Abstract

Fernando Rodriguez-Villegas has been studying hypergeometric families of Calabi–Yau manifolds, and from his investigations he has found (numerically) many possible supercongruences. For example, he conjectures for every odd prime $p$ that

$$\sum_{n=0}^{p-1} \binom{2n}{n}^2 16^{-n} \equiv \left( -\frac{4}{p} \right)^n \pmod{p^2}.$$

Here, we use the theory of Gaussian hypergeometric series, the properties of the $p$-adic $\Gamma$-function, and a strange combinatorial identity to prove this conjecture.

MSC: 11F85; 11L10

Keywords: Supercongruences

1. Introduction

In [RV] Rodriguez-Villegas observed (numerically) some supercongruences between the truncation of a hypergeometric function associated to a Calabi–Yau manifold at a prime $p$ and the number of its $\mathbb{F}_p$-points. Supercongruences of this type were first observed by Beukers [B] in connection with the Apéry numbers used in the
proof of the irrationality of \( \zeta(3) \). He conjectured that if \( p \) is an odd prime, then

\[
\sum_{j=0}^{p-1} \left( \frac{p-1}{2} + j \right)^2 \left( \frac{p-1}{2} \right)^2 j \equiv a(p) \pmod{p^2},
\]

where the integers \( a(n) \) are defined by

\[
\sum_{n=1}^{\infty} a(n)q^n := q \prod_{n=1}^{\infty} \frac{1 - q^{2n}}{(1 - q^4)^4(1 - q^{4n})^4}.
\]

This result was proved by Ahlgren and Ono in [A-O]. Beukers’ supercongruence was also proved in the case when \( p \mid a(p) \) by Ishikawa [I].

Theorem 1 establishes one of Rodriguez-Villegas’ conjectures (see [RV, (44)]). This is the \( d = 2 \) hypergeometric weight system \( \gamma = 2[2] - 4[1] \) (see [RV, (36)]).

**Theorem 1.** If \( p \) is an odd prime, then

\[
\sum_{n=0}^{p-1} \binom{2n}{n} 16^{-n} \equiv \left( \frac{-4}{p} \right) \quad \pmod{p^2}.
\]

Throughout we let \( \phi_p(x) \) denote the Legendre symbol \( \left( \frac{n}{p} \right) \). Since \( p^2 \mid \binom{2n}{n}^2 \) for all \( n \) with \( \frac{p-1}{2} < n < p \), it suffices to prove

\[
\sum_{n=0}^{p-1} \binom{2n}{n} 16^{-n} \equiv \left( \frac{-4}{p} \right) \quad \pmod{p^2}. \quad (1.1)
\]

We prove Theorem 1 by using Gaussian hypergeometric series, the \( p \)-adic \( \Gamma \)-function, and a strange combinatorial identity. We note that the method of proof follows the method used in [A,A-O].

2. Proof of Theorem 1

Unless otherwise stated, \( p \) will be a prime with \( p \geq 5 \). Theorem 1 is easy to check at \( p = 3 \). We denote by \( \mathbb{F}_p \) the finite field with \( p \) elements. We extend all multiplicative characters \( \chi \) of \( \mathbb{F}_p^\times \) to \( \mathbb{F}_p \) by setting \( \chi(0) := 0 \). If \( A \) and \( B \) are two characters of \( \mathbb{F}_p \), then we define \( \left( \frac{A}{B} \right) \) in terms of a Jacobi sum by

\[
\left( \frac{A}{B} \right) := \frac{B(-1)}{p} J(A, \bar{B}) = \frac{B(-1)}{p} \sum_{x \in \mathbb{F}_p} A(x) \bar{B}(1 - x). \quad (2.1)
\]
Let $A_0, A_1, \ldots, A_n$, and $B_1, B_2, \ldots, B_n$ be characters of $\mathbb{F}_p$. We follow Greene [G] and define the Gaussian hypergeometric series over $\mathbb{F}_p$ by

$$\sum_{\chi} \left( \begin{array}{c} A_0 \chi \\ B_1 \chi \\ \vdots \\ A_n \chi \\ B_n \chi \end{array} \right) \frac{x^j}{p} = \frac{p}{p-1} \sum \left( \begin{array}{c} A_0 \chi \\ A_1 \chi \\ \vdots \\ A_n \chi \\ B_1 \chi \\ \vdots \\ B_n \chi \end{array} \right) \chi(x),$$

(2.2)

where the sum runs over all characters $\chi$ of $\mathbb{F}_p$.

