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Structure on the set of closure operations of a commutative ring

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ABSTRACT

We investigate the algebraic structure on the set of closure operations of a ring. We show the set of closure operations is not a monoid under composition for a discrete valuation ring. Even the set of semiprime operations over a DVR is not a monoid; however, it is the union of two monoids, one being the left but not right act of the other. We also determine all semiprime operations over the ring $K[[t^2, t^3]]$.

1. Introduction

Let $I \mapsto I_c$ be an operation on the set of ideals of a ring R. Consider the following properties where I and J are ideals and b is a regular element:

(a) $I \subseteq I_c$.
(b) If $I \subseteq J$, then $I_c \subseteq J_c$.
(c) $(I_c)_c = I_c$.
(d) $I_c J_c \subseteq (IJ)_c$.
(e) $(bI)_c = bI_c$.

If $I \to I_c$ satisfies (a)–(c) above, we call $I \to I_c$ a *closure operation*. If $I \to I_c$ is a closure operation and also satisfies (d) above, we call $I \to I_c$ a *semiprime operation*. If $I \to I_c$ is semiprime and also satisfies (e), then we say $I \to I_c$ is a *prime operation*.

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The definition of prime operation or $\prime$-operation for the set of fractional ideals of an integral domain was given by Krull in his 1935 book, Idealtheorie [Kr1]. In his original definition, he actually added a sixth property (f) $I_c + J_c \subseteq (I+J)_c$. Then in his 1936 paper [Kr2], he discusses the integral completion or b-operation in terms of $\prime$-operations on the set of fractional ideals and mentions that he left out the properties (g) $R = R_c$ and (h) $(I_c \cap J_c)_c = I_c \cap J_c$. In fact, Sakuma [Sa] shows in 1957 that when looking at prime operations on the set of fractional ideals of a domain, properties (d), (f) and (h) are consequences of properties of (a), (b), (c), (e) and (g). In 1964, Petro [Pe] called the operations satisfying properties (a)–(d) on the set of fractional ideals semiprime operations. The first reference to integral closure strictly on the set of ideals of a commutative ring seems to be Northcott and Rees' 1954 paper on reductions [NR]. In 1969, Kirby [Ki] seems to be the first to discuss general closure operations on the set of ideals over a commutative ring with identity. The terms prime and semiprime operation were reintroduced on the set of ideals of a commutative ring by Ratliff in his 1989 paper [Ra] on Δ-closures of ideals. Heinzer, Ratliff and Rush [HRR] also use the term semiprime operation when referring to the basically full closure on the set of m-primary submodules of a module over a local ring (R, m).

There are many well-known closure operations defined on a commutative ring, such as: integral closure, tight closure if the ring contains a field [HH], Δ-closure [Ra], basically full closure [HRR], etc. It is known that all of these closure operations are contained in the integral closure, excluding the Δ-closure. However, if Δ does not contain any ideals which are contained in a minimal prime, then the Δ-closure is also contained in the integral closure. Otherwise, the relationship between these other closures is not as well understood. Knowing the structure on the set of closure operations may shed some light on this relationship.

Abstractly, closure operations are elements of the monoid of maps from the set of ideals, $\mathfrak{I}$, of a ring to itself, $M_{\mathfrak{I}} = \{f : \mathfrak{I} \to \mathfrak{I}\}$ satisfying the above properties. For example, C_R is the set of maps satisfying (a)–(c), S_R is the set of maps satisfying (a)–(d) and P_R is the set of maps satisfying (a)–(e). C_R, S_R and P_R are all partially ordered sets, but otherwise these sets are in general poorly behaved. In Section 2, we will give examples showing that C_R is not even a monoid in the nice case that R is a discrete valuation ring. Then in Section 3 we show that S_R for a discrete valuation ring R is almost a monoid. In fact, S_R is the union of two submonoids of $M_{\mathfrak{I}}$, one a left but not a right act of the other. Also we show that P_R is a monoid. We are also able to extend our results to semiprime and prime operations over a Dedekind domain. In Section 4, we consider closure operations over the semigroup ring $K[\![t^2, t^3]\!]$ and determine all the semiprime operations over $K[\![t^2, t^3]\!]$.

2. Preliminaries

Recall that $(S, \circ)$ is a *semigroup* if $\circ$ is an associative binary operation on S. We say that a semigroup $(S, \circ)$ is a *monoid* if there is a unique identity element e in S such that $es = se = s$ for all $s \in S$. In particular, the whole number $\mathbb{N}_0 = \{0, 1, 2, \ldots, n, \ldots\}$ is a monoid under addition, with identity 0. For a more thorough reference on semigroups see [Gi] or [Na].

Let R be a commutative ring, $\mathfrak{I} = \{I \subseteq R \mid I \text{ an ideal of } R\}$ and $M_{\mathfrak{I}} = \{f : \mathfrak{I} \to \mathfrak{I}\}$. $M_{\mathfrak{I}}$ is clearly a monoid under composition of maps, with identity the identity map $e : \mathfrak{I} \to \mathfrak{I}$, and function composition is associative. C_R will be the subset of $M_{\mathfrak{I}}$ consisting of closure operations. Hence the f_c in C_R are the set of maps satisfying the following three properties: (a) $f_c(I) \supseteq I$, (b) f_c preserves inclusions in R, and (c) $f_c \circ f_c = f_c$. S_R will be the set of semiprime operations of R, i.e. S_R are the maps in C_R which also satisfy $f_c(I)f_c(J) \subseteq f_c(IJ)$. P_R will be the set of prime operations of R, i.e. maps in S_R which also satisfy (e) $f_c(bI) = bf_c(I)$. We note that if C_R, S_R or P_R are monoids, by property (c), they will be band monoids.

Definition 2.1. A monoid is a band monoid if every element is idempotent.

We will say $f_{c_1} \leqslant f_{c_2}$ for two different closure operations if $f_{c_1}(I) \subseteq f_{c_2}(I)$ for all $I \in \mathfrak{I}$.

Proposition 2.2. *C_R, S_R and P_R are partially ordered sets.*

The proof is straightforward as the ideals of R are partially ordered under containment.

Now let us consider the algebraic structure of C_R, R a commutative ring. Unfortunately, C_R is not a submonoid under composition even for a discrete valuation ring.

Example 2.3. C_R, where (R, P) is a discrete valuation ring, is not a monoid. The ideals of R have the form P^i for all $i \geqslant 0$ and (0). Let $f_n : \mathfrak{I} \to \mathfrak{I}$ and $g_n : \mathfrak{I} \to \mathfrak{I}$ be defined as follows

$$f_n(P^i) = \begin{cases} P^i & \text{for } i \leqslant n, \\ P^n & \text{for } i > n, \end{cases} \quad \text{and} \quad g_n(P^i) = \begin{cases} R & \text{for } i \leqslant n, \\ P^n & \text{for } i > n \end{cases}$$

and $f_n(0) = (0) = g_n(0)$. If $m > n$, then

$$f_n \circ g_m(P^i) = \begin{cases} R & \text{for } i < m, \\ P^n & \text{for } i > m. \end{cases}$$

This fails property (c) as $(f_n \circ g_m) \circ (f_n \circ g_m)(P^i) = R$ for all i.

We will see in the next section that g_n in the above example is not a semiprime operation, because semiprime operations are not allowed to have any finite jumps.

In Example 2.3 we see that the maps f_n and g_n are bounded maps on the ideals of R. This prompts the following definition for closure operations of commutative rings:

Definition 2.4. We say a closure operation f_c is bounded on a commutative ring R if for every maximal ideal m of R, there is an m-primary ideal I such that for all m-primary $J \subseteq I$, $f_c(J) = I$. If this is not the case, we will say that f_c is an unbounded closure operation.

We define bounded in this way for m-primary ideals, because it would be hard to come up with a precise statement for all ideals.

3. Algebraic structure on S_R and P_R when R is a Dedekind domain

It seems unlikely that S_R and P_R are submonoids of $M_{\mathfrak{I}}$ for a general commutative ring R, but in the case that R is a discrete valuation ring, P_R is the trivial submonoid of $M_{\mathfrak{I}}$ and S_R decomposes into the union of two submonoids whose only common element is the identity. We use the following definition to explain their relationship.

Definition 3.1. Let S be a monoid and A any set, then we say A is a left (right) S-act if there is a map $\delta : S \times A \to A$ $(\delta : A \times S \to A)$ satisfying $\delta(st, a) = \delta(s, \delta(t, a))$ $(\delta(a, st) = \delta(\delta(a, s), t))$ for every $a \in A$ and $s, t \in S$ and $\delta(e, a) = a$ $(\delta(a, e) = a)$ for all $a \in A$ where e is the identity of S.

