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Projections of cones and the arithmetical rank of toric varieties☆

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Abstract

Let I_M and I_N be defining ideals of toric varieties such that I_M is a projection of I_N, i.e. $I_N \subseteq I_M$. We give necessary and sufficient conditions for the equality $I_M = rad(I_N + (f_1, \ldots, f_s))$, where $f_1, \ldots, f_s$ belong to I_M. Also, a method for finding toric varieties which are set-theoretic complete intersection is given. Finally, we apply our method in the computation of the arithmetical rank of certain toric varieties and provide the defining equations of the above toric varieties.

MSC: 14M25; 14M10

1. Introduction

One of the classical problems of Algebraic Geometry with a long history, see [3, Chapter 15], is to determine the minimal number of equations needed to define set-theoretically an algebraic variety over an algebraically closed field. Even more difficult is to provide minimal sets of equations that define the algebraic variety. The problem is open even for very simple cases, like the Macaulay curve (t^4, t^3u, tu^3, u^4) in the three-dimensional projective space. This article addresses these two problems for toric varieties and in several cases we are

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able to compute the minimal number, but also provide the equations that define the variety set-theoretically.

Let $M = (a_{i,j})$ be an $m \times n$ matrix with integer entries $a_{i,j}$, such that every column has atleast one non-zero entry. Let K be a field and let $A = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ be the set of vectors in $\mathbb{Z}^m$, where $\mathbf{a}_i = (a_{1,i}, \dots, a_{m,i})$ for $1 \leqslant i \leqslant n$. The *toric ideal* I_M associated with M is the kernel of the K-algebra homomorphism

$$\phi : K[x_1, \dots, x_n] \to K[t_1, \dots, t_m, t_1^{-1}, \dots, t_m^{-1}]$$

given by

$$\phi(x_i) = \mathbf{t}^{\mathbf{a}_i} := t_1^{a_{1,i}} \cdots t_m^{a_{m,i}} \quad \text{for all } i = 1, \dots, n.$$

The ideal I_M is prime and therefore $rad(I_M) = I_M$. A difference of two monomials is called a *binomial*. Every vector $\mathbf{u}$ in $\mathbb{Z}^n$ can be written uniquely as $\mathbf{u} = \mathbf{u}_+ - \mathbf{u}_-$, where $\mathbf{u}_+ = (u_{+,1}, \dots, u_{+,n})$ and $\mathbf{u}_- = (u_{-,1}, \dots, u_{-,n})$ are non-negative and have disjoint support. If D is an $m \times n$ matrix with rational entries and columns $\{\mathbf{d}_1, \dots, \mathbf{d}_n\}$, then the kernel of D is

$$ker(D) = \{\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{Q}^n | u_1\mathbf{d}_1 + \cdots + u_n\mathbf{d}_n = \mathbf{0}\}.$$

Set $ker_{\mathbb{Z}}(D) = ker(D) \cap \mathbb{Z}^n$. The height $ht(I_M)$ of I_M equals the rank of the lattice $ker_{\mathbb{Z}}(M)$ (see [12]).

Lemma 1.1 (*Sturmfels [12]*). *The toric ideal I_M is generated by the binomials $\mathbf{x}^{\mathbf{u}_+} - \mathbf{x}^{\mathbf{u}_-}$, where $\mathbf{u}$ belongs to $ker_{\mathbb{Z}}(M)$.*

We grade the polynomial ring $K[x_1, \dots, x_n]$ by setting $deg_A(x_i) = \mathbf{a}_i$ for $i = 1, \dots, n$. We define the *A-degree* of the monomial $\mathbf{x}^{\mathbf{u}}$ to be

$$deg_A(\mathbf{x}^{\mathbf{u}}) := u_1\mathbf{a}_1 + \cdots + u_n\mathbf{a}_n \in \mathbb{N}(A),$$

where $\mathbb{N}(A)$ is the semigroup generated by A. Every toric ideal I_M is A-homogeneous, since it is generated by binomials and every binomial $\mathbf{x}^{\mathbf{u}_+} - \mathbf{x}^{\mathbf{u}_-}$ is A-homogeneous.

The *toric variety* X_M associated with M is the set $V(I_M) \subset K^n$ of zeroes of I_M in the sense of [12], which also includes non-normal varieties. The *toric set* $\Gamma(M)$ determined by M is the subset of K^n defined parametrically by $x_i = t_1^{a_{1,i}} \cdots t_m^{a_{m,i}}$ for all i, i.e. it is the set of points that can be expressed in the form

$$(t_1^{a_{1,1}} \cdots t_m^{a_{m,1}}, \dots, t_1^{a_{1,i}} \cdots t_m^{a_{m,i}}, \dots, t_1^{a_{1,n}} \cdots t_m^{a_{m,n}})$$

for some t_i in K. Note that $\Gamma(M)$ is a subset of X_M. When $m = 1$ and $a_{1,1} < a_{1,2} < \cdots < a_{1,n}$ are positive integers, the g.c.d. of which equals 1, then $\Gamma(M)$ is known as a monomial curve and I_M as the ideal of the monomial curve.

We associate to the toric variety X_M the rational polyhedral cone $\sigma = pos_{\mathbb{Q}}(A) := \{\sum_{i=1}^{n} d_i\mathbf{a}_i | d_i \in \mathbb{Q} \text{ and } d_i \geqslant 0\}$. The dimension of σ is equal to the dimension of the vector space $\mathbb{Q}A = \{\sum_{i=1}^{n} d_i\mathbf{a}_i | d_i \in \mathbb{Q}\}$ and also is equal to the dimension of X_M.

In this paper we consider the following two problems related to the toric ideal I_M:

(I) Given a toric ideal I_N, such that $I_N \subseteq I_M$, and a set of binomials $\{f_1, \ldots, f_s\}$ in I_M, formulate a criterion for deciding the equality $I_M = rad(I_N + (f_1, \ldots, f_s))$.

(II) Find the smallest number of polynomials needed to generate I_M up to radical. This problem is more general than the corresponding problem in Algebraic Geometry of the determination of the minimum number of equations needed to define a toric variety X_M set-theoretically, over an algebraically closed field. This number is called *arithmetical rank* of X_M and will be denoted by $ara(X_M)$. The generalized Krull's principal ideal theorem provides a lower bound for the arithmetical rank of X_M, namely the height of I_M. When $ht(I_M) = ara(X_M)$ the ideal I_M (and the variety X_M as well) is called a *set-theoretic complete intersection*.

Problem (I) was studied by Eliahou–Villarreal in [7] in the special case that $I_N = (0)$. There they give a necessary and sufficient condition for the equality $rad(f_1, \ldots, f_s) = I_M$. More precisely they prove that:

Theorem 1.2 (*Eliahou and Villarreal [7], Villarreal [15]*). *Let $\{f_1, \ldots, f_s\}$ be a set of binomials in the toric ideal I_M. Set $J = (f_1, \ldots, f_s)$ and $G = \langle \widehat{f_1}, \ldots, \widehat{f_s} \rangle \subset ker_{\mathbb{Z}}(M)$, where for a binomial $f = \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in K[x_1, \ldots, x_n]$ we let $\widehat{f} = \mathbf{u} - \mathbf{v} \in \mathbb{Z}^n$. If $char(K) = p \neq 0$ (resp. $char(K) = 0$), then $rad(J) = I_M$ if and only if*:

(a) *$p^k ker_{\mathbb{Z}}(M) \subset G$ for some $k \in \mathbb{N}$ (resp. $ker_{\mathbb{Z}}(M) = G$),*
(b) *$rad(J, x_i) = rad(I_M, x_i)$, for all $i = 1, \ldots, n$.*

We generalize (see Theorem 4.4) this result in terms of projections of ideals. Our criterion can be used also to determine different binomial generators for the radical of the ideal of a toric variety (see Example 4.6). However in the case that we can make a good choice of a projection I_N, minimal binomial generators up to radical for the ideal I_M of the toric variety are derived.