As an immediate consequence of the results in [G], Section 3, we have the following expression for $\phi_p(-1)$. We state it without proof.

**Proposition 2.1** (Greene [G]). If $\varepsilon_p$ is the trivial character modulo $p$ and $p$ is an odd prime, then

$$p \cdot 2 F_1 \left( \begin{array}{c} \phi_p, \\ \varepsilon_p \end{array} | 1 \right) = -\phi_p(-1).$$

Along with Proposition 2.1, we require the following curious combinatorial identity.

**Lemma 2.2.** For $m$ a positive integer, we define $H(m) := 1 + \frac{1}{2} + \cdots + \frac{1}{m}$. We further define $H(0) := 0$. For every positive integer $n$ we then have that

$$\sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \binom{n}{k} (1 + 2kH(n+k) - 2kH(k)) = (-1)^n(2n + 1).$$

**Proof.** If we define $F(n)$ by

$$F(n) := \sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \binom{n}{k} (1 + 2kH(n+k) - 2kH(k)),$$

the lemma follows easily from the following recursive relation:

$$(2 + n)^2 F(n) + (3 + 6n + 2n^2)F(n + 1) + (1 + n)^2 F(n + 2) = 0.$$ 

This relation was discovered using a software package developed by Carsten Schneider [S] which implements a summation algorithm of Karr [K].

Before we prove Theorem 1, we begin with some preliminaries on Gauss sums, Jacobi sums, and the $p$-adic gamma function. These facts can be found in, for
example, [A-O]. We let \( \pi \in \mathbb{C}_p \) be a fixed root of \( x^{p-1} + p = 0 \), and we let \( \zeta_p \) be the unique \( p \)th root of unity in \( \mathbb{C}_p \) such that \( \zeta_p \equiv 1 + \pi (\text{mod } \pi^2) \). For a multiplicative character \( \chi : \mathbb{F}_p^\times \to \mathbb{C}_p \), which has been extended to zero by \( \chi(0) := 0 \), we then define the Gauss sum to be

\[
g(\chi) := \sum_{x=0}^{p-1} \chi(x) \zeta_p^x.
\]

(2.3)

**Proposition 2.3.** If \( \chi, \chi_1, \) and \( \chi_2 \) are characters of \( \mathbb{F}_p \), then the following are true:

1. We have \( g(\chi)g(\bar{\chi}) = \chi(-1)p \).
2. If \( \chi_1 \) and \( \chi_2 \) are not both trivial and if \( \chi_1 \chi_2 = \epsilon \), then \( J(\chi_1, \chi_2) = -\chi_1(-1) \).
3. If \( \chi_1 \chi_2 \neq \epsilon \), then

\[
J(\chi_1, \chi_2) = \frac{g(\chi_1)g(\chi_2)}{g(\chi_1 \chi_2)}.
\]

The \( p \)-adic gamma function is defined on the ring \( \mathbb{Z}_p \) of \( p \)-adic integers by

\[
\Gamma_p(n) := (-1)^n \prod_{j<n, p \nmid j} j \quad \text{for } n \in \mathbb{N}
\]

and then by

\[
\Gamma_p(x) := \lim_{n \to x} \Gamma_p(n) \quad \text{for } x \in \mathbb{Z}_p.
\]

**Proposition 2.4.** Let \( p \geq 5 \) be a prime. If \( x, y, z \in \mathbb{Z}_p \) and \( |z| \leq |p| \), where \( |\cdot| \) is the \( p \)-adic norm, then the following are true:

1. We have \( \frac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} -x & \text{if } |x| = 1, \\ -1 & \text{if } |x| < 1. \end{cases} \)
2. If \( 0 \leq n \leq p - 1 \), then \( n! = (-1)^{n+1} \Gamma_p(n + 1) \).
3. If \( n \geq 1 \) and \( x \equiv y (\text{mod } p^n) \), then \( \Gamma_p(x) \equiv \Gamma_p(y) (\text{mod } p^n) \).
4. We have \( \Gamma'_p(x + z) \equiv \Gamma'_p(x) (\text{mod } p) \).
5. We have \( \Gamma_p(x + z) \equiv \Gamma_p(x) + z \Gamma'_p(x) (\text{mod } p^2) \).
6. Furthermore, if \( x \in \mathbb{Z}_p \), and \( R(x) \) denotes the representative of \( x \mod p \) in the set \( \{1, \ldots, p\} \), then we have \( \Gamma_p(x) \Gamma_p(1 - x) = (-1)^{R(x)} \).