Proposition 3.2. *When (R, P) is a discrete valuation ring, S_R can be decomposed into the union of two submonoids*

$$M_0 = \{e\} \cup \left\{ f_m \in M_{\mathfrak{I}} \;\middle|\; f_m(P^i) = \begin{cases} P^i & \text{for } 0 \leqslant i < m, \\ P^m & \text{for } i \geqslant m \end{cases} \text{ and } f_m(0) = (0) \right\}$$

and

$$M_f = \{e\} \cup \left\{ g_m \in M_{\mathfrak{I}} \;\middle|\; g_m(P^i) = \begin{cases} P^i & \text{for } 0 \leqslant i < m, \\ P^m & \text{for } i \geqslant m \end{cases} \text{ and } g_m(0) = P^m \right\}$$

where M_f is a left M_0-act but not a right M_0-act under composition.

Before proving the proposition, we need the following lemma:

Lemma 3.3. *Let f_c be a semiprime operation on the discrete valuation ring (R, P). Then if f_c is constant for P^i on a finite interval $m \leqslant i \leqslant n$ for $m < n$, then there exists a $j \leqslant m$ such that $f_c(P^i) = P^j$ for all $i \geqslant j$.*

Proof. The ideals of R have the form P^i and they are totally ordered. Being a closure operation, $f_c(P^i) = P^j \supseteq P^i$, where $j \leqslant i$, since f_c must be increasing on the ideals of R.

Suppose f_c is constant for P^i, where $m \leqslant i \leqslant n$, $m < n$. For all such i suppose that $f_c(P^i) = P^j$. Then $f_c(P^j) = f_c(f_c(P^i)) = f_c(P^i) = P^j$. Thus $P^m \subseteq f_c(P^m) = P^j$ and $j \leqslant m$. Since f_c is increasing we see that $f_c(P^i) = P^j$ for all $j \leqslant i < m$.

We know $f_c(P^n) = P^j$ by assumption. If we show that $f_c(P^{n+1}) = P^j$, then by induction, $f_c(P^i) = P^j$ for all $i \geqslant j$. Then once again, the fact that f_c is increasing implies that $f_c(P^{n+1}) = P^k \subseteq f_c(P^j) = P^j$ for $j \leqslant k \leqslant n+1$. Since f_c is a closure operation, $f_c(f_c(P^{n+1})) = f_c(P^k) = P^k \subseteq f_c(P^j) = P^j$. So either $f_c(P^{n+1}) = P^j$ or $f_c(P^{n+1}) = P^{n+1}$. Suppose the latter. Since, f_c is a semiprime operation, then $f_c(P^i)f_c(P^k) \subseteq f_c(P^{i+k})$ for all i and k; however, $f_c(P)f_c(P^n) \supseteq P^{j+1}$ properly contains $f_c(P^{n+1}) = P^{n+1}$. Thus $f_c(P^{n+1}) = P^j$. $\square$

Proof of Proposition 3.2. The ideals of a discrete valuation ring (R, P) are either of the form P^i for $i \geqslant 0$ or (0) and they are totally ordered $R \supseteq P \supseteq P^2 \supseteq \cdots \supseteq P^m \supseteq \cdots \supseteq (0)$.

By Lemma 3.3, we know that any semiprime operation f_c on R which is constant on some finite interval has the property that $f_c(P^i) = P^m$ for all $i \geqslant m$ for some m. I claim that for $i \leqslant m$, $f_c(P^i) = P^i$. Suppose not, then $f_c(P^i) = P^k$ for some $k \leqslant i$ since f_c is increasing. Then for $k \leqslant j \leqslant i$, $P^k = f_c(P^k) \subseteq f_c(P^j) \subseteq f_c(P^i) = P^k$. If $k < i$ then by Lemma 3.3 $f_c(P^i) = P^k$ on interval $i \geqslant k$ contradicting the fact that for $i \geqslant m$, $f_c(P^i) = P^m$.

Note, in the case where

$$f_c(P^i) = \begin{cases} P^i & \text{for } i < m, \\ P^m & \text{for } i \geqslant m, \end{cases} \qquad f_c(0) \subseteq \bigcap_{i \geqslant 0} f_c(P^i) = P^m.$$

Thus $f_c(0) = (0)$ or $f_c(0) = P^m$ since $f_c(P^n) = P^m$ for $n \geqslant m$. Hence, $f_c = f_m$ or $f_c = g_m$ as defined in the statement of the proposition.

Now, suppose that f_c is a semiprime operation which is not constant on any such interval $m \leqslant i \leqslant n$ with $m < n$. Suppose $f_c(P^i) = P^k$ for $k < i$. Then $P^k = f_c(P^k) \subseteq f_c(P^j) \subseteq f_c(P^i) = P^k$ for all $k \leqslant j \leqslant i$ which contradicts that fact that f_c is not constant on any interval. Hence, $f_c(P^i) = P^i$ for all $i \geqslant 0$. Since $f_c(0) \subseteq f_c(P^i) = P^i$ for all $i \geqslant 0$, then $f_c(0) \subseteq \bigcap_{i \geqslant 0} P^i = (0)$. Hence, f_c must be the identity map.

Clearly $f_m \circ f_n = f_{\min(m,n)}$ and $g_m \circ g_n = g_{\min(m,n)}$ both imply that the corresponding sets of semiprime operations in S_R, M_0 and M_f are submonoids of $M_{\mathfrak{I}}$. That M_f is a left M_0-act can be seen by $f_n \circ g_m = g_{\min(m,n)}$. However, for $m > n$, $g_m \circ f_n(0) = P^m$ and $(g_m \circ f_n) \circ (g_m \circ f_n) = g_n$ which implies $g_m \circ f_n$ is not a closure operation. Thus, M_f is not a right M_0-act and $S_R = M_0 \cup M_f$ is not a submonoid. $\square$

For every $n \geqslant 0$, $M_n = \{e\} \cup \{f_n\} \cup \{g_n\}$ also form finite submonoids of $M_{\mathfrak{I}}$ contained in S_R, interrelating M_0 and M_f.

Proposition 3.4. *The only element of P_R when (R, P) is a discrete valuation ring is the identity.*

Proof. Let $(b) = P$. If f_c is prime, then $bf_c(P^i) = f_c(bP^i) = f_c(P^{i+1})$. Note if f_c was either f_m or g_m in the above proof, then $bf_c(P^m) = bP^m \subsetneq P^m = f_c(P^{m+1}) = f_c(bP^m)$. This contradicts the assumption of primeness. Thus $P_R = \{e\}$. $\square$

If R is a Dedekind domain which is not necessarily local then for every maximal ideal $\mathfrak{m}$ in R, $R_{\mathfrak{m}}$ is a discrete valuation ring. We know the structure $S_{R_{\mathfrak{m}}}$, and can build the structure of S_R from $S_{R_{\mathfrak{m}}}$.

Given a Dedekind domain R with maximal ideals P_i, $i \in \Lambda$. Consider the monoid given by $\coprod_{i\in\Lambda} \mathbb{N}_0$, the coproduct of $\mathbb{N}_0$ (i.e. the set of all functions $\phi : \Lambda \to \mathbb{N}_0$ such that $\phi(i) = 0$ for all but finitely many $i \in \Lambda$). Suppose $\phi(i_j) = m_j \neq 0$ for $i_1, i_2, \dots, i_s$ and $\phi(i) = 0$ all other i. This ϕ corresponds to the ideal $P_{i_1}^{m_1} P_{i_2}^{m_2} \cdots P_{i_s}^{m_s}$. The function $\phi \equiv 0$ in $\coprod_{i\in\Lambda} \mathbb{N}_0$ corresponds to the unit ideal R.

As the non-negative integers play a major role in identifying the semiprime operations in a discrete valuation, certain subsets of the semigroup $\mathbb{N}_0^\Lambda = \coprod_{i\in\Lambda} \mathbb{N}_0$ will determine the semiprime operations of a Dedekind domain with maximal ideals $P_i, i \in \Lambda$. All the nonzero ideals in a Dedekind domain are finite products of the P_i, i.e. $I = P_{i_1}^{m_1} \cdots P_{i_r}^{m_r}$.

To determine these subsets, first consider the semilocal principal ideal domain R with two maximal ideals P and Q, the ideals of R are $P^i Q^j$, $i, j \geqslant 0$, which corresponds to the lattice point (i, j) in $\mathbb{N}_0^2$. Suppose that for some semiprime operation f_c defined on (R, P, Q),

$$f_c(P^i) = \begin{cases} P^i & \text{for } i < m, \\ P^m & \text{for } i \geqslant m \end{cases} \quad \text{and} \quad f_c(Q^j) = \begin{cases} Q^j & \text{for } j < n, \\ Q^n & \text{for } j \geqslant n. \end{cases}$$

As f_c is semiprime, we know that

$$f_c(P^i) f_c(Q^j) \subseteq f_c(P^i Q^j) \subseteq f_c(P^i) \cap f_c(Q^j) = f_c(P^i) f_c(Q^j)$$

as $P^i Q^j \subseteq P^i$ and $P^i Q^j \subseteq Q^j$. Thus

$$f_c(P^i Q^j) = \begin{cases} P^m Q^n & \text{if } i \geqslant m \text{ and } j \geqslant n, \\ P^m Q^j & \text{if } i \geqslant m \text{ and } 0 \leqslant j < n, \\ P^i Q^n & \text{if } 0 \leqslant i < m \text{ and } j \geqslant n, \\ P^i Q^j & \text{if } 0 \leqslant i < m \text{ and } 0 \leqslant j < n. \end{cases}$$

We define the *identity rectangle* B of a semiprime operation f_c on the lattice (R, P, Q) to be the (i, j) such that $f_c(P^i Q^j) = P^i Q^j$.