Basic ingredient of our approach is the notion of projections of toric ideals. This notion, although it was never before explicitly defined, has been used for the first time by Herzog in [9] to prove that the ideal of the monomial curve $(t^{a_1}, t^{a_2}, t^{a_3})$ is set-theoretic complete intersection. In [13], the same notion has been used to prove that, when $char(K) = 0$, smooth monomial curves are not binomial set-theoretic complete intersections, except for the twisted cubic. Finally in [14], Thoma used this notion to deduce that certain ideals of monomial curves $(t^{a_1}, \ldots, t^{a_n})$ are set-theoretic complete intersections. The techniques developed there cannot always be applied, for example the ideal of the monomial curve $(t^4, t^6, t^{11}, t^{13})$ was generally unknown whether or not it is a set-theoretic complete intersection. Our method begins with a toric ideal which is set-theoretic complete intersection and it produces a large number of toric ideals which are set-theoretic complete intersections. This method also provides the defining equations of the toric variety.

In Section 2 we introduce the basic notion of this paper, the notion of projections of toric ideals, and present its connection with the geometric notion of projections of cones.

In Section 3 we give necessary and sufficient conditions for the equality $rad(I_N + (f_1, \ldots, f_s)) = I_M$, where I_M is a projection of I_N and $f_1, \ldots, f_s$ belong to I_M.

In Section 4 we study the previous equality in the special case that $f_1, \ldots, f_s$ are binomials.

In Section 5 we develop a method for finding toric ideals which are set-theoretic complete intersections.

In Section 6 we apply the theory developed in Section 5 in the computation of the exact value of the arithmetical rank of certain toric ideals. Among other results, we prove that the ideal of the monomial curve (t^4, t^6, t^a, t^b) is set-theoretic complete intersection, so for $a = 11, b = 13$ the ideal of the monomial curve $(t^4, t^6, t^{11}, t^{13})$ is set-theoretic complete intersection.

2. Projections of toric ideals

We consider the toric ideals I_M, I_N associated with the $m \times n$ matrix $M = (a_{i,j})$ and $l \times n$ matrix $N = (b_{i,j})$, respectively. Let $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$, $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, where $\mathbf{a}_i = (a_{1,i}, \ldots, a_{m,i})$ and $\mathbf{b}_i = (b_{1,i}, \ldots, b_{l,i})$ for $1 \leqslant i \leqslant n$.

Definition 2.1. We say that I_M is a *projection* of I_N if $I_N \subseteq I_M$.

Let $\widehat{\pi} : \mathbb{Q}^l \to \mathbb{Q}^m$ be a rational affine map with $\widehat{\pi}(pos_{\mathbb{Q}}(B)) = pos_{\mathbb{Q}}(A)$. We call

$$\pi := \widehat{\pi}|_{pos_{\mathbb{Q}}(B)} : pos_{\mathbb{Q}}(B) \to pos_{\mathbb{Q}}(A)$$

a *projection of cones*. The next theorem makes the connection between the algebraic notion of projections of toric ideals and the geometric notion of projections of cones.

Theorem 2.2. *The following are equivalent*:

(a) *I_M is a projection of I_N.*
(b) *The lattice $ker_{\mathbb{Z}}(N)$ is a subset of the lattice $ker_{\mathbb{Z}}(M)$.*
(c) *Every B-homogeneous ideal in $K[x_1, \ldots, x_n]$ is also A-homogeneous.*
(d) *There is a projection of cones $\pi : pos_{\mathbb{Q}}(B) \to pos_{\mathbb{Q}}(A)$ given by $\pi(\mathbf{b}_i) = \mathbf{a}_i$ for all $i = 1, \ldots, n$.*
(e) *There is an $m \times l$ matrix D with rational entries such that $DN = M$.*

Proof. (a) ⇒ (b) Let $\mathbf{u} = \mathbf{u}_+ - \mathbf{u}_-$ be an element of $ker_{\mathbb{Z}}(N)$. Then the binomial $\mathbf{x}^{\mathbf{u}_+} - \mathbf{x}^{\mathbf{u}_-}$ belongs to I_N which, from the assumption, is a subset of I_M. Consequently $\mathbf{u}$ is in $ker_{\mathbb{Z}}(M)$.

(b) ⇒ (c) Let $I \subseteq K[x_1, \ldots x_n]$ be a B-homogeneous ideal and $\mathbf{x}^{\mathbf{u}}$, $\mathbf{x}^{\mathbf{v}}$ two monomials of a B-homogeneous generator f of I, where $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$. Set $g = \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$. We have

$$u_1\mathbf{b}_1 + \cdots + u_n\mathbf{b}_n = v_1\mathbf{b}_1 + \cdots + v_n\mathbf{b}_n,$$

which implies that $\widehat{g}$ belongs to the lattice $ker_{\mathbb{Z}}(N)$. As $ker_{\mathbb{Z}}(N) \subseteq ker_{\mathbb{Z}}(M)$, we obtain that the vector $\widehat{g}$ belongs to $ker_{\mathbb{Z}}(M)$ and so

$$u_1\mathbf{a}_1 + \cdots + u_n\mathbf{a}_n = v_1\mathbf{a}_1 + \cdots + v_n\mathbf{a}_n.$$

Hence I is A-homogeneous.

(c) ⇒ (d) It is enough to consider the case $dim(pos_{\mathbb{Q}}(B)) = l$. Let $\{\mathbf{b}_{j_1}, \dots, \mathbf{b}_{j_l}\}$ be a base for the $\mathbb{Q}$-vector space $\mathbb{Q}^l$ and define $\widehat{\pi} : \mathbb{Q}^l \to \mathbb{Q}^m$ by $\widehat{\pi}(\mathbf{b}_{j_i}) = \mathbf{a}_{j_i}$ for all $i = 1, \dots, l$. Set $\pi = \widehat{\pi}|_{pos_{\mathbb{Q}}(B)}$. Obviously, $\pi(\mathbf{b}_{j_i}) = \mathbf{a}_{j_i}$ for every $i = 1, \dots, l$. Let $\mathbf{b}_{j_i} = \sum_{k=1}^{l} \lambda_k \mathbf{b}_{j_k} \in pos_{\mathbb{Q}}(B)$ for some $\lambda_k \in \mathbb{Q}$ and $i \in \{l+1, \dots, n\}$. Clear the denominators to obtain an equality $v\mathbf{b}_{j_i} = \sum_{k=1}^{l} \xi_k \mathbf{b}_{j_k}$, where v and $\xi_1, \dots, \xi_l$ are integers. Suppose that v is positive, the case $v < 0$ is essentially the same. The ideal $I = (x_{j_1}^{\xi_{1,-}} \cdots x_{j_l}^{\xi_{l,-}} x_{j_i}^{v} - x_{j_1}^{\xi_{1,+}} \cdots x_{j_l}^{\xi_{l,+}})$ is B-homogeneous and therefore A-homogeneous. Thus $v\mathbf{a}_{j_i} = \sum_{k=1}^{l} \xi_k \mathbf{a}_{j_k}$, which leads to $\mathbf{a}_{j_i} = \sum_{k=1}^{l} \lambda_k \mathbf{a}_{j_k}$. Hence $\pi(\mathbf{b}_{j_i}) = \mathbf{a}_{j_i}$, for every $i \in \{1, \dots, n\}$.