We define \( G(x) := \frac{\Gamma'_p(x)}{\Gamma_p(x)} \). Then if \( x \in \mathbb{Z}_p \), we have \( G(x) \in \mathbb{Z}_p \). Using Proposition 2.4(1), we have

\[
G(x + 1) - G(x) = \frac{1}{x} \quad \text{if } x \in \mathbb{Z}_p, \quad |x| = 1.
\]

(2.4)
We are now able to state a relationship between Gauss sums and the $p$-adic gamma function. Let $\omega$ denote the Teichmüller character. This is a primitive character defined uniquely by the property that $\omega(x) \equiv x \pmod{p}$ for $x = 0, \ldots, p - 1$. The formula of Gross–Koblitz [Gr-Ko] states that

$$g(\omega j) = -\pi^j \Gamma_p \left( \frac{j}{p-1} \right), \quad 0 \leq j \leq p - 2. \quad (2.5)$$

We have two propositions which will also be needed in the proof of Theorem 1.

**Proposition 2.5.** If $p$ is an odd prime, then

$$1 - \phi_p(-1) \sum_{j=1}^{p-1} \frac{\Gamma_p(\frac{1}{2} + j)^2}{\Gamma_p(1 + j)^2} = \sum_{j=0}^{p-1} \binom{2j}{j} 16^{-j}. \quad (2.6)$$

**Proof.** By repeated use of Proposition 2.4(1) and the fact $\Gamma_p(\frac{1}{2})^2 = -\phi_p(-1)$, we have for $1 \leq j \leq \frac{p-1}{2}$ that

$$\Gamma_p \left( \frac{1}{2} + j \right)^2 = \left( \frac{2j - 1}{2} \right)^2 \left( \frac{2j - 3}{2} \right)^2 \cdots \left( \frac{1}{2} \right)^2 \Gamma_p \left( \frac{1}{2} \right)^2 = - \phi_p(-1) \frac{(2j)^2}{j!^2} 16^{-j}. \quad (2.5)$$

By Proposition 2.4(2),

$$\Gamma_p(1 + j)^2 = (j!)^2.$$

The proposition follows by accounting for $j = 0$. \hfill \Box

**Proposition 2.6.** If $p$ is an odd prime, then

$$1 - \phi_p(-1) \sum_{j=1}^{p-1} \frac{\Gamma_p(\frac{1}{2} + j)^2}{\Gamma_p(1 + j)^2} \left( 1 + 2j \left\{ H \left( \frac{p-1}{2} + j \right) - H(j) \right\} \right) \equiv 0 \pmod{p}. \quad (2.6)$$

**Proof.** We begin by proving that if $p$ is an odd prime and $1 \leq j \leq \frac{p-1}{2}$, then

$$-\phi_p(-1)(-1)^j \binom{\frac{p-1}{2} + j}{j} \binom{\frac{p-1}{2}}{j} \equiv \frac{\Gamma_p(\frac{1}{2} + j)^2}{\Gamma_p(1 + j)^2} \pmod{p}. \quad (2.6)$$
Using Proposition 2.4(2), we have
\[
- \phi_p(-1)(-1)^j \frac{(\frac{p}{2} - 1 + j)!}{j!(\frac{p}{2} - j)!} = - \phi_p(-1)(-1)^j \frac{1}{\Gamma_p(1+j)^j(\frac{p}{2} - j)^j} \Gamma_p(\frac{p}{2} + 1) + j)
\]
\[
= - \phi_p(-1)(-1)^j \frac{\Gamma_p(\frac{p}{2} + 1) + j)}{\Gamma_p(1+j)^j(\frac{p}{2} - j)^j} \Gamma_p(\frac{p}{2} - j)}
\]