In general, where Λ is not necessarily a two-element set, we denote the ideal corresponding to $\phi \in \mathbb{N}_0^\Lambda$ by $I(\phi)$. Similarly we can define an identity Λ-box for R with maximal ideals indexed by Λ.

Definition 3.5. The *identity Λ-box B_Λ* of the semiprime operation f_c over a Dedekind domain R is the set of all $\phi \in \coprod_{i\in\Lambda} \mathbb{N}_0$ such that $f_c(I(\phi)) = I(\phi)$.

For simplicity we will denote ϕ_i^j to be the element of $\coprod_{i\in\Lambda} \mathbb{N}_0$ such that $\phi(i) = j$ and $\phi(\lambda) = 0$ for all $\lambda \neq i$. All elements are of the form $\phi_{i_1}^{j_1} + \phi_{i_2}^{j_2} + \cdots + \phi_{i_r}^{j_r} := \phi_{i_1 i_2 \cdots i_r}^{j_1 j_2 \cdots j_r}$ for distinct i_k. Note that the identity Λ-box B_Λ of f_c could be bounded if for every $i \in \Lambda$ there is a finite m with $f_c(I(\phi_i^j)) = I(\phi_i^m)$ for $j \geqslant m$. For each $i_h \in \Lambda$, define

$$m_h = \begin{cases} m & \text{if } f_c(I(\phi_{i_h}^j)) = I(\phi_{i_h}^m) \text{ for } j \geqslant m, \\ \infty & \text{otherwise.} \end{cases}$$

In fact, all semiprime operations on the ideals of $\coprod_{i\in\Lambda} \mathbb{N}_0$ satisfy the equations

$$f_{B_\Lambda}\big(I(\phi_{i_1 i_2 \cdots i_r}^{j_1 j_2 \cdots j_r})\big) = \begin{cases} I(\phi_{i_1 i_2 \cdots i_r}^{j_1 j_2 \cdots j_r}) & \text{if } \phi_{i_1, i_2, \dots, i_r}^{j_1, j_2, \dots, j_r} \in B_\Lambda, \\ I(\phi_{i_1 i_2 \cdots i_r}^{k_1 k_2 \cdots k_r}) & \text{if } \phi_{i_1, i_2, \dots, i_r}^{j_1, j_2, \dots, j_r} \notin B_\Lambda \text{ and } k_l = m_l \neq \infty \\ \quad \text{for some } l \text{ and } k_h = j_h \text{ for all } h \text{ with } k_h \leqslant m_h. \end{cases}$$

If B_Λ and C_Λ are any two identity Λ-boxes, clearly, $B_\Lambda \cap C_\Lambda$ is also an identity Λ-box and the action of $f_{B_\Lambda} \circ f_{C_\Lambda}$ on nonzero ideals of R is the same as that of $f_{B_\Lambda \cap C_\Lambda}$.

Since the semiprime operations of a Dedekind domain correspond to elements of $\coprod_{i\in\Lambda}\mathbb{N}_0\cup\{\infty\}$ under partial ordering, when B_Λ is bounded with a finite number $i\in\Lambda$ with $m_i\neq 0$, there are two types of semiprime operations f_{B_Λ} and g_{B_Λ}. The only difference is that $f_{B_\Lambda}(0)=(0)$ and $g_{B_\Lambda}(0)=P_{i_1}^{m_1}P_{i_2}^{m_2}\cdots P_{i_r}^{m_r}$, where $\{i_1,\dots,i_r\}$ is exactly the set of all $i_j\in\Lambda$ with $m_j<\infty$.

Let us define two subsets of $M_\mathfrak{I}$:

- $M_f=\{e\}\cup\{g_{B_\Lambda}\in M_\mathfrak{I}\mid g_{B_\Lambda}(0)=P_{i_1}^{m_1}P_{i_2}^{m_2}\cdots P_{i_r}^{m_r}$ for some primes P_{i_j}, $j=1,\dots,r\}$: the set of closure operations for which the zero ideal is not closed (along with the identity).
- $M_0=\{e\}\cup\{f_{B_\Lambda}\in M_\mathfrak{I}\mid f_{B_\Lambda}(0)=(0)\}$: the set of closure operations for which the zero ideal is closed.

Suppose now that B_Λ and C_Λ are two identity Λ-boxes with both B_Λ and C_Λ bounded. Then $B_\Lambda\cap C_\Lambda$ is also bounded and is also an identity Λ-box and $f_{B_\Lambda}\circ f_{C_\Lambda}=f_{B_\Lambda\cap C_\Lambda}$ and $g_{B_\Lambda}\circ g_{C_\Lambda}=g_{B_\Lambda\cap C_\Lambda}$. This shows that M_0 and M_f are submonoids of $M_\mathfrak{I}$.

Lastly, suppose that B_Λ and C_Λ are two identity Λ-boxes with C_Λ bounded. Then $B_\Lambda\cap C_\Lambda\subsetneq C_\Lambda$ is also bounded as above and is also an identity Λ-box. Note that, $f_{B_\Lambda}\circ g_{C_\Lambda}=g_{B_\Lambda\cap C_\Lambda}$ but $g_{C_\Lambda}\circ f_{B_\Lambda}\neq g_{B_\Lambda\cap C_\Lambda}$ since $g_{C_\Lambda}\circ f_{B_\Lambda}(0)=\bigcap_{i\in\Lambda}P_i^m$ where $\phi_i^m\in C_\Lambda\neq B_\Lambda\cap C_\Lambda$ which is not a closure operation. This shows that M_f is a left M_0-act, but not a right M_0-act.

We have just proved:

Proposition 3.6. *When R is a Dedekind domain, S_R can be decomposed into the union of two submonoids $M_0=\{e\}\cup\{f_{B_\Lambda}\in M_\mathfrak{I}\}$ and $M_f=\{e\}\cup\{g_{B_\Lambda}\in M_\mathfrak{I}\}$ where M_f is a left M_0-act but not a right M_0-act under composition.*

Proposition 3.7. *The only element of P_R when R is a Dedekind domain is the identity.*

Proof. Suppose $(b_i)=P_i$. If f_c is prime, then $b_if_c(I)=f_c(b_iI)$ for all I. In particular, $b_if_c(P^j)=f_c(b_i(P_i)^j)$ for all $j\geqslant 0$. Note if f_c was either f_{B_Λ} or g_{B_Λ} and P_i is a prime such that $f_c(P_i^j)=P_i^{m_i}$ for $j\geqslant m_i$ then $b_if_c(P_i^{m_i})=P_i^{m_i+1}\subsetneq P_i^{m_i}=f_c(b_iP_i^{m_i})$. This contradicts the assumption of primeness. Thus $P_R=\{e\}$. □

4. S_R and P_R when $R=K[[t^2,t^3]]$

Although $K[[t^2,t^3]]$ is a local ring, the ideal structure in $K[[t^2,t^3]]$ is not totally ordered as in the case of a discrete valuation ring. All ideals in $K[[t^2,t^3]]$ are either generated by one element t^n+at^{n+1} where $a\in K$ or two elements of the form (t^n,t^{n+1}). I would like to thank Hwa Young Lee for pointing out that I was ignoring the ideals (t^i+at^{i+1}), with $a\neq 0$ in a previous version of this paper. She shared with me some of the ideas from her developing thesis including some theorems which she proved which can be summed up in the following proposition. The proof here is my own.

Proposition 4.1. *Each nonzero nonunit ideal of $R=K[[t^2,t^3]]$ can either be expressed as a principal ideal in the form (t^n+at^{n+1}), $a\in K$, $n\geqslant 2$, or as a two generated ideal (t^n,t^{n+1}) for $n\geqslant 2$.*

Proof. Suppose $0\neq f\in R$. Thus, after multiplying by a nonzero element of K, $f=t^n+a_1t^{n+1}+a_2t^{n+2}+\cdots$ for $n\geqslant 2$. We will show that $t^m\in(f)$ for $m\geqslant n+2$. Hence, $t^n+a_1t^{n+1}\in(f)$. Similarly, $t^m\in(t^n+a_1t^{n+1})$ for $m\geqslant n+2$. Hence, $f\in(t^n+a_1t^{n+1})$.

Let $g\in K[[t]]$. Note that $t^{m-n}g\in K[[t^2,t^3]]$. Hence, if g is a unit in $K[[t]]$, then $t^{m-n}g^{-1}\in K[[t^2,t^3]]$ also. In $K[[t]]$, $f=t^n(1+a_1t+a_2t^2+\cdots)=t^ng$. Note that $t^{m-n}g^{-1}f=t^m$. Similarly $t^m\in(t^n+a_1t^{n+1})$. Since $f-(t^n+a_1t^{n+1})=a_2t^{n+2}+a_3t^{n+3}+\cdots\in(f)\cap(t^n+a_1t^{n+1})$, we see that $(t^n+a_1t^{n+1})=(f)$. Hence, all principal ideals of $K[[t^2,t^3]]$ have the form (t^n+at^{n+1}).