(d) ⇒ (e) The matrix D is the matrix of $\widehat{\pi}$ in the canonical bases of $\mathbb{Q}^l$ and $\mathbb{Q}^m$.

(e) ⇒ (b) Let $\mathbf{u} = (u_1, \dots, u_n)$ be an element of $ker_{\mathbb{Z}}(N)$. Then $D(N\mathbf{u}^{\mathbf{T}}) = \mathbf{0}^{\mathbf{T}}$, and therefore $M\mathbf{u}^{\mathbf{T}} = \mathbf{0}^{\mathbf{T}}$. So $ker_{\mathbb{Z}}(N) \subseteq ker_{\mathbb{Z}}(M)$.

(b) ⇒ (a) Let $f = \mathbf{x}^{\mathbf{u}_+} - \mathbf{x}^{\mathbf{u}_-}$ be a binomial generator of I_N, where $\widehat{f}$ belongs to $ker_{\mathbb{Z}}(N)$. As $ker_{\mathbb{Z}}(N) \subseteq ker_{\mathbb{Z}}(M)$, we take that $\widehat{f}$ is in $ker_{\mathbb{Z}}(M)$. Hence f belongs to I_M. □

Corollary 2.3. *If I_M is a projection of I_N, then $ht(I_N) \leqslant ht(I_M)$.*

Proof. We have $ker_{\mathbb{Z}}(N) \subseteq ker_{\mathbb{Z}}(M)$ and therefore $rank(ker_{\mathbb{Z}}(N)) \leqslant rank(ker_{\mathbb{Z}}(M))$, by Theorem 2.2. Thus, $ht(I_N) = rank(ker_{\mathbb{Z}}(N)) \leqslant rank(ker_{\mathbb{Z}}(M)) = ht(I_M)$. □

3. Set-theoretic generation

Let I_M, I_N be toric ideals such that I_M is a projection of I_N, where M is an $m \times n$ integer matrix and $N = (b_{i,j})$ is an $l \times n$ matrix with non-negative integer entries. Set $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$, where $\mathbf{b}_i = (b_{1,i}, \dots, b_{l,i})$ for $i = 1, \dots, n$. Note that if $pos_{\mathbb{Q}}(B)$ is strongly convex, i.e. $\mathbf{0}$ is the only invertible element of $\mathbb{N}(B)$, then we can choose an appropriate matrix N with non-negative integer entries. The toric ideal I_N is the kernel of the K-algebra homomorphism

$$\phi : K[x_1, \dots, x_n] \to K[t_1, \dots, t_l]$$

given by

$$\phi(x_i) = \mathbf{t}^{\mathbf{b}_i} \quad \text{for all } i = 1, \dots, n.$$

We shall denote by $(\phi(I_M))^e$ the ideal $\phi(I_M)K[t_1, \dots, t_l]$.

Theorem 3.1. *Assume that $\Gamma(N) = V(I_N)$ in $\bar{K}^n$. Let $\{f_1, \dots, f_s\}$ be a set of polynomials in I_M and let $J = (f_1, \dots, f_s)$. Then*

$$I_M = rad(I_N + J)$$

if and only if

$$rad((\phi(I_M))^e) = rad(\phi(f_1), \dots, \phi(f_s)).$$

Proof. Let us first assume that

$$rad((\phi(I_M))^e) = rad(\phi(f_1), \dots, \phi(f_s)).$$

We claim that

$$I_M = rad(I_N + J).$$

By Hilbert's Nullstellensatz it is enough to prove that any point $\mathbf{x}=(x_1, \dots, x_n)$ of $V(I_N, J)$ in $\bar{K}^n$ belongs to $V(I_M)$. This $\mathbf{x}$ belongs also to $V(I_N)=\Gamma(N)$. The last statement means that there exist $T_i \in \bar{K}$ for $1 \leqslant i \leqslant l$ such that $x_i = \mathbf{T}^{\mathbf{b}_i}$. Note that $f(\mathbf{x}) = \phi(f)(T_1, \dots, T_l)$ for every $f \in K[x_1, \dots, x_n]$, since $\mathbf{x}$ is of the above form. In addition, the point $(T_1, \dots, T_l)$ belongs to $V(\phi(f_1), \dots, \phi(f_s)) = V((\phi(I_M))^e)$, because $f_i(\mathbf{x}) = 0$ for every $i \in \{1, \dots, s\}$. Let f be a polynomial in I_M. Then $\phi(f)$ belongs to $(\phi(I_M))^e$ and therefore $\phi(f)(T_1, \dots, T_l) = 0$. Thus $f(\mathbf{x}) = 0$, which implies the required relation. Conversely assume that $I_M = rad(I_N + J)$. We will prove that

$$rad((\phi(I_M))^e) = rad((\phi(I_N))^e + (\phi(f_1), \dots, \phi(f_s))).$$

It is enough to prove the assertion for a generator $f = \phi(g)$ of $(\phi(I_M))^e$, where $g \in I_M$. We have $g^k = h_1 + h_2$ for some $k \in \mathbb{N}$, where $h_1 \in I_N$ and $h_2 \in J$. Therefore $f^k = \phi(g^k) = \phi(h_1) + \phi(h_2)$, which means that f belongs to $rad((\phi(I_N))^e + (\phi(f_1), \dots, \phi(f_s)))$. But $I_N = ker(\phi)$, so $rad((\phi(I_M))^e) = rad(\phi(f_1), \dots, \phi(f_s))$. □

Remark 3.2. Every toric variety can always be expressed as an appropriate toric set over an algebraically closed field, for details see [10]. The proof of this fact is constructive and also an algorithm is given there to find this toric set. Therefore, in any case, the condition $\Gamma(N) = V(I_N)$ in $\bar{K}^n$ can be achieved, by choosing an appropriate matrix N for any toric ideal.

Remark 3.3. We cannot omit from the assumptions of Theorem 3.1 the fact that $V(I_N)$ coincides with the toric set $\Gamma(N)$. Let I_N be the toric ideal associated with the matrix

$$N = \begin{pmatrix} 2 & 1 & 0 & 0 & 1 & 2 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{pmatrix},$$

and let I_M the toric ideal associated with the matrix

$$M = \begin{pmatrix} 3 & 3 & 2 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 & 3 & 3 \end{pmatrix}.$$

Notice that I_M is a projection of I_N and that the toric variety $V(I_N)$ does not coincide with $\Gamma(N)$(see [10]). The toric ideal $I_M \subset K[x_1, \dots, x_6]$ is minimally generated by the following 12 binomials : $x_1^2 - x_2x_3x_4$, $x_3^3 - x_1x_2$, $x_4^3 - x_5x_6$, $x_6^2 - x_3x_4x_5$, $x_3^2x_5 - x_2x_4^2$, $x_3x_5^2 - x_4^2x_6$, $x_1x_3^2 - x_2^2x_4$, $x_3^2x_4 - x_2x_6$, $x_3^2x_5 - x_1x_6$, $x_3x_4^2 - x_1x_5$, $x_2x_5 - x_3x_6$, $x_1x_4 - x_3x_6$. Let $\phi : K[x_1, \dots, x_6] \to K[t_1, t_2, t_3]$ be the K-algebra homomorphism with $I_N = ker(\phi)$. The ideal $(\phi(I_M))^e$ is minimally generated by the binomials : $t_1^4t_2^2 - t_1t_2^5t_3^3$, $t_2^6t_3^3 - t_1^3t_2^3$, $t_2^3t_3^6 - t_1^3t_3^3$, $t_1^4t_3^2 - t_1t_2^3t_3^5$, $t_2^5t_3^4 - t_1^3t_2^2t_3$, $t_1t_2^4t_3^4 - t_1^4t_2t_3$, $t_2^4t_3^5 - t_1^3t_2t_3^2$. Therefore

