \(i \in J\) such that \(a_i > b\).

**Proof.** Let us first assume that \(X^* = (x_j^*)_{j \in J} \in \mathcal{D}_1\) is a maximal element of \(\mathcal{D}_1\), then from \(b \neq 1\) it follows that there exists an \(i_0 \in J\) such that \(x_{i_0}^* \neq 1\). We claim that \(x_j^* = 1\) for any \(j \in J \setminus \{i_0\}\). Otherwise, assume that there also exists a \(j \in J \setminus \{i_0\}\) such that \(x_j^* \neq 1\). Since \(x_{i_0}^*\) and \(x_j^*\) belong to \([0, 1]\), without loss of generality, suppose \(x_j^* \geq x_{i_0}^*\), it then follows from Proposition 5.2.1 that
\[
b = \bigwedge_{j \in J} x_j^* = \bigwedge_{k \in J, k \neq j} x_k^*,
\]
which contradicts Corollary 5.2.1. So, \(b = \bigwedge_{k \in J} x_k^* = 1 \land \cdots \land 1 \land x_{i_0}^* \land 1 \land \cdots = x_{i_0}^*\). Further, according to Corollary 5.2.1 we have
\[
b = x_{i_0}^* = a_{i_0} x_{i_0}^* = a_{i_0} \alpha b.
\]
If \(a_{i_0} \leq b\), then \(b = a_{i_0} \alpha b = 1\) by Lemma 2.2, it contradicts the hypothesis that \(b \neq 1\). Thus, \(a_{i_0} > b\).
Conversely, if there exists an \(i_0 \in J\) such that \(a_{i_0} > b\), then consider \(X^{**} = (x_j^{**})_{j \in J}\) with
\[
x_j^{**} = \begin{cases} b & j = i_0, \\ 1 & j \neq i_0. \end{cases}
\]
By Lemma 2.1, it is easy to verify that \(X^{**} \in \mathcal{D}_1\) and \(X^{**}\) is a maximal element in \(\mathcal{D}_1\). \(\square\)

In general, if \(L\) is not linear, then Proposition 5.2.3 may not be true.

**Example 5.2.1.** It is easy to see that \((1, 0.6, 0.5, 1)\) is a maximal solution of \((0.3, 0.7, 0.6, 0.4)\) on \((0, 1)^2, \leq\) even if
\[
a_1 = (0.3, 0.7) \not\succ b = (0.5, 0.6) \quad \text{and} \quad a_2 = (0.6, 0.4) \not\succ b = (0.5, 0.6).
\]
Corollary 5.2.2. Let $L = [0, 1]$, $\mathcal{X}_1 \neq \emptyset$ and $b \neq 1$. If $X^* = (x^*_j)_{j \in J}$ is a maximal element in $\mathcal{X}_1$, then there exactly exists a $j \in J$ such that $x^*_j = b$.

Corollary 5.2.3. Let $L = [0, 1]$ and $\mathcal{X}_1 \neq \emptyset$. If there exist maximal elements in $\mathcal{X}_1$, then all maximal elements of $\mathcal{X}_1$ are of the form (6).

Proposition 5.2.4. Let $L = [0, 1]$ and $\mathcal{X}_1 \neq \emptyset$. Then a necessary and sufficient condition for existence of a maximal element $F_X$ in $\mathcal{X}_1$ such that $F_X \geq X$ for any $X = (x_j)_{j \in J} \in \mathcal{X}_1$ is that there exists a $j \in J$ such that $b = a_j \alpha x_j$.

Proof. Necessity. If $b = 1$, then it is trivial. Now, suppose $b \neq 1$, it then follows from Lemma 2.2, Corollaries 5.2.1 and 5.2.2 that there exists a $j \in J$ such that $b = a_j \alpha x_j$.

Sufficiency. Follows immediately from Proposition 5.2.2 with $B = \{b\}$. □

Notice that Proposition 5.2.3 is essentially different from Proposition 5.2.4.

Example 5.2.2. Consider equation $\frac{1}{4} = (1 \alpha x_1) \wedge (\bigwedge_{n=2}^{\infty} \frac{1}{4} \alpha x_n)$, from Proposition 5.2.3 and Corollary 5.2.3 it follows that $(\frac{1}{4}, 1, 1, \ldots, 1, \ldots)$ is a maximal solution of the equation. For $X = (1, \frac{1}{4}(1 + \frac{1}{2}), \ldots, \frac{1}{4}(1 + \frac{1}{n+1}), \ldots)^T$, we have $(1 \alpha 1) \wedge (\frac{1}{4}(1 + \frac{1}{2}) \alpha \frac{1}{4}(1 + \frac{1}{2})) \wedge (\frac{1}{4}(1 + \frac{1}{3}) \alpha \frac{1}{4}(1 + \frac{1}{3})) \wedge \cdots \wedge (\frac{1}{4}(1 + \frac{1}{n+1}) \alpha \frac{1}{4}(1 + \frac{1}{n+1})) \wedge \cdots = 1 \wedge \frac{1}{4}(1 + \frac{1}{2}) \wedge \cdots \wedge \frac{1}{4}(1 + \frac{1}{n+1}) \wedge \cdots = \bigwedge_{n=2}^{\infty} \frac{1}{4}(1 + \frac{1}{n+1}) = \frac{1}{4}$.

Thus, $X$ is a solution of the equation, but there does not exist a maximal solution $X^*$ such that $X^* \geq X$ by Proposition 5.2.4.

6. Conclusions

In this paper, in finite domains, the solution set of an inf-$\alpha$ composite fuzzy relational equation (an equation system, respectively) is determined when each component on the right-hand side of the equation and the equation system has irredundant finite meet-decomposition. Many important necessary and sufficient conditions about the maximal solutions are obtained in finite and infinite domains. These contribute to our future research on a general inf-$\alpha$ composite fuzzy relational equation $A_{I \times J} \circ X_{J \times K} = B_{I \times K}$ with fuzzy relations $A_{I \times J}$ and $B_{I \times K}$ given, and $X_{J \times K}$ unknown.

In addition, by comparing the results in Sections 4 and 5 with those of Wang [5–8] which discuss sup-inf composite fuzzy relational equations, some of them show certain duality even if the advantages of the inf-$\alpha$ composite fuzzy relational equations have been explained by Di Nola et al. [4,10]. On the other hand, some properties cannot be dual. For example, Proposition 5.2.4 cannot be dually obtained from the proposition below:

Proposition 6.1. Let $L = [0, 1]$ and $\mathcal{X} = \{X : A \circ X = b\} \neq \emptyset$. Then a necessary and sufficient condition for the existence of a maximal element $F_X$ in $\mathcal{X}$ such that $F_X \leq X$ for any $X = (x_j)_{j \in J} \in \mathcal{X}$ is that there exists a $j \in J$ such that $b \leq a_j \wedge x_j$.

Proof. Refer to Theorem 3.1 of Wang [8]. □

References