Suppose, I is not principal. As $t^m\in(t^n+at^{n+1})$ for $m\geqslant n+2$, then I can be generated by at most 2 elements of the form $(t^n+at^{n+1},t^m+bt^{m+1})$ where $m=n$ or $m=n+1$. If $m=n$, then $t^{n+1}\in I$ which also implies that $t^n\in I$. Hence $I=(t^n,t^{n+1})$. If $m=n+1$, then $t^{n+2}\in(t^n+at^{n+1})\subseteq I$ as in the principal case above. However, $t^{n+1} = t^{n+1} + bt^{n+2} - bt^{n+2} \in I$ and once again $t^n \in I$. Hence, $I = (t^n, t^{n+1})$. $\square$

In fact the ideals are woven in the following way:

$$\begin{tikzcd}
 & & & (t^3+at^4) \arrow[dr, dash] & & (t^5+at^6) \arrow[dr, dotted, dash] & \\
R \arrow[r, dash] & m \arrow[r, dash] \arrow[dr, dash] & (t^3,t^4) \arrow[r, dash] \arrow[ur, dash] & m^2 \arrow[r, dash] \arrow[dr, dash] & (t^5,t^6) \arrow[r, dash] \arrow[ur, dash] & m^3 \arrow[r, dash] \arrow[r, dotted, dash, bend right] & \cdots (0) \\
 & & (t^2+at^3) \arrow[ur, dash] & & (t^4+at^5) \arrow[ur, dash] & &
\end{tikzcd}$$

where each line segment in the above diagram indicates $\supseteq$.

In the case of a discrete valuation ring (R, P), integral closure is the identity map on ideals of R. For $K[[t^2, t^3]]$, the integral closure of ideals of the form $(t^i + at^{i+1})$ is $\overline{(t^i + at^{i+1})} = (t^i, t^{i+1})$ whereas the ideals of the form (t^i, t^{i+1}) are all integrally closed. Looking at the above diagram, we see that the chain of ideals in the center are all integrally closed. However, the principal ideals are not. Clearly there are now many more closure operations for $K[[t^2, t^3]]$. In fact, the semiprime operations which are not bounded abound. To shorten the expressions appearing in the proofs we will denote the principal ideals $P_{i,a} := (t^i + at^{i+1})$ and $M_i := (t^i, t^{i+1})$.

Proposition 4.2. *In $K[[t^2, t^3]]$, for all $i \geqslant 2$ and all $a \in K$, the map*

$$f_i^{int}(I) := \begin{cases} M_i & \text{if } I = P_{i,a}, \\ I & \text{if } I \neq P_{i,a} \end{cases}$$

is a closure operation which is not semiprime.

Proof. Clearly $f_i^{int}(I) \supseteq I$ for all I and if $I \subseteq J$, $f_i^{int}(I) \subseteq f_i^{int}(J)$. As $f_i^{int}(I) = I$ whenever $I \neq P_{i,a}$, and

$$f_i^{int} \circ f_i^{int}(P_{i,a}) = f_i^{int}(P_{i,a}) = M_i = f_i^{int}(M_i),$$

then f_i^{int} is a closure operation.

As $M_j P_{k,a} = M_{j+k}$, the only ideals which are proper factors of $P_{m,a}$ are of the form $P_{j,b}$, $j \leqslant m-2$ and $b \in K$. If $j+k = m$ with $j, k \geqslant 2$, then

$$f_i^{int}(P_{j,a}) f_i^{int}(P_{k,b}) = \begin{cases} P_{m,a+b} & \text{if } j \neq i \text{ and } k \neq i, \\ M_m & \text{if } j = i \text{ or } k = i. \end{cases}$$

If $m \geqslant i + j$, $j \geqslant 2$, $f_i^{int}(P_{m,a+b}) = P_{m,a+b}$ and $M_m \nsubseteq P_{m,a+b}$. Thus f_i^{int} is not a semiprime operation. $\square$

We observe in the proof, that if we want such a closure operation which maps $P_{i,a}$ to M_i to be semiprime we also need $P_{m,a}$ to map to M_m for $m \geqslant i+2$. Hence, we have the following:

Corollary 4.3. *Let $S \neq \emptyset$ and T, possibly empty, be subsets of the field K. Over $K[[t^2, t^3]]$ for all $i \geqslant 2$, the maps*

$$f_{i,S,T}^{int}(I) = \begin{cases} I & \text{if } I \supseteq M_{i+1},\ I = P_{i,a},\ a \notin S \text{ or } I = P_{i+1,b},\ b \notin T, \\ \bar{I} & \text{if } I \subseteq M_{i+2},\ I \supseteq P_{i,a},\ a \in S \text{ or } I \subseteq P_{i+1,b},\ b \in T \end{cases}$$

are semiprime operations.

Proof. Clearly $f^{int}_{i,S,T}$ are also closure operations and from the proof of above, they are semiprime. $\square$

Lemma 4.4. *If f_c is a semiprime operation on $K[[t^2,t^3]]$ and $M_j = f_c(M_{j+2})$ for some j, then f_c is a bounded semiprime operation.*

Proof. As $M_j \supseteq M_{j+1} \supseteq M_{j+2}$, then

$$M_j = f_c(M_{j+2}) \subseteq f_c(M_{j+1}) \subseteq f_c(M_j) = f_c\big(f_c(M_{j+2})\big) = M_j.$$

We will use induction to show that $f_c(M_{j+n}) = M_j$ for $n \geqslant 0$. Assume that $f_c(M_{j+k}) = M_j$ for $2 \leqslant k \leqslant n$. Since $M_{j+n+1} = P_{2,0}M_{j+n-1}$,

$$f_c(M_{j+n+1}) \supseteq f_c(P_{2,0}) f_c(M_{j+n-1}) \supseteq P_{2,0} f_c(M_{j+k-2}) = M_{j+2} \supseteq M_{j+n+1}.$$

Applying f_c to the chain, $f_c(M_{j+n+1}) \supseteq f_c(M_{j+2}) = M_j \supseteq f_c(M_{j+n+1})$. As the right-hand and left-hand sides of the chains are equal, we obtain $f_c(M_{j+n+1}) = M_j$.

For any $a \in K$ and $k \geqslant 0$, we have $M_{j+k} \supseteq P_{j+k,a} \supseteq M_{j+k+2}$. Applying f_c to the chain and using the fact that $f_c(M_{j+k}) = M_j$ for $k \geqslant 0$ we obtain $f_c(P_{j+k,a}) = M_j$.

Since the above arguments show if $0 \neq I \subseteq M_j$, $f_c(I) = M_j$, by the definition of bounded, we see that f_c is a bounded semiprime operation. $\square$

Lemma 4.5. *If f_c is a semiprime operation on $K[[t^2,t^3]]$ and $f_c(M_j) = f_c(M_{j+2})$ for some j, then f_c is a bounded semiprime operation.*

Proof. We can break the proof down into the following two cases:

(1) $f_c(M_j) = M_k$, $k \leqslant j$ or
(2) $f_c(M_j) = P_{k,a}$ for some $a \in K$ and $k \leqslant j-2$.

In case (1), $M_k = f_c(M_k) \supseteq f_c(M_{k+1}) \supseteq f_c(M_{k+2}) \supseteq f_c(M_{j+2}) = M_k$. By Lemma 4.4, f_c is bounded.

In case (2), we need to show that for any nonzero ideal $I \subseteq P_{k,a}$, $f_c(I) = P_{k,a}$. Clearly, if $M_{j+2} \subseteq I \subseteq P_{k,a}$, then $P_{k,a} = f_c(M_j) = f_c(M_{j+2}) = f_c(I)$. We will see by induction that $f_c(M_{j+n}) = P_{k,a}$ for $n \geqslant 2$. Assume that $f_c(M_{j+i}) = P_{k,a}$ for $2 \leqslant i \leqslant n$. Since $M_{j+n+1} = P_{2,0}M_{j+n-1}$, we have

$$f_c(M_{j+n+1}) = f_c(P_{2,0}M_{j+n-1}) \supseteq f_c(P_{2,0}) f_c(M_{j+n-1}) \supseteq P_{k+2,a} \supseteq M_{j+n+1}.$$

Note that $M_{j+2} \subseteq P_{k+2,a} \subseteq P_{k,a}$. Hence, $f_c(P_{k+2,a}) = P_{k,a}$ which implies after applying f_c to the above chain of containments that $f_c(M_{j+n+1}) = P_{k,a}$. Hence, $f_c(M_{j+n}) = P_{k,a}$ for $n \geqslant 0$.