Using Proposition 2.4(6), we have
\[
\Gamma_p\left(\frac{1}{2} - j + \frac{p}{2}\right) \Gamma_p\left(\frac{1}{2} + j - \frac{p}{2}\right) = (-1)^{\frac{p+1}{2} - j} = - \phi_p(-1)(-1)^j.
\]

Hence,
\[
- \phi_p(-1)(-1)^j \frac{(\frac{p}{2} - 1 + j)!}{j!(\frac{p}{2} - j)!} = \frac{\Gamma_p(\frac{1}{2} + j + \frac{p}{2}) \Gamma_p(\frac{1}{2} + j - \frac{p}{2})}{\Gamma_p(1+j)^j(\frac{p}{2} - j)^j}
\]
\[
= \frac{\Gamma_p(\frac{1}{2} + j)^2}{\Gamma_p(1+j)^j} (\text{mod } p),
\]

where the last congruence follows from Proposition 2.4(3). This proves (2.6).

Accounting for \( j = 0 \), we have
\[
1 - \phi_p(-1) \sum_{j=1}^{\frac{p-1}{2}} \frac{\Gamma_p(\frac{1}{2} + j)^2}{\Gamma_p(1+j)^j} \left(1 + 2j \left\{ H\left(\frac{p}{2} + 1 + j) - H(j) \right) \right\} \right)
\]
\[
= \sum_{j=0}^{\frac{p-1}{2}} (-1)^j \left(\frac{\frac{p}{2} - 1 + j}{j}\right) \left(\frac{\frac{p}{2} - 1}{j}\right) \left(1 + 2j \left\{ H\left(\frac{p}{2} + 1 + j\right) - H(j) \right) \right) (\text{mod } p).
\]

Substituting \( n = \frac{p-1}{2} \) into Lemma 2.2 yields the proposition. □

**Proof of Theorem 1.** By Proposition 2.1, we have
\[
\left(\frac{-4}{p}\right) = -p \cdot \sum \Phi_p \Phi_p | 1 \right) \right).
\]

Using (2.1), (2.2), and the fact that \( \left(\frac{p}{\chi}\right) \chi(-1) = \left(\frac{\phi_p}{\chi}\right) \), we have
\[
\left(\frac{-4}{p}\right) = -\frac{p^2}{p-1} \sum \left(\Phi_p \Phi_p | \chi \right) \left(\Phi_p \Phi_p | \phi_p \phi_p \chi \right) \chi(1) = \frac{1}{1-p} \sum \chi J(\phi_p, \chi)^2.
\]
By Proposition 2.3, we have
\[
\left( -\frac{4}{p} \right) = \frac{1}{1-p} \sum_{\chi} J(\phi_p, \chi)^2 = \frac{1}{1-p} \left\{ 1 + \phi_p(-1) \sum_{\chi \neq \phi_p} \frac{g(\chi)^2}{g(\phi_p \chi)^2} \right\}.
\]

If \( \omega \) is the Teichmüller character, then
\[
\left( -\frac{4}{p} \right) = \frac{1}{1-p} \left\{ 1 + \phi_p(-1) \frac{p-3}{2} \sum_{j=0} g(\sigma^j)^2 + \phi_p(-1) \frac{p-2}{2} \sum_{j=0} g(\sigma^j)^2 \right\}.
\]

By the Gross–Koblitz formula (2.5), we have
\[
\left( -\frac{4}{p} \right) = \frac{1}{1-p} \left\{ 1 - \phi_p(-1) \frac{p-3}{2} \sum_{j=0} \frac{\Gamma_p(j)}{\Gamma_p(\frac{1}{2} + \frac{j}{p-1})^2} - \phi_p(-1) \frac{p-2}{2} \sum_{j=0} \frac{\Gamma_p(j)}{\Gamma_p(\frac{1}{2} - \frac{j}{p-1})^2} \right\}.
\]

Hence
\[
\left( -\frac{4}{p} \right) \equiv \frac{1}{1-p} \left\{ 1 - \phi_p(-1) \frac{p-3}{2} \sum_{j=0} \frac{\Gamma_p(j)}{\Gamma_p(\frac{1}{2} + \frac{j}{p-1})^2} \right\} \pmod{p^2}.
\]

