A Note on Algebra and Macroscopic Observation

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We want here to point out a possible use of algebra in the field of macroscopic observation theory. Let us first notice that many physical entities can be described by a function $f(M, t)$ of space and time and that, whatever observation device may be used, it is only a function $g(P, s)$, more or less closely related to $f(M, t)$, which can be observed. This is due to many reasons. The observation device, which we suppose of macroscopic nature, does not work permanently, so parts of the evolution of the phenomenon are lost. Portions of space may be outside of the scope of the observation device and so may remain unobserved. The recording of a signal involves distortions and time lag, an optical system does not give an exact image of the object. So, generally speaking, we can represent the transition from $f(M, t)$ to $g(P, s)$ by

$$g(P, s) = \mathcal{C}f(M, t),$$

where $\mathcal{C}$ is an operator.

Let us consider only two special classes of linear $\mathcal{C}$ operators. The operators involved when the spatio-temporal domain of observation is bounded, or more generally when there is attenuation by a factor which is a function of space and time, will be called $\alpha$ operators and we have

$$g(P, s) = \alpha f(M, t) = A(P, s)f(P, s).$$

The other $\mathcal{C}$ operators we are interested in will be called $\mathcal{Z}$ operators and defined as follows

$$g(P, s) = \mathcal{Z}f(M, t) = \int k(P - M, s - t)f(M, t) \, dM \, dt.$$
In other terms, if $F(\Omega, \omega)$ and $G(\Omega, \omega)$ are the Fourier transforms (when they exist) of $f(M, t)$ and $g(M, t)$, we have
\[ G(\Omega, \omega) = Z(\Omega, \omega)F(\Omega, \omega), \]
where $Z(\Omega, \omega)$ is the Fourier transform of the kernel $k$, $\Omega$ symbolizes the "space frequencies," and $\omega$ is the time frequency.

Obviously the set of operators $\alpha$ (or $Z$) is a commutative ring with respect to sum and product and this has been studied in the case of the one-dimensional $Z$ operators known as electrical filters (Zadeh and Miller, 1952). But to what extent can we speak of sum and product of observation devices associated to $\alpha$ (or $Z$) operators? Let us consider two optical systems. It is well known (Duffieux, 1946, and other authors) that they can be represented by $Z$ operators which we shall call $Z_1$ and $Z_2$. It is possible to align these systems in such a way that the image given by the first system plays the role of an optical object to the second. In this way we realize physically the product of operators $Z_1$ and $Z_2$ and obtain a new optical system, the "product" of the two others, described by
\[ Z_3 = Z_2Z_1, \quad \text{with} \quad Z_3(\Omega) = Z_2(\Omega)Z_1(\Omega). \]
Examples of association of observation devices described by operators $\alpha$ can be given too. A transparent and shaded screen can be described by a transparency function $A(P)$. Two screens of this kind, closely superimposed, give a new transparent screen described by a transparency function equal to the product of the transparency functions of the given screens. If $\alpha_1$ and $\alpha_2$ are the operators associated to the screens the resultant one is represented by
\[ \alpha_3 = \alpha_2\alpha_1, \quad \text{with} \quad A_3(P) = A_2(P)A_1(P). \]
It is possible to give physical examples of the sum of operators $\alpha$ (or $Z$) and consequently of the "sum of two observation devices." The simultaneous use of stroboscopic apparatus which can be described by functions $A_1(t)$ and $A_2(t)$ (equal to zero everywhere except in short intervals of time) is equivalent to a stroboscopic device represented by the sum of $A_1(t)$ and $A_2(t)$
\[ \alpha_3 = \alpha_2 + \alpha_1, \quad \text{with} \quad A_3(t) = A_2(t) + A_1(t). \]
The examples given above are very simple. Nevertheless they suggest
that an algebraical treatment of some classes of chains of observation
devices is possible and probably useful.

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References