Since $M_{k+n} \supseteq P_{k+n,b} \supseteq M_{k+n+2}$, applying f_c to this chain of containments and noting that $f_c(M_{k+n}) = P_{k,a}$ for all $n \geqslant 2$, we conclude that $f_c(P_{k+n,b}) = P_{k,a}$. Now we have seen that for all nonzero $I \subseteq P_{k,a}$, $f_c(I) = P_{k,a}$. Hence, f_c is bounded. $\square$

Lemma 4.6. *If f_c is a semiprime operation on $K[[t^2,t^3]]$ and $f_c(M_j) = f_c(M_{j+1})$ for some j, then f_c is a bounded semiprime operation.*

Proof. Note that for $j \geqslant 2$, if $R = f_c(M_j)$ then $f_c(M_{2j}) = f_c(M_j^2) \supseteq f_c(M_j)^2 = R$. Since $M_j \supseteq M_{j+2} \supseteq M_{2j}$, then $R = f_c(M_j) = f_c(M_{j+2}) = f_c(M_{2j})$. By Lemma 4.5 we can conclude that f_c is bounded.

Also for $j \geqslant 3$ if $I = f_c(M_j) \supseteq M_{j-1} \supseteq M_j$ then $I = f_c(M_{j-1}) = f_c(M_{j+1})$. By Lemma 4.5 we can conclude that f_c is bounded. That leaves us with the cases:

(1) $f_c(M_j) = M_j$ for $j \geqslant 2$ or
(2) $f_c(M_j) = P_{j-2,a}$ for some $a \in K$ and $j \geqslant 4$.

In case (1), consider $f_c(M_{2j+2}) \supseteq f_c(M_{j+1}^2) \supseteq f_c(M_{j+1})^2 = M_{2j} \supsetneq M_{2j+2}$. Applying f_c, we now see that $f_c(M_{2j}) = f_c(M_{2j+2})$. Again, Lemma 4.5 yields that f_c is bounded.

In case (2), $f_c(M_{j-1}) \supseteq f_c(M_j) = P_{j-2,a}$ and $f_c(M_{j-1}) \supseteq M_{j-1}$. Thus

$$f_c(M_{j-1}) \supseteq P_{j-2,a} + M_{j-1} = M_{j-2} \supseteq M_{j-1} \text{ implies } f_c(M_{j-1}) = f_c(M_{j-2}).$$

Now we are in the same set up as our lemma but two steps up. If $f_c(M_{j-2}) = M_{j-2}$ we are done by case (1) above. Otherwise, $f_c(M_{j-2}) = f_c(M_{j-1}) = P_{j-4,b}$, for some $b \in K$.

Now $P_{2j-8,2b} = f_c(M_{j-1})^2 \subseteq f_c(M_{2j-2}) \subseteq f_c(M_{2j-4}) \subseteq f_c(P_{2j-8,2b})$. If we apply f_c to this chain of containments we see that $f_c(M_{2j-2}) = f_c(M_{2j-4})$. Again f_c is bounded by Lemma 4.5. □

Lemma 4.7. *If f_c is a semiprime operation on $K[[t^2, t^3]]$ and $f_c(M_j) = f_c(P_{j-2,b})$ for some $j \geqslant 4$ and $b \in K$, then f_c is a bounded semiprime operation.*

Proof. As in the proof of Lemma 4.6, $f_c(M_j) = f_c(P_{j-2,b})$ implies that $f_c(M_{j-1}) = f_c(M_{j-2})$. We now conclude by Lemma 4.6 that f_c is also bounded. □

The following theorem describes the unbounded semiprime operations over $K[[t^2, t^3]]$.

Theorem 4.8. *Let S be a nonempty subset of K, T any subset. If f_c is an unbounded semiprime operation over $K[[t^2, t^3]]$, then f_c is either the identity or*

$$f^{int}_{i,S,T}(I) = \begin{cases} I & \text{if } I \supseteq P_{i,a},\ a \notin S \text{ or } I \supseteq P_{i+1,b},\ b \notin T, \\ \bar{I} & \text{if } I \subseteq M_{i+2},\ I = P_{i,a},\ a \in S \text{ or } I = P_{i+1,b},\ b \in T. \end{cases}$$

Proof. Suppose f_c is an unbounded semiprime operation over $K[[t^2, t^3]]$ which is not the identity. Then $f_c(I) \neq I$ for some nonzero ideal I.

If $I = M_j$ for some $j \geqslant 2$, then by Lemmas 4.5, 4.6 and 4.7, f_c would be bounded, contradicting the unbounded assumption. Thus I must be a principal ideal.

If $f_c(P_{k,a}) = f_c(P_{k+2,b})$ for some k, then $f_c(P_{k,a}) = f_c(M_{k+2}) = f_c(P_{k+2,b})$ and Lemma 4.7 implies that f_c is bounded, contradicting the unboundedness assumption.

If $f_c(P_{k,a}) = f_c(M_{k-1})$ for some k, then $f_c(P_{k,a}) = f_c(M_k) = f_c(M_{k-1})$ which is bounded by Lemma 4.6.

Thus $f_c(P_{k,a}) = M_k$. Let $W = \{k \mid f_c(P_{k,a}) = M_k$ for some $a \in K$ and some $k \geqslant 2\}$. Since W is nonempty subset of the positive integers there is a smallest $j \geqslant 2$ in W. Let $S = \{a \in K \mid f_c(P_{j,a}) = M_j\}$. Since $P_{n,b} = P_{j,a}P_{n-j,b-a}$ for all $b \in K$, $a \in S$ and all $n \geqslant j+2$, then $f_c(P_{n,b}) \supseteq f_c(P_{j,a})f_c(P_{n-j,b-a}) \supseteq M_j P_{n-j,b-a} = M_n \supseteq P_{n,b}$. Applying f_c to the chain of containments, we see that $M_n = f_c(P_{n,b})$ for all $b \in K$ and $n = j$ or $n \geqslant j+2$.

Note for all $b \in K$, $f_c(P_{j+1,b}) \subseteq f_c(M_{j+1}) = M_{j+1}$, thus $f_c(P_{j+1,b}) = P_{j+1,b}$ or $f_c(P_{j+1,b}) = M_{j+1}$. If $T = \{b \in K \mid f_c(P_{j+1,b}) = M_{j+1}\}$ then $f_c = f^{int}_{j,S,T}$ as defined in the statement of the theorem. □

The bounded semiprime operations are given by the following theorem:

Theorem 4.9. *The only bounded semiprime operations on $K[[t^2, t^3]]$ are of the forms*

$$f^{f}_{m,a}(I) = \begin{cases} I & \text{for } I \supseteq P_{m,a}, \\ M_{m-1} & \text{for } I = P_{m-1,b},\ \forall b \in K, \\ M_m & \text{for } I = P_{m,b},\ b \neq a,\ I = P_{m+1,d},\ \forall d \in K \text{ or } I = M_{m+1}, \\ P_{m,a} & \text{for nonzero } I \subseteq P_{m,a}, \\ (0) & \text{if } I = (0), \end{cases}$$

$$g^f_{m,a}(I) = \begin{cases} I & \text{for } I \supseteq P_{m,a}, \\ M_{m-1} & \text{for } I = P_{m-1,b},\ \forall b \in K, \\ M_m & \text{for } I = P_{m,b},\ b \neq a,\ I = P_{m+1,d},\ \forall d \in K \text{ or } I = M_{m+1}, \\ P_{m,a} & \text{for } I \subseteq P_{m,a} \end{cases}$$

for $m \geqslant 2$ and $a \in K$,

$$f^f_{n,S,T,m}(I) = \begin{cases} I & \text{for } I \supseteq P_{n,a},\ a \notin S \text{ or } I \supseteq P_{n+1,b},\ b \notin T, \\ \bar{I} & \text{for } P_{m-1,d} \subseteq I \subseteq J,\ J = P_{n,a},\ a \in S \text{ or } J = P_{n+1,b},\ b \in T, \\ M_m & \text{for nonzero } I \subseteq M_m, \\ (0) & \text{if } I = (0), \end{cases}$$

$$g^f_{n,S,T,m}(I) = \begin{cases} I & \text{for } I \supseteq P_{n,a},\ a \notin S \text{ or } I \supseteq P_{n+1,b},\ b \notin T, \\ \bar{I} & \text{for } P_{m-1,d} \subseteq I \subseteq J,\ J = P_{n,a},\ a \in S \text{ or } J = P_{n+1,b},\ b \in T, \\ M_m & \text{for } I \subseteq M_m \end{cases}$$

for $m - 1 \geqslant n \geqslant 2$, $S \neq \emptyset$ and if $m = n + 1$, $T = K$,

$$f^f_{n,S,T,m'}(I) = \begin{cases} I & \text{for } I \supseteq P_{n,a},\ a \notin S \text{ or } I \supseteq P_{n+1,b},\ b \notin T, \\ \bar{I} & \text{for } M_{m-2} \subseteq I \subseteq J,\ J = P_{n,a},\ a \in S \text{ or } J = P_{n+1,b},\ a \in S,\ b \in T, \\ M_{m-2} & \text{for } I = P_{m-1,d}, M_{m-1}, \\ M_m & \text{for nonzero } I \subseteq M_m, \\ (0) & \text{if } I = (0), \end{cases}$$

$$g^f_{n,S,T,m'}(I) = \begin{cases} I & \text{for } I \supseteq P_{n,a},\ a \notin S \text{ or } I \supseteq P_{n+1,b},\ b \notin T, \\ \bar{I} & \text{for } M_{m-2} \subseteq I \subseteq J,\ J = P_{n,a},\ a \in S \text{ or } J = P_{n+1,b},\ a \in S,\ b \in T, \\ M_{m-2} & \text{for } I = P_{m-1,d}, M_{m-1}, \\ M_m & \text{for } I \subseteq M_m \end{cases}$$

for $m - 2 \geqslant n \geqslant 2$, $S \neq \emptyset$.