$rad((\phi(I_M))^e) = rad(t_2^6t_3^3 - t_1^3t_2^3, t_2^3t_3^6 - t_1^3t_3^3)$, since a power of the other generators of $(\phi(I_M))^e$ belongs to the ideal $(t_2^6t_3^3 - t_1^3t_2^3, t_2^3t_3^6 - t_1^3t_3^3)$. Observe that $t_2^6t_3^3 - t_1^3t_2^3 = \phi(x_3^3 - x_1x_2)$ and $t_2^3t_3^6 - t_1^3t_3^3 = \phi(x_4^3 - x_5x_6)$. But $I_M \neq rad(I_N + (x_3^3 - x_1x_2, x_4^3 - x_5x_6))$, since I_N does not have any monic binomial in the variable x_6 and therefore no power of the binomial $x_6^2 - x_3x_4x_5 \in I_M$ belongs to the ideal $I_N + (x_3^3 - x_1x_2, x_4^3 - x_5x_6)$. Let

$$D = \begin{pmatrix} 2 & 1 & 0 & 0 & 1 & 2 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 2 & 2 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 1 & 0 & 0 & 1 & 2 & 2 \end{pmatrix}.$$

Then $\Gamma(D) = V(I_N)$, a proof of this fact can be found in [10]. Let $\psi : K[x_1, \ldots, x_6] \to K[t_1, \ldots, t_6]$ be the K-algebra homomorphism with $ker(\psi) = I_D = I_N$. We have $rad((\psi(I_M))^e) \neq rad(\psi(x_3^3 - x_1x_2), \psi(x_4^3 - x_5x_6))$, since no power of $\psi(x_6^2 - x_3x_4x_5)$ belongs to the ideal $(\psi(x_3^3 - x_1x_2), \psi(x_4^3 - x_5x_6))$.

Definition 3.4. The toric ideal I_M is called set-theoretic complete intersection on I_N if there are polynomials $f_1, \ldots, f_s$ in I_M, where s is equal to the difference of the heights of I_M and I_N, satisfying $I_M = rad(I_N + (f_1, \ldots, f_s))$.

The next corollary is directly derived from Theorem 3.1.

Corollary 3.5. *Keep the assumptions of Theorem* 3.1. *The toric ideal I_M is set-theoretic complete intersection on I_N if and only if the radical of the ideal $(\phi(I_M))^e$ is equal to the radical of an ideal generated by $ht(I_M) - ht(I_N)$ elements of the form $\phi(f)$, where $f \in I_M$. Moreover, if I_N is set-theoretic complete intersection and I_M is set-theoretic complete intersection on I_N then I_M is set-theoretic complete intersection.*

4. Binomial generation

In this section we will give an equivalent condition for the equality $I_M = rad(I_N + (f_1, \ldots, f_s))$ when $f_1, \ldots, f_s$ are binomials in I_M.

We consider the toric ideals I_M and I_N, where M is an $m \times n$ integer matrix and $N = (b_{i,j})$ is an $l \times n$ integer matrix with non-negative entries. Assume that I_M is a projection of I_N and let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, where $\mathbf{b}_i = (b_{1,i}, \ldots, b_{l,i})$ for $i = 1, \ldots, n$. The toric ideal I_N is the kernel of the K-algebra homomorphism

$$\phi : K[x_1, \ldots, x_n] \to K[t_1, \ldots, t_l]$$

defined by

$$\phi(x_i) = \mathbf{t}^{\mathbf{b}_i} \quad \text{for all } i = 1, \ldots, n.$$

Given a lattice $L \subset \mathbb{Z}^l$ the ideal

$$I_L := (\{\mathbf{t}^{\mathbf{z}_+} - \mathbf{t}^{\mathbf{z}_-} | \mathbf{z} = \mathbf{z}_+ - \mathbf{z}_- \in L\}) \subset K[t_1, \ldots, t_l]$$

is called *lattice ideal*. The height of I_L equals the rank of the lattice L (see [4]). For a prime number p we denote by $L : p^\infty$ the lattice

$$\{\mathbf{u} \in \mathbb{Z}^l | p^k \mathbf{u} \in L \ \text{ for some } \ k \in \mathbb{N}\}.$$

Let $\mathbb{Z}B = \{\mathbf{y}_\mathbf{u} := u_1 \mathbf{b}_1 + \cdots + u_n \mathbf{b}_n | \mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{Z}^n\}$ be the lattice spanned by B and let

$$N(ker_\mathbb{Z}(M)) = \{\mathbf{y}_\mathbf{u} = u_1 \mathbf{b}_1 + \cdots + u_n \mathbf{b}_n | \mathbf{u} = (u_1, \ldots, u_n) \in ker_\mathbb{Z}(M)\} \subset \mathbb{Z}B.$$

Note that if $ker_\mathbb{Z}(M) = \langle \mathbf{u}_1, \ldots, \mathbf{u}_r \rangle$, then $N(ker_\mathbb{Z}(M)) = \langle \mathbf{y}_{\mathbf{u}_1}, \ldots, \mathbf{y}_{\mathbf{u}_r} \rangle$.

Lemma 4.1. *The lattice ideal $I_{N(ker_\mathbb{Z}(M))}$ coincides with the ideal $(\phi(I_M))^e : (t_1 \cdots t_l)^\infty$.*

Proof. Let $f = x_1^{u_1} \cdots x_n^{u_n} - x_1^{v_1} \cdots x_n^{v_n}$ be a binomial in I_M, where $\widehat{f} = (u_1 - v_1, \ldots, u_n - v_n) \in ker_\mathbb{Z}(M)$. Then $\phi(f) = \mathbf{t}^{u_1 \mathbf{b}_1} \cdots \mathbf{t}^{u_n \mathbf{b}_n} - \mathbf{t}^{v_1 \mathbf{b}_1} \cdots \mathbf{t}^{v_n \mathbf{b}_n}$. Set $\mathbf{z} = u_1 \mathbf{b}_1 + \cdots + u_n \mathbf{b}_n - (\mathbf{y}_{\widehat{f}})_+ = v_1 \mathbf{b}_1 + \cdots + v_n \mathbf{b}_n - (\mathbf{y}_{\widehat{f}})_-$ and observe that $\mathbf{z} \in \mathbb{N}^l$. We have $\phi(f) = \mathbf{t}^\mathbf{z}(\mathbf{t}^{(\mathbf{y}_{\widehat{f}})_+} - \mathbf{t}^{(\mathbf{y}_{\widehat{f}})_-})$ and therefore $(\phi(I_M))^e \subseteq I_{N(ker_\mathbb{Z}(M))} \subseteq (\phi(I_M))^e : (t_1 \cdots t_l)^\infty$. Clearly, $(\phi(I_M))^e : (t_1 \cdots t_l)^\infty \subseteq I_{N(ker_\mathbb{Z}(M))} : (t_1 \cdots t_l)^\infty \subseteq ((\phi(I_M))^e : (t_1 \cdots t_l)^\infty) : (t_1 \cdots t_l)^\infty$. But $I_{N(ker_\mathbb{Z}(M))} = I_{N(ker_\mathbb{Z}(M))} : (t_1 \cdots t_l)^\infty$ (see [4]) and $(\phi(I_M))^e : (t_1 \cdots t_l)^\infty = ((\phi(I_M))^e : (t_1 \cdots t_l)^\infty) : (t_1 \cdots t_l)^\infty$. Thus $I_{N(ker_\mathbb{Z}(M))} = (\phi(I_M))^e : (t_1 \cdots t_l)^\infty$. □

Proposition 4.2. *Let G be a sublattice of $N(ker_\mathbb{Z}(M))$. If $char(K) = p \neq 0$ (resp. $char(K) = 0$), then the following two conditions are equivalent*:

(a) $rad(I_{N(ker_\mathbb{Z}(M))}) = rad(I_G)$,

(b) $p^k N(ker_\mathbb{Z}(M)) \subset G$ *for some* $k \in \mathbb{N}$ (*resp.* $N(ker_\mathbb{Z}(M)) = G$).