Using Proposition 2.4(3) and the fact that \( \frac{j}{p-1} \equiv -j - jp \pmod{p^2} \), we have
\[
\left( -\frac{4}{p} \right) \equiv \frac{1}{1-p} \left\{ 1 - \phi_p(-1) \frac{p-3}{2} \sum_{j=0} \frac{\Gamma_p(-j - jp)^2}{\Gamma_p(\frac{1}{2} - \frac{j}{p-1})^2} \right\} \pmod{p^2}.
\]

Replacing \( j \) with \( \frac{p-1}{2} - j \), we have
\[
\left( -\frac{4}{p} \right) \equiv (1 + p) \left\{ 1 - \phi_p(-1) \frac{p-1}{2} \sum_{j=1} \frac{\Gamma_p(j + j p)^2}{\Gamma_p(1 + j + j p)^2} \right\} \pmod{p^2}.
\]
Using Proposition 2.4(5) we write this as

\[
\left( \frac{-4}{p} \right) \equiv (1 + p) \left\{ 1 - \phi_p(-1) \sum_{j=1}^{p-1} \frac{(\Gamma_p(\frac{1}{2} + j) + jp\Gamma_p'(\frac{1}{2} + j))^2}{\Gamma_p(1 + j) + j\Gamma_p'(1 + j)^2} \right\} \pmod{p^2}
\]

\[
\equiv (1 + p) \left\{ 1 - \phi_p(-1) \sum_{j=1}^{p-1} \frac{\Gamma_p(\frac{1}{2} + j)^2(1 + jpG(\frac{1}{2} + j))^2}{\Gamma_p(1 + j)^2(1 + jpG(1 + j))^2} \right\} \pmod{p^2}
\]

\[
\equiv (1 + p) \left\{ 1 - \phi_p(-1) \sum_{j=1}^{p-1} \frac{\Gamma_p(\frac{1}{2} + j)^2(1 + 2jpG(\frac{1}{2} + j))}{\Gamma_p(1 + j)^2(1 + 2jpG(1 + j))} \right\} \pmod{p^2}.
\]

Multiplying the numerator and denominator of the summand by \(1 - 2jpG(1 + j)\), we have

\[
\left( \frac{-4}{p} \right) \equiv (1 + p) \left\{ 1 - \phi_p(-1) \sum_{j=1}^{p-1} \frac{\Gamma_p(\frac{1}{2} + j)^2}{\Gamma_p(1 + j)^2} \left(1 + 2jp \left\{ G\left(\frac{1}{2} + j\right) - G(1 + j) \right\} \right) \right\} \pmod{p^2}.
\]

By (3) and (4) of Proposition 2.4, we find that

\[
G\left(\frac{1}{2} + j\right) - G(1 + j) \equiv G\left(\frac{p + 1}{2} + j\right) - G(1 + j) \pmod{p}.
\]

For nonnegative integers \(n\) we define

\[
H(n) := 1 + \frac{1}{2} + \cdots + \frac{1}{n}, \quad \text{for } n \geq 1,
\]

\[
H(0) := 0.
\]

Hence using (2.4) repeatedly, we obtain

\[
G\left(\frac{1}{2} + j\right) - G(1 + j) \equiv H\left(\frac{p - 1}{2} + j\right) - H(j) \pmod{p}.
\]

Consequently, we can write

\[
\left( \frac{-4}{p} \right) \equiv 1 - \phi_p(-1) \sum_{j=1}^{p-1} \frac{\Gamma_p(\frac{1}{2} + j)^2}{\Gamma_p(1 + j)^2} \\
+ p \left\{ 1 - \phi_p(-1) \sum_{j=1}^{p-1} \frac{\Gamma_p(\frac{1}{2} + j)^2}{\Gamma_p(1 + j)^2} \left(1 + 2j \left\{ H\left(\frac{p - 1}{2} + j\right) - H(j) \right\} \right) \right\} \pmod{p^2}.
\]
Using Propositions 2.5 and 2.6 we have

\[
\left( -\frac{4}{p} \right) \equiv \sum_{j=0}^{p-1} \binom{2j}{j}^2 16^{-j} \pmod{p^2}.
\]

This is (1.1), and so Theorem 1 is proved. \qed

References