Proof. If f_c is a bounded semiprime operation, then for small nonzero I, either

(1) $f_c(I) = P_{m,a}$ for $I \subseteq P_{m,a}$ for some $a \in K$ or
(2) $f_c(I) = M_m$ for $I \subseteq M_m$

for some $m \geqslant 2$.

Case (1): If $f_c(I) = P_{m,a}$ for $I \subseteq P_{m,a}$, then for f_c to be semiprime, we see as in the proof of Proposition 4.1 that $f_c(P_{i,b}) = P_{i,b}$ for $2 \leqslant i \leqslant m-2$ and all $b \in K$, since the only factors of $P_{m,a}$ are $P_{i,b}$ for $2 \leqslant i \leqslant m-2$ and any $b \in K$. Note that $f_c(M_i) \subseteq f_c(P_{i-2,b}) = P_{i-2,b}$ for all $4 \leqslant i \leqslant m-2$ and $b \in K$. Hence, $f_c(M_i) \subseteq \bigcap_{b \in K} P_{i-2,b} = M_i$ for $4 \leqslant i \leqslant m-2$. $M_2 \supseteq M_3$ contain only the unit principal ideal R. Let $i = 2, 3$, then $M_j f_c(M_i) \subseteq f_c(M_i) f_c(M_j) \subseteq f_c(M_{i+j}) = M_{i+j}$, for $2 \leqslant j \leqslant m-2-i$. This set of containments implies that $f_c(M_i) \subseteq M_{i+j} : M_j = M_i$. Hence $f_c(M_i) = M_i$ for $i = 2, 3$.

Since $M_{m+3} \subseteq P_{m,a}$, we see that $f_c(M_{m+3}) = P_{m,a}$. Applying f_c to the following chain of containments: $M_{m+3} \subseteq P_{m+1,b} \subseteq M_{m+1}$, we see that $P_{m,a} \subseteq f_c(P_{m+1,b}) \subseteq f_c(M_{m+1})$. However, $P_{m+1,b} \subseteq M_{m+1}$ are both incomparable with $P_{m,a}$. Thus $M_m \subseteq f_c(P_{m+1,b}) \subseteq f_c(M_{m+1})$. Since $P_{2,d} P_{m+1,b} = P_{m+3,b+d}$ for all $d \in K$, we see that

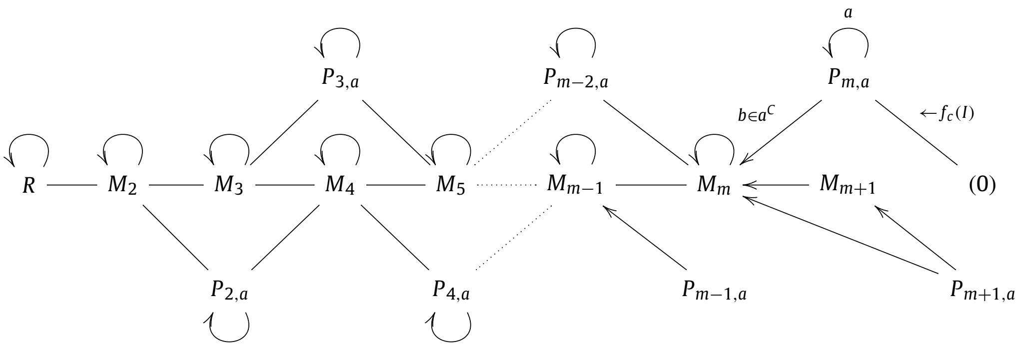
$$P_{2,d} f_c(P_{m+1,b}) = f_c(P_{2,d}) f_c(P_{m+1,b}) \subseteq f_c(P_{m+3,b+d}) = P_{m,a}.$$

Thus $f_c(P_{m+1,b}) \subseteq P_{m-2,a-d}$ for all $d \in K$. Since $\bigcap_{d \in K} P_{m-2,a-d} = M_m$, we see that $f_c(P_{m+1,b}) = M_m$. Now since $P_{m+1,b} \subseteq M_{m+1} \subseteq M_m$ then we easily see that $f_c(M_{m+1}) = M_m$ also.

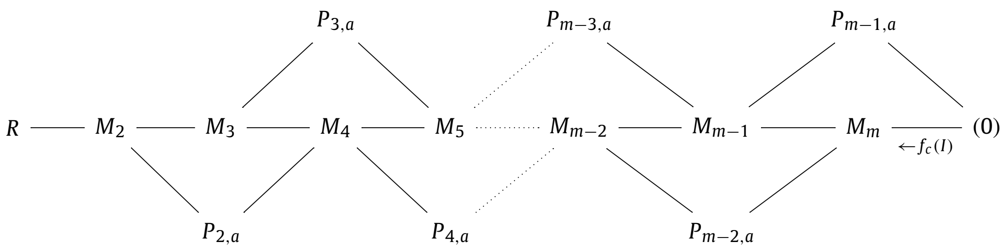
As $M_{m+1} \subseteq P_{m-1,d}$ for all $d \in K$ and $f_c(M_{m+1}) = M_m$, we can see that $M_{m-1} = M_m + P_{m-1,d} \subseteq f_c(P_{m-1,d}) \subseteq f_c(M_{m-1})$. Applying f_c to the chain of containments, we observe that $f_c(P_{m-1,d}) = f_c(M_{m-1})$. Noting that $f_c(M_{m-1}) \subseteq f_c(P_{m-3,d}) = P_{m-3,d}$ for all $d \in K$, we can conclude that

$f_c(M_{m-1}) \subseteq \bigcap_{d\in K} P_{m-3,d} = M_{m-1}$. Putting this fact together with the equality above we see that $f_c(P_{m-1,d}) = f_c(M_{m-1}) = M_{m-1}$.

Putting all the above arguments together we see that f_c must be $f^f_{m,a}$ or $g^f_{m,a}$ depending on whether or not $f_c(0)$ is (0) or $P_{m,a}$. The following diagram represents f_c. The arrows represent the f_c-closure of the indicated ideals.



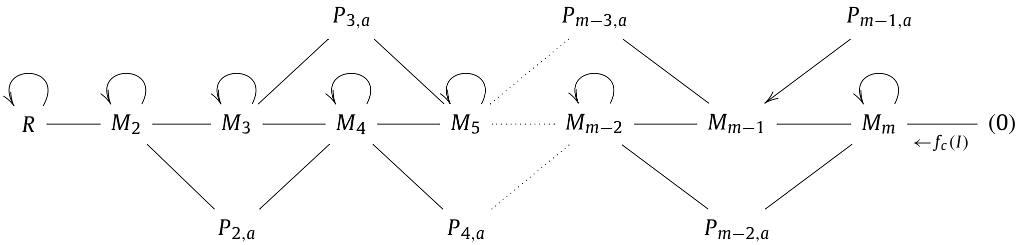
Case (2): Suppose $f_c(I) = M_m$ for all $I \subseteq M_m$ and $m \geqslant 2$. The closure of ideals in the following diagram still needs to be determined.



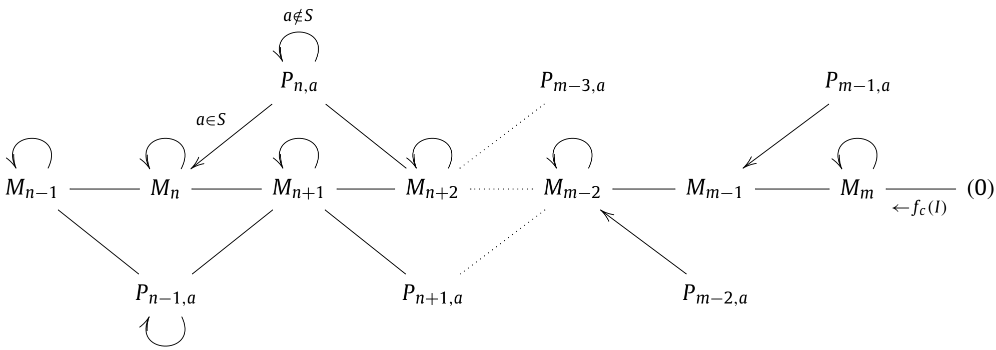
Only the $P_{m-1,b}$ are not comparable to M_m. Since $M_{m+1} \subseteq P_{m-1,b}$ and $f_c(M_{m+1}) = M_m$ then $f_c(P_{m-1,b}) = f_c(M_{m-1})$. We will get back to this later; however, in the next diagram we will indicate this with an arrow from the $P_{m-1,a}$'s to M_{m-1} and omitting the line from (0) to the $P_{m-1,a}$'s.