Proof. Suppose first that $rad(I_{N(ker_\mathbb{Z}(M))}) = rad(I_G)$. By [4, Corollary 2.2] it follows that in characteristic zero $I_{N(ker_\mathbb{Z}(M))} = I_G$ and so $N(ker_\mathbb{Z}(M)) = G$, since a binomial $f = \mathbf{t}^\mathbf{u} - \mathbf{t}^\mathbf{v}$ lies in a lattice ideal I_L if and only if $\widehat{f}$ belongs to L. Also in characteristic $p \neq 0$ it holds $I_{N(ker_\mathbb{Z}(M)):p^\infty} = I_{G:p^\infty}$ and so $N(ker_\mathbb{Z}(M)) : p^\infty = G : p^\infty$. Note that in [4] the lattice $L : p^\infty$ is denoted by $Sat_p(L)$. Suppose that $N(ker_\mathbb{Z}(M)) = \langle \mathbf{y}_{\mathbf{u}_1}, \ldots, \mathbf{y}_{\mathbf{u}_r} \rangle$. We have $N(ker_\mathbb{Z}(M)) \subset N(ker_\mathbb{Z}(M)) : p^\infty = G : p^\infty$. Hence for every $i = 1, \ldots, r$ there exist $k_i \in \mathbb{N}$ such that $p^{k_i} \mathbf{y}_{\mathbf{u}_i} \in G$. By choosing k the maximum of all k_i we take $p^k N(ker_\mathbb{Z}(M)) \subset G$. Conversely, it is clear that in characteristic zero $I_{N(ker_\mathbb{Z}(M))} = I_G$ and therefore $rad(I_{N(ker_\mathbb{Z}(M))}) = rad(I_G)$. We restrict now our attention in the case $char(K) = p \neq 0$. It is enough to show that $N(ker_\mathbb{Z}(M)) : p^\infty = G : p^\infty$.Obviously $G : p^\infty \subset N(ker_\mathbb{Z}(M)) : p^\infty$. Let $\mathbf{u} \in N(ker_\mathbb{Z}(M)) : p^\infty$. Then there exist $d \in \mathbb{N}$ such that $p^d \mathbf{u}$ is in $N(ker_\mathbb{Z}(M))$, so from the hypothesis $p^{d+k} \mathbf{u}$ belongs to G. Thus $\mathbf{u} \in G : p^\infty$ and therefore $rad(I_{N(ker_\mathbb{Z}(M))}) = rad(I_G)$. □

Remark 4.3. From the proof of the above Proposition we can see that in characteristic $p \neq 0$ condition (b) is equivalent with the condition $N(ker_\mathbb{Z}(M)) : p^\infty = G : p^\infty$.

Theorem 4.4. *Assume that $\Gamma(N) = V(I_N)$ in $\bar{K}^n$. Let $\{f_1, \ldots, f_s\}$ be a set of binomials in I_M. Set $J = (f_1, \ldots, f_s)$ and $G = \langle \mathbf{y}_{\widehat{f_1}}, \ldots, \mathbf{y}_{\widehat{f_s}} \rangle \subset N(ker_{\mathbb{Z}}(M))$. If $char(K) = p \neq 0$ (resp. $char(K) = 0$), then $I_M = rad(I_N + J)$ if and only if*:

(a) $p^k N(ker_{\mathbb{Z}}(M)) \subset G$ *for some* $k \in \mathbb{N}$ (*resp.* $N(ker_{\mathbb{Z}}(M)) = G$),
(b) $rad((\phi(I_M))^e, t_i) = rad(\phi(f_1), \ldots, \phi(f_s), t_i)$ *for all* $i = 1, \ldots, l$.

Proof. Suppose that $I_M = rad(I_N + J)$, then from Theorem 3.1 $rad((\phi(I_M))^e) = rad(\phi(f_1), \ldots, \phi(f_s))$. Clearly,

$$rad((\phi(I_M))^e, t_i) = rad(\phi(f_1), \ldots, \phi(f_s), t_i)$$

for all $i = 1, \ldots, l$. In addition $rad((\phi(I_M))^e) : (t_1 \cdots t_l)^\infty = rad(\phi(f_1), \ldots, \phi(f_s)) : (t_1 \cdots t_l)^\infty$ and therefore

$$rad((\phi(I_M))^e : (t_1 \cdots t_l)^\infty) = rad((\phi(f_1), \ldots, \phi(f_s)) : (t_1 \cdots t_l)^\infty).$$

So, from Lemma 4.1, we obtain the equality $rad(I_{N(ker_{\mathbb{Z}}(M))}) = rad(I_G)$. Now Proposition 4.2 assures that in characteristic zero $N(ker_{\mathbb{Z}}(M)) = G$, and in positive characteristic $p^k N(ker_{\mathbb{Z}}(M)) \subset G$ for some $k \in \mathbb{N}$. Conversely suppose that (a) and (b) hold. By Proposition 4.2 $rad(I_{N(ker_{\mathbb{Z}}(M))}) = rad(I_G)$, which implies that $rad((\phi(I_M))^e : (t_1 \cdots t_l)^\infty) = rad((\phi(f_1), \ldots, \phi(f_s)) : (t_1 \cdots t_l)^\infty)$. If I is any ideal of $K[t_1, \ldots, t_l]$, then by Lemma 3.2 in [4] the radical of I satisfies

$$rad(I) = rad(I : (t_1 \cdots t_l)^\infty) \cap rad(I, t_1) \cap \cdots \cap rad(I, t_l).$$

Applying this formula to $(\phi(I_M))^e$ we obtain that

$$rad((\phi(I_M))^e) = rad(\phi(f_1), \ldots, \phi(f_s)).$$

Hence $I_M = rad(I_N + J)$. □

Remark 4.5. In the special case that $I_N = (0)$ we take Theorem 2.5 in [7].

Example 4.6. In [5] Eliahou studied the binomial generation of the radical of the ideal of a monomial curve. Our theory will provide different binomial generators arising from different projections. For example, let $a \geqslant 7$ be an odd integer and let $M_a = (4, 6, a, a+2)$. The toric ideal I_{M_a} is a projection of the toric ideal I_{D_a} associated with the matrix

$$D_a = \begin{pmatrix} a-2 & a-4 & 2 & 0 \\ 0 & 2 & a-4 & a-2 \end{pmatrix}.$$

Note that $I_{D_a} = rad(x_2^{a-2} - x_1^{a-4}x_4^2, x_3^{a-2} - x_1^2 x_4^{a-4}, x_1x_4 - x_2x_3)$. Set $f_1 = x_1^{a+2} - x_4^4$, $f_2 = x_4^2 - x_1x_3^2$ and $G = \langle (a+2, -4) \rangle$. We have $V((\phi(I_{M_a}))^e, t_i) \cap \bar{K}^2 = V(\phi(f_1), \phi(f_2), t_i) \cap \bar{K}^2 = \{\mathbf{0}\}$,since $\phi(f_1) = t_1^{(a-2)(a+2)} - t_2^{4(a-2)}$ and therefore $t_1 = 0$ if and only if $t_2 = 0$. Thus $rad((\phi(I_{M_a}))^e, t_i) = rad(\phi(f_1), \phi(f_2), t_i)$ for $i = 1, 2$. Let $\mathbf{u} = (u_1, \ldots, u_4) \in ker_{\mathbb{Z}}(M_a)$. Then $\mathbf{y_u} = (u_1 + u_2 + u_3 + u_4)(a+2, -4)$ and therefore $D_a(ker_{\mathbb{Z}}(M_a)) = G$. Now Theorem 4.4 assures that $I_{M_a} = rad(x_2^{a-2} - x_1^{a-4}x_4^2, x_3^{a-2} - x_1^2x_4^{a-4}, x_1x_4 - x_2x_3, x_1^{a+2} - x_4^4, x_4^2 - x_1x_3^2)$.