First, we will see that $f_c(M_2) = M_2$. Suppose $f_c(M_2) = R$, then $f_c(M_{m-2}) = f_c(M_2) f_c(M_{m-2}) \subseteq f_c(M_m) = M_m \subseteq M_{m-2}$. Applying f_c to this chain of containments, we see that $M_m = f_c(M_m) = f_c(M_{m-2})$ which is a contradiction since $M_{m-2} \nsubseteq M_m$.

We now show that $f_c(M_n) = M_n$ for $2 < n \leqslant m-2$. Suppose $M_n \subsetneq f_c(M_n) = I$ where $I \supseteq M_{n-1}$ or $I \supseteq P_{n-2,d}$ for some d. Once again, we decompose $M_m = M_n M_{m-n}$. Noting that $P_{n-2,d} M_{m-n} = M_{m-2}$ and $M_{n-1} M_{m-n} = M_{m-1}$ and $M_{m-1} \subseteq M_{m-2}$, we see that $M_{m-1} \subseteq f_c(M_n) f_c(M_{m-n}) \subseteq f_c(M_n) f_c(M_{m-n}) \subseteq f_c(M_m) = M_m \subseteq M_{m-1}$. As above, this implies that $M_m = f_c(M_m) = f_c(M_{m-1})$ which gives a contradiction. These arguments imply that all ideals along the central line in the above figure excluding possibly M_{m-1} are f_c-closed which I will indicate by a loop in the diagram.

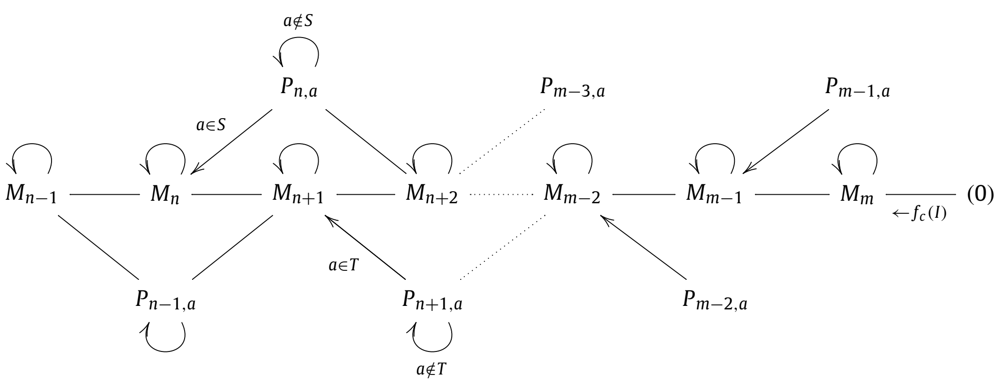


Now, we will determine $f_c(P_{k,b})$ for $2 \leqslant k \leqslant m-2$. Since $f_c(P_{k,b}) \subseteq f_c(M_k) = M_k$ for all $b \in K$ and $2 \leqslant k \leqslant m-2$, we see that $f_c(P_{k,b})$ may equal $P_{k,b}$ or M_k. Suppose that $f_c(P_{k,b}) = M_k$ for some $b \in K$ with $2 \leqslant k \leqslant m-2$. Assume n is the smallest $2 \leqslant n \leqslant m-2$ satisfying this property for some $b \in K$ and define $S = \{b \in K \mid f_c(P_{n,b}) = M_n\}$. Observing that $P_{k,d} = P_{j,a}P_{k-j,b}, a+b=d$ for $2 \leqslant j \leqslant k-n$ and $f_c(P_{k,d}) \supseteq f_c(P_{j,a})f_c(P_{k-j,b}) = M_k$, we conclude that $f_c(P_{k,d}) = M_k$ for all $d \in K$ and $n+2 \leqslant k \leqslant m-2$. For each of these $P_{k,d}$'s, we indicate that the closure is M_k in the following diagram by indicating an arrow between $P_{k,d}$ and M_k and omitting the line between the $P_{k,d}$'s and M_{k+2}. I have left off the ideals containing M_{n-1} since all of these ideals are now known to be f_c-closed.



At this point, there are two ambiguities. What is $f_c(M_{m-1})$ and what is $f_c(P_{n+1,b})$ for $b \in K$? Since $f_c(M_{m-1}) \subseteq f_c(M_{m-2}) = M_{m-2}$ and $f_c(M_{m-1}) \subseteq f_c(P_{m-3,b})$ for all $b \in K$. If $f_c(P_{m-3,d}) = P_{m-3,d}$ for some $d \in K$, then $f_c(M_{m-1}) \subseteq M_{m-2} \cap P_{m-3,d} = M_{m-1}$. Hence, $f_c(M_{m-1}) = f_c(P_{m-1,b}) = M_{m-1}$. Otherwise, $f_c(P_{m-1,b}) = f_c(M_{m-1})$ could be M_{m-1} or M_{m-2}.

In the case that $f_c(P_{m-1,b}) = f_c(M_{m-1}) = M_{m-1}$, let $S = \{b \in K \mid f_c(P_{n,b}) = M_n\}$ as above and $T = \{b \in K \mid f_c(P_{n+1,b}) = M_{n+1}\}$, then $f_c = f^f_{n,S,T,m}$ or $f_c = g^f_{n,S,T,m}$ depending on where f_c maps (0). From the previous diagram, I have added the loop at M_{m-1} to indicate that M_{m-1} is f_c-closed.



In the case that $f_c(P_{m-1,b}) = f_c(M_{m-1}) = M_{m-2}$, let $S = \{b \in K \mid f_c(P_{n,b}) = M_n\}$ as above and $T = \{b \in K \mid f_c(P_{n+1,b}) = M_{n+1}\}$, then $f_c = f^f_{n,S,T,m'}$ or $f_c = g^f_{n,S,T,m'}$ depending on where f_c maps (0). Unlike the previous diagram, there is not a loop at M_{m-2}, since it is not f_c-closed, but an arrow from both M_{m-1} and the $P_{m-1,a}$'s to indicate their f_c-closure.

$$\begin{tikzcd}
 & & P_{n,a} \arrow[loop above, "a\notin S"] \arrow[dl, "a\in S"'] \arrow[dr, dash] & & P_{m-3,a} & & P_{m-1,a} \arrow[dll] \arrow[dl] & \\
M_{n-1} \arrow[loop above] \arrow[r, dash] \arrow[dr, dash] & M_n \arrow[loop above] \arrow[r, dash] & M_{n+1} \arrow[loop above] \arrow[r, dash] & M_{n+2} \arrow[loop above] \arrow[r, dotted, dash] \arrow[ur, dotted, dash] & M_{m-2} \arrow[loop above] & M_{m-1} \arrow[l] \arrow[r, dash] & M_m \arrow[loop above] \arrow[r, dash, "\leftarrow f_c(I)"'] & (0) \\
 & P_{n-1,a} \arrow[loop below] \arrow[ur, dash] & & P_{n+1,a} \arrow[loop below, "a\notin T"'] \arrow[ul, "a\in T"] \arrow[ur, dotted, dash] & & P_{m-2,a} \arrow[ul] & &
\end{tikzcd}$$

$\square$

Surprisingly, the semiprime operations of the form $f^f_{n,S,T,m'}$ and $g^f_{n,S,T,m'}$ do not commute with some of the other semiprime operations for a nonzero ideal. For example,

$$f^f_{n,S,T,m'} \circ f^f_{n,S,T,m-1'}(M_m) = M_{m-2} \quad \text{but} \quad f^f_{n,S,T,m-1'} \circ f^f_{n,S,T,m'}(M_m) = M_{m-1}.$$

Also,

$$f^f_{n,S,T,m'} \circ f^f_{m-1,a}(M_{m+1}) = M_{m-2} \quad \text{but} \quad f^f_{m-1,a} \circ f^f_{n,S,T,m'}(M_{m+1}) = M_{m-1}.$$

This makes it hard to decompose the semiprime operations of $R = K[\![t^2, t^3]\!]$, S_R, into the union of submonoids of $M_{\mathfrak{I}}$ like we did in the Dedekind case.

We make the following definition:

Definition 4.10. Let R be a one-dimensional semigroup ring defined by $S \subseteq \mathbb{N}_0$. Let f_c is a bounded semiprime operation and J be the unique ideal with $f_c(I) = J$ for all $(0) \neq I \subseteq J$ and $n \geqslant 1$ be the conductor of S. Suppose $\mathfrak{a}$ is an ideal which is incomparable to J and $f_c(\mathfrak{a}) \supseteq J$ and $\mathfrak{a} = \mathfrak{a}_0 \subseteq \mathfrak{a}_1 \subseteq \cdots \subseteq \mathfrak{a}_k = f_c(\mathfrak{a})$ is a composition series for $f_c(\mathfrak{a})/\mathfrak{a}$ for $k \geqslant n$ with $\mathfrak{a}_i \supseteq J$ for all $i > 0$. Then we say f_c is an exceptional semiprime operation.