But also I_{M_a} is a projection of the toric ideal $I_{N_a}=(x_1^{(a+1)/2}-x_3x_4, x_1^3-x_2^2)$ associated with the matrix

$$N_a=\begin{pmatrix} 2 & 3 & a+1 & 0 \\ 2 & 3 & 0 & a+1 \end{pmatrix}.$$

Let $\psi : K[x_1,\dots,x_4] \to K[t_1,t_2]$ be the K-algebra homomorphism with $I_{N_a}=ker(\psi)$. Set $g_1=x_3^{a+2}-x_4^a$, $g_2=x_1x_4-x_2x_3$ and $H=\langle(a+2,-a)\rangle$. Using the same arguments as before we take $rad((\psi(I_{M_a}))^e, t_i)=rad(\psi(g_1), \psi(g_2), t_i)$ for $i=1,2$. If $\mathbf{u}=(u_1,\dots,u_4)\in ker_{\mathbb{Z}}(M_a)$, then $\mathbf{y_u}=(2u_1+3u_2+(a+1)/2\,u_3+(a+1)/2\,u_4)(a+2,-a)$ and therefore $N_a(ker_{\mathbb{Z}}(M_a))=H$. Consequently $I_{M_a}=rad(x_1^{(a+1)/2}-x_3x_4, x_1^3-x_2^2, x_3^{a+2}-x_4^a, x_1x_4-x_2x_3)$.

5. Set-theoretic complete intersection

Let $I_N\subset K[x_1,\dots,x_n]$ be a toric ideal of height $r_1\geqslant 1$ associated with an $l\times n$ integer matrix N with non-negative entries. Let B be the set of columns of N. Suppose that $\Gamma(N)=V(I_N)$ in $\bar{K}^n$. We consider a lattice $L=ker_{\mathbb{Z}}(D)$ in $\mathbb{Z}^l$, where D is an $m\times l$ rational matrix such that the matrix $M=DN$ has integer entries. The last statement means that I_M is a projection of I_N. Let r_2 be the height of I_M and $\phi : K[x_1,\dots,x_n]\to K[t_1,\dots,t_l]$ the K-algebra homomorphism with $I_N=ker(\phi)$.

Lemma 5.1. *The dimension of the $\mathbb{Q}$-vector space $ker(D)\cap\mathbb{Q}B$ equals the difference r_2-r_1.*

Proof. Let $\{\mathbf{u}_1,\dots,\mathbf{u}_{r_1}\}$ be a basis of $ker(N)$ and $\{\mathbf{u}_1,\dots,\mathbf{u}_{r_1},\mathbf{u}_{r_1+1},\dots,\mathbf{u}_{r_2}\}$ a basis of $ker(M)$. For the sake of simplicity the symbol $\mathbf{y}_i$ will represent $\mathbf{y}_{\mathbf{u}_i}$. We will show that

$$ker(D)\cap\mathbb{Q}B=\mathbb{Q}\{\mathbf{y}_{r_1+1},\dots,\mathbf{y}_{r_2}\}.$$

Obviously $\mathbb{Q}\{\mathbf{y}_{r_1+1},\dots,\mathbf{y}_{r_2}\}\subseteq ker(D)\cap\mathbb{Q}B$. Let $\mathbf{v}\in ker(D)\cap\mathbb{Q}B$, then $\mathbf{v}=\mathbf{y_z}$ for some vector $\mathbf{z}\in\mathbb{Q}^n$. The vector $\mathbf{z}$ belongs to $ker(M)$, since $M=DN$. Thus $\mathbf{z}=\sum_{i=1}^{r_2}\kappa_i\mathbf{u}_i$ for some rationals $\kappa_1,\dots,\kappa_{r_2}$. Consequently $\mathbf{v}=\sum_{i=r_1+1}^{r_2}\kappa_i\mathbf{y}_i\in\mathbb{Q}\{\mathbf{y}_{r_1+1},\dots,\mathbf{y}_{r_2}\}$. It remains to show that the set $\{\mathbf{y}_{r_1+1},\dots,\mathbf{y}_{r_2}\}$ is linearly independent. Every relation of the form $\sum_{i=r_1+1}^{r_2}\kappa_i\mathbf{y}_i=\mathbf{0}$ implies that the vector $\sum_{i=r_1+1}^{r_2}\kappa_i\mathbf{u}_i$ belongs to $ker(N)$, so there exist some λ_i such that $\sum_{i=r_1+1}^{r_2}\kappa_i\mathbf{u}_i=\sum_{i=1}^{r_1}\lambda_i\mathbf{u}_i$. But the set $\{\mathbf{u}_1,\dots,\mathbf{u}_{r_2}\}$ is linearlyindependent, so all the k_i are equal to zero. □

Remark 5.2. The rank of the lattice $L\cap\mathbb{Z}B$ is equal to the dimension of $ker(D)\cap\mathbb{Q}B$. Also $L\cap\mathbb{Z}B$ coincides with the lattice $N(ker_{\mathbb{Z}}(M))$, so

$$ht(I_{N(ker_{\mathbb{Z}}(M))})=r_2-r_1.$$

Theorem 5.3. *Set $s=r_2-r_1$. If there are polynomials $f_1,\dots,f_s$ in I_M such that $rad(I_{N(ker_{\mathbb{Z}}(M))})=rad(\phi(f_1),\dots,\phi(f_s))$, then I_M is set-theoretic complete intersection on I_N.*

Proof. We have that $rad((\phi(I_M))^e) \subseteq rad(I_{N(ker_{\mathbb{Z}}(M))})$. From the assumption $rad(I_{N(ker_{\mathbb{Z}}(M))}) = rad(\phi(f_1), \ldots, \phi(f_s))$, so $rad((\phi(I_M))^e) = rad(\phi(f_1), \ldots, \phi(f_s))$ and therefore $I_M = rad(I_N + (f_1, \ldots, f_s))$. □

Combining Corollary 3.5 and Theorem 5.3 we get the following Corollary:

Corollary 5.4. *Set $s = r_2 - r_1$. If I_N is set-theoretic complete intersection and $rad(I_{N(ker_{\mathbb{Z}}(M))}) = rad(\phi(f_1), \ldots, \phi(f_s))$ for some polynomials $f_1, \ldots, f_s$ in I_M, then I_M is set-theoretic complete intersection.*

Example 5.5. In this example we will use the previous results to prove that the toric ideal I_M of height 4 associated with the matrix

$$M = \begin{pmatrix} 7 & 0 & 0 & 5 & 4 & 5 & 2 \\ 0 & 7 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 & 4 & 3 \end{pmatrix},$$

is set-theoretic complete intersection. Let $I_N = (x_5^3 - x_1x_4, x_7^5 - x_3x_6^2) \subset K[x_1, \ldots, x_7]$ be the toric ideal of height 2 associated with the matrix

$$N = \begin{pmatrix} 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 2 \end{pmatrix}.$$

Note that I_M is a projection of I_N. Suppose that $\phi : K[x_1, \ldots, x_7] \to K[t_1, \ldots, t_5]$ is the K-algebra homomorphism with $I_N = ker(\phi)$. The set of vectors

$$\{(1, 0, 0, 1, -3, 0, 0), (0, 0, 1, 0, 0, 2, -5)\}$$

constitutes a base for $ker_{\mathbb{Z}}(N)$ and the set

$$\{(1, 0, 0, 1, -3, 0, 0), (0, 0, 1, 0, 0, 2, -5), (1, 0, 0, 0, 0, -3, 4), (0, -1, 0, 4, -5, 0, 0)\}$$

constitutes a base for $ker_{\mathbb{Z}}(M)$. Thus

$$N(ker_{\mathbb{Z}}(M)) = \langle (3, 0, 4, 0, -7), (-5, -3, 0, 7, 0) \rangle$$

and therefore $I_{N(ker_{\mathbb{Z}}(M))} = (t_5^7 - t_1^3t_3^4, t_4^7 - t_1^5t_2^3)$. Let $f_1 = x_4^7 - 3x_1x_2x_4^4x_5^2 + 3x_1^3x_2^2x_4^2x_5 - x_1^5x_2^3 \in I_M$ and $f_2 = x_6^7 - 5x_1x_6^4x_7^4 + 10x_1^2x_3x_6^3x_7^3 - 10x_1^3x_3^2x_6^2x_7^2 + 5x_1^4x_3^3x_6x_7 - x_1^5x_3^4 \in I_M$. We have $(t_4^7 - t_1^5t_2^3)^3 = \phi(f_1)$ and $(t_5^7 - t_1^3t_3^4)^5 = \phi(f_2)$. So $rad(I_{N(ker_{\mathbb{Z}}(M))}) = rad(\phi(f_1), \phi(f_2))$, which implies that I_M is the set-theoretic complete intersection of $x_5^3 - x_1x_4$, $x_7^5 - x_3x_6^2$, $x_4^7 - 3x_1x_2x_4^4x_5^2 + 3x_1^3x_2^2x_4^2x_5 - x_1^5x_2^3$, $x_6^7 - 5x_1x_6^4x_7^4 + 10x_1^2x_3x_6^3x_7^3 - 10x_1^3x_3^2x_6^2x_7^2 + 5x_1^4x_3^3x_6x_7 - x_1^5x_3^4$.

We will compute the ideal $I_{N(ker_{\mathbb{Z}}(M))}$ in the special case that $s=1$. Let

$$N=\begin{pmatrix} b_1 & 0 & \dots & 0 & d_{1,1} & \dots & d_{r_1,1} \\ 0 & b_2 & \dots & 0 & d_{1,2} & \dots & d_{r_1,2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & b_l & d_{1,l} & \dots & d_{r_1,l} \end{pmatrix},$$

where $b_1,\dots,b_l$ are positive integers and $d_{i,j}$ are non-negative integers such that, for all $i=1,\dots,r_1$, at least one of $d_{i,1},\dots,d_{i,l}$ is non-zero. From Corollary 2 in [10] we have $\Gamma(N)=V(I_N)$ in $\bar{K}^n$. The symbol $|N|$ will represent the greatest common divisor of the subdeterminants of N of order l. We assume that $L=\langle \mathbf{a}\rangle$ and set $w=|N|/\left|(N\mathbf{a}^{\mathbf{T}})\right|$, where $(N\mathbf{a}^{\mathbf{T}})$ is the augmented matrix. Given a vector $\mathbf{u}$ in $\mathbb{Z}^l$, the binomial $\mathbf{t}^{\mathbf{u}_+}-\mathbf{t}^{\mathbf{u}_-}$ will be denoted by $F(\mathbf{u})$.

Theorem 5.6. *The lattice ideal $I_{N(ker_{\mathbb{Z}}(M))}$ is equal to the ideal generated by $F(w\mathbf{a})$. Moreover, if I_N is set-theoretic complete intersection and there exists $g\in I_M$ such that $rad(\phi(g))=rad(F(w\mathbf{a}))$, then I_M is set-theoretic complete intersection.*

Proof. We have $L\cap\mathbb{Z}B=\langle w\mathbf{a}\rangle$, since every system of the form $\mathbf{y_u}=c\mathbf{a}$ has a solution if and only if c is an integer multiple of w ([1, Theorem 1]).Thus $N(ker_{\mathbb{Z}}(M))=\langle w\mathbf{a}\rangle$ and therefore $I_{N(ker_{\mathbb{Z}}(M))}=(F(w\mathbf{a}))$. Also, from Corollary 5.4, I_M is set-theoretic complete intersection. $\square$

Remark 5.7. In Example 4.6 the choice of $f_2=x_4^2-x_1x_3^2$ (resp. $g_2=x_1x_4-x_2x_3$) was made by solving the system $\mathbf{y_u}=(a+2,-4)$ (resp. $\mathbf{y_u}=(a+2,-a)$).

6. Applications

In this section we will present some applications of the theory developed in Section 5.

We consider the toric ideal of height $d-1$ associated with the $(m+1)\times(m+d)$ integer matrix

$$N_d=\begin{pmatrix} d & d-1 & d-2 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 2 & \dots & d-1 & d & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & d & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & d \end{pmatrix},$$

where $d>1$. The toric ideal $I_{N_d}\subset K[x_1,\dots,x_{m+d}]$ is set-theoretic complete intersection, for details see [11,14]. Theorem 6.1 will generalize this result. Let c_1 be a positive integer, $c_2,\dots,c_{m+1}$ be non-negative integers with $c_1\geqslant dc_2$ and let $L=ker_{\mathbb{Z}}(D)$ for

$$D=\begin{pmatrix} \frac{c_1}{d} & \frac{c_1-dc_2}{d} & 0 & \dots & 0 \\ 0 & c_3 & \frac{c_1}{d} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & c_{m+1} & 0 & \dots & \frac{c_1}{d} \end{pmatrix}.$$

Set $\mathbf{g} = \gcd(c_1, c_1 - dc_2, dc_3, \dots, dc_{m+1})$ and let $c_1^\star = c_1/\mathbf{g}$, $(c_1 - dc_2)^\star = (c_1 - dc_2)/\mathbf{g}$ and $(dc_i)^\star = dc_i/\mathbf{g}$ for $i = 3, \dots, m+1$. Let $\mathbf{a} = ((c_1 - dc_2)^\star, -c_1^\star, (dc_3)^\star, \dots, (dc_{m+1})^\star)$. Observe that $L = \langle \mathbf{a} \rangle$.

Theorem 6.1. *The toric ideal of height d associated with the* $m \times (m+d)$ *matrix*

$$M_{c_1,\dots,c_{m+1},d} = \begin{pmatrix} c_1 & c_1 - c_2 & c_1 - 2c_2 & \dots & c_1 - dc_2 & 0 & \dots & 0 \\ 0 & c_3 & 2c_3 & \dots & dc_3 & c_1 & \dots & 0 \\ \dots & \dots & \dots\dots & \dots & \dots & \dots & \dots & \dots \\ 0 & c_{m+1} & 2c_{m+1} & \dots & dc_{m+1} & 0 & \dots & c_1 \end{pmatrix}$$

is set-theoretic complete intersection.

Proof. Without loss of generality we can assume that the greatest common divisor of the elements of $M_{c_1,\dots,c_{m+1},d}$ is equal to 1. Let $\phi : K[x_1, \dots, x_{m+d}] \to K[t_1, \dots, t_{m+1}]$ be the K-algebra homomorphism with $I_{N_d} = ker(\phi)$. Note that $I_{M_{c_1,\dots,c_{m+1},d}}$ is a projection of I_{N_d}. For the integer $\mathbf{g}$ we have $\mathbf{g}/d$, since $\mathbf{g}/dc_i$ for all $i = 2, \dots, m+1$ and $\gcd(\mathbf{g}, c_2, \dots, c_{m+1}) = 1$. In this case $|N_d| = d^m$, $\left|(N_d \mathbf{a}^{\mathbf{T}})\right| = d^m/\mathbf{g}$ and $w = \mathbf{g}$. Also $F(w\mathbf{a}) = t_2^{c_1} - t_1^{c_1 - dc_2} t_3^{dc_3} \cdots t_{m+1}^{dc_{m+1}}$. We have $(t_2^{c_1} - t_1^{c_1 - dc_2} t_3^{dc_3} \cdots t_{m+1}^{dc_{m+1}})^d = \phi(f)$ for

$$\begin{aligned} f = {} & x_{d+1}^{c_1} - \binom{d}{1} x_d^{c_1 - dc_2} x_{d+1}^{(d-1)c_2} x_{d+2}^{c_3} \cdots x_{d+m}^{c_{m+1}} \\ & + \binom{d}{2} x_{d-1}^{c_1 - dc_2} x_{d+1}^{(d-2)c_2} x_{d+2}^{2c_3} \cdots x_{d+m}^{2c_{m+1}} - \\ & \cdots + (-1)^{d-1} \binom{d}{d-1} x_2^{c_1 - dc_2} x_{d+1}^{c_2} x_{d+2}^{(d-1)c_3} \cdots x_{d+m}^{(d-1)c_{m+1}} \\ & + (-1)^d x_1^{c_1 - dc_2} x_{d+2}^{dc_3} \cdots x_{d+m}^{dc_{m+1}}. \end{aligned}$$

Notice that f belongs to $I_{M_{c_1,\dots,c_{m+1},d}}$. Consequently, from Theorem 5.6, the toric ideal $I_{M_{c_1,\dots,c_{m+1},d}}$ is set-theoretic complete intersection. □

Next we prove that the toric ideal associated with the row matrix $M_{a,b} = (a, a+2b, 2a+3b, 2a+5b)$ is set-theoretic complete intersection. Especially, when $a=4$, $b=1$ we deduce that the ideal of the monomial curve $(t^4, t^6, t^{11}, t^{13})$ is set-theoretic complete intersection. We consider the toric ideal associated to the matrix

$$N = \begin{pmatrix} 5 & 1 & 4 & 0 \\ 0 & 2 & 3 & 5 \end{pmatrix}.$$

The toric ideal $I_N \subset K[x_1, \dots, x_4]$ is the set-theoretic complete intersection of $x_3^2 - x_1 x_2^3$ and $x_2^5 - 2x_2 x_3 x_4 + x_1 x_4^2$. Let $L = ker_{\mathbb{Z}}(E)$ for $E = (a/5, (2a+5b)/5)$. Set $\mathbf{h} = \gcd(a, 2a+5b)$, $a^\star = a/\mathbf{h}$ and $(2a+5b)^\star = (2a+5b)/\mathbf{h}$. Note that $L = \langle \mathbf{a} \rangle$, where $\mathbf{a} = (-(2a+5b)^\star, a^\star)$. In addition $I_{M_{a,b}}$ is a projection of I_N.

Theorem 6.2. *For every positive integers* a, b *the ideal of the monomial curve* $(t^a, t^{a+2b}, t^{2a+3b}, t^{2a+5b})$ *is set-theoretic complete intersection.*

Proof. Let $\phi : K[x_1, x_2, x_3, x_4] \to K[t_1, t_2]$ be the K-algebra homomorphism with $I_N = ker(\phi)$. For the integer $\mathbf{h}$ we have $\mathbf{h}/a$ and $\mathbf{h}/5b$, so $\mathbf{h}/5$ since $gcd(\mathbf{h}, b) = 1$. Here $|N| = 5$, $\left|(N\mathbf{a}^{\mathbf{T}})\right| = 5/\mathbf{h}$ and $w = \mathbf{h}$. Also $F(w\mathbf{a}) = t_1^{2a+5b} - t_2^a$. When $a = 1$ the toric ideal associated with the matrix $M_{1,b}$ is obviously set-theoretic complete intersection. Suppose that $a > 1$, which implies that $a = 2\mu + 3v$ for some non-negative integers μ, v. We have $(t_1^{2a+5b} - t_2^a)^5 = \phi(f)$, where $f = x_1^{4\mu+6v+5b} - 5x_1^{3\mu+4v+4b}x_2^{\mu}x_3^{v} + 10x_1^{2\mu+2v+3b}x_2^{2\mu}x_3^{2v} - 10x_1^{\mu+2b}x_2^{3\mu}x_3^{3v} + 5x_1^{v+b}x_2^{v}x_3^{\mu}x_4^{\mu+2v} - x_4^{2\mu+3v}$ belongs to $I_{M_{a,b}}$. Therefore, from Theorem 5.6, the ideal of the curve $(t^a, t^{a+2b}, t^{2a+3b}, t^{2a+5b})$ is set-theoretic complete intersection. □

Finally, we prove that the toric ideal associated with the matrix $M_{a,b} = (4, 6, a, b)$ is set-theoretic complete intersection.

Theorem 6.3. *For every positive integers a, b the ideal of the monomial curve (t^4, t^6, t^a, t^b) is set-theoretic complete intersection.*

Proof. Suppose that $b = a + k$, where k is a positive integer. If a or b is even, then the semigroup $\mathbb{N}(4, 6, a, b)$ is symmetric by Proposition 2.1 in [8] and therefore $I_{M_{a,b}}$ is set-theoretic complete intersection (see [2]). It remains to examine the case a is odd and k is even. When $k \geqslant 4$, the semigroup $\mathbb{N}(4, 6, a, b)$ is symmetric and the result is straightforward. Therefore we have to deal only with the case $k = 2$. Since $a > 1$, there is a non-negative integer μ and a positive integer v such that $a = 2\mu + 3v$. We consider the toric ideal $I_{N_a} = ker(\psi)$ associated with the matrix N_a of Example 4.6. In this example it was proved that $N_a(ker_{\mathbb{Z}}(M_{a,a+2})) = \langle (a+2, -a) \rangle$ and so $I_{N_a(ker_{\mathbb{Z}}(M_{a,a+2}))} = (t_1^{a+2} - t_2^a)$. Set

$$f = \sum_{0 \leqslant i \leqslant \frac{a+1}{2}} (-1)^i \binom{a+1}{i} x_1^{i\mu} x_2^{iv} x_3^{a+2-2i} + \sum_{\frac{a+1}{2} < i \leqslant a+1} (-1)^i \binom{a+1}{i} \times x_1^{(a+1-i)(\mu+1)} x_2^{v(a+1-i)} x_4^{2i-a-2}$$

and observe that $f \in I_{M_{a,a+2}}$. We have $(t_1^{a+2} - t_2^a)^{a+1} = \psi(f)$ and therefore $I_{M_{a,a+2}}$ is the set-theoretic complete intersection of $f, x_1^{(a+1)/2} - x_3x_4, x_1^3 - x_2^2$. □

Remark 6.4. The last theorem provides the polynomials that minimally generate up to radical the ideal of the Eliahou's curve (t^4, t^6, t^7, t^9), see also [6] for a proof that the above ideal is set-theoretic complete intersection. Also it provides a different minimal polynomial generating set, than the one obtained in Theorem 6.2, up to radical for the ideal of the monomial curve $(t^4, t^6, t^{11}, t^{13})$.

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