Note that the semiprime operations $f^f_{n,S,T,m'}$ and $g^f_{n,S,T,m'}$ are exceptional bounded semiprime operations since $P_{m-1,b} \subseteq M_{m-1} \subseteq M_{m-2}$ is a composition series for $f_c(P_{m-1,b})/P_{m-1,b}$ of length 2 which is the conductor of $\langle 2, 3\rangle$ the semigroup associated to $R = K[\![t^2, t^3]\!]$.

Combining the results of Theorems 4.8 and 4.9 and looking at compositions of the maps obtained in the theorems we see that the non-exceptional semiprime operations can be decomposed as in the Dedekind case:

Theorem 4.11. *Let $R = K[\![t^2, t^3]\!]$ and E be the set of exceptional semiprime operations of R. Then the complement of E in S_R, $S_R \setminus E$, is the union of the monoids*

$$M_0 = \{e\} \cup \left\{ f^{int}_{n,S,T}, f^f_{n,a}, f^f_{n,S,T,m} \right\}$$

and

$$M_f = \{e\} \cup \left\{ g^f_{n,a}, g^f_{n,S,T,m} \right\}$$

where M_f is a left M_0-act but not a right M_0-act under composition.

Proof. Above we saw by example that the semiprime operations $f^f_{n,S,T,m'}$ and $g^f_{n,S,T,m'}$ were exceptional. To see the remaining bounded semiprime operations are not exceptional, we need to find all nonzero ideals $\mathfrak{a}$ which are not comparable to the ideal J for each bounded semiprime operation f_c for which $f_c(I)=J$.

For both $f_c = f^f_{n,S,T,m}$ and $f_c = g^f_{n,S,T,m}$, the J in the definition is M_m. The only ideals which are incomparable to M_m are $P_{m-1,a}$ for all $a \in K$ and $f_c(P_{m-1,a}) = M_{m-1} \supseteq M_m$ and $P_{m-1,a} \subseteq M_{m-1}$ is a composition series for $f_c(P_{m-1,a})/P_{m-1,a}$. Thus $f^f_{n,S,T,m}$ and $g^f_{n,S,T,m}$ are not exceptional.

For both $f_c = f^f_{m,a}$ and $f_c = g^f_{m,a}$, the J in the definition is $P_{m,a}$. The ideals which are incomparable to $P_{m,a}$ are $P_{m,b}$ for $b \neq a$, M_{m+1}, $P_{m+1,b}$ for all $b \in K$ and $P_{m-1,b}$ for all $b \in K$. Note that $f_c(P_{m,b}) = M_m \supseteq P_{m,a}$ and $P_{m,b} \subseteq M_m$ is a composition series for $f_c(P_{m,b})/P_{m,b}$. Also $f_c(M_{m+1}) = M_m \supseteq P_{m,a}$ and $M_{m+1} \subseteq M_m$ is a composition series for $f_c(M_{m+1})/M_{m+1}$. Similarly, $f_c(P_{m+1,b}) = M_m \supseteq P_{m,a}$ and $P_{m+1,b} \subseteq M_{m+1} \subseteq M_m$ is a composition series for $f_c(P_{m+1,b})/P_{m+1,b}$ and only M_m is comparable to $P_{m,a}$. Lastly, $f_c(P_{m-1,b}) = M_{m-1} \supseteq P_{m,a}$ and $P_{m-1,b} \subseteq M_{m-1}$ is a composition series for $f_c(P_{m-1,b})/P_{m-1,b}$. Now by definition both $f^f_{m,a}$ and $g^f_{m,a}$ are not exceptional.

Now we look at all compositions of semiprime operations in M_0. Throughout, we will denote $K \setminus \{a\} = a^C$. The compositions are as follows:

$$\text{(M1)}\quad f^{int}_{m,S,T} \circ f^{int}_{n,U,V} = f^{int}_{n,U,V} \circ f^{int}_{m,S,T} = \begin{cases} f^{int}_{m,S,T} & \text{if } m+2 \leqslant n,\\ f^{int}_{m,S,T\cup U} & \text{if } m+1 = n,\\ f^{int}_{m,S\cup U,T\cup V} & \text{if } m = n,\\ f^{int}_{n,U,V\cup S} & \text{if } n+1 = m,\\ f^{int}_{n,U,V} & \text{if } n+2 \leqslant m; \end{cases}$$

$$\text{(M2)}\quad f^{f}_{n,S,T,m} \circ f^{int}_{l,U,V} = f^{int}_{l,U,V} \circ f^{f}_{n,S,T,m} = \begin{cases} f^{f}_{n,S,T,m} & \text{if } n+1 \leqslant m < l,\\ f^{f}_{n,S,T\cup U,m} & \text{if } n+1 = l \leqslant m,\\ f^{f}_{n,S\cup U,T\cup V,m} & \text{if } n = l \leqslant m-1,\\ f^{f}_{l,U,S\cup V,m} & \text{if } l+1 = n \leqslant m-1,\\ f^{f}_{l,U,V,m} & \text{if } l+1 < n \leqslant m-1; \end{cases}$$

$$\text{(M3)}\quad f^{f}_{m,a} \circ f^{int}_{l,U,V} = f^{int}_{l,U,V} \circ f^{f}_{m,a} = \begin{cases} f^{f}_{m,a} & \text{if } m < l, m = l,\ a \notin U \text{ or } l = m-1,\ a \notin V,\\ f^{f}_{m-1,K,K,m} & \text{if } m = l,\ a \in U,\\ f^{f}_{m-1,U,K,m} & \text{if } l = m-1,\ a \in V,\\ f^{f}_{l,U,V,m} & \text{if } l < m-1; \end{cases}$$

$$\text{(M4)}\quad f^{f}_{n,S,T,m} \circ f^{f}_{l,U,V,k} = f^{f}_{l,U,V,k} \circ f^{f}_{n,S,T,m} = \begin{cases} f^{f}_{n,S,T,m} & \text{if } n+1 < m, l,\\ f^{f}_{n,S,T\cup U,m} & \text{if } n+1 = l \leqslant m < k,\\ f^{f}_{n,S,T\cup U,k} & \text{if } n+1 = l \leqslant k-1 \leqslant m-1,\\ f^{f}_{n,S\cup U,T\cup V,m} & \text{if } n = l < m-1 < k-1,\\ f^{f}_{n,S\cup U,T\cup V,k} & \text{if } n = l < k-1 \leqslant m-1,\\ f^{f}_{l,U,S\cup V,m} & \text{if } l+1 = n \leqslant m-1 < k-1,\\ f^{f}_{n,U,S\cup V,k} & \text{if } l+1 = n < k \leqslant m,\\ f^{f}_{l,U,V,k} & \text{if } l+1 < m, k; \end{cases}$$

$$\text{(M5)}\ \ f^f_{n,S,T,m}\circ f^f_{l,a}=f^f_{l,a}\circ f^f_{n,S,T,m}=\begin{cases} f^f_{n,S,T,m} & \text{if } m\leqslant l,\\ f^f_{n,S,T,l} & \text{if } n+1<l\leqslant m,\\ f^f_{n,K,T\cup a^c,l} & \text{if } n+1=l\leqslant m,\\ f^f_{n-1,K,K,n} & \text{if } n=l\leqslant m-1,\\ f^f_{l,a} & \text{if } l<n;\end{cases}$$

$$\text{(M6)}\ \ f^f_{n,a}\circ f^f_{m,b}=f^f_{m,b}\circ f^f_{n,a}=\begin{cases} f^f_{n,a} & \text{if } n+1<m,\\ f^f_{n-1,K,K,n} & \text{if } m\leqslant n\leqslant m+1,\\ f^f_{m-1,K,K,m} & \text{if } n+1=m,\\ f^f_{m,b} & \text{if } m+1<n.\end{cases}$$

Clearly, M_0 is a monoid and similar compositions show that M_f is a monoid. To see that M_f is a left M_0-act but not a right M_0-act we look at the mixed compositions.

$$\text{(L1)}\ \ g^f_{n,S,T,m}\circ f^{int}_{l,U,V}=f^{int}_{l,U,V}\circ g^f_{n,S,T,m}=\begin{cases} g^f_{n,S,T,m} & \text{if } n+1\leqslant m,l,\\ g^f_{n,S,T\cup U,m} & \text{if } n+1=l\leqslant m,\\ g^f_{n,S\cup U,T\cup V,m} & \text{if } n=l\leqslant m-1,\\ g^f_{l,U,S\cup V,m} & \text{if } l+1=n\leqslant m-1,\\ g^f_{l,U,V,m} & \text{if } l<n\leqslant m-1;\end{cases}$$

$$\text{(L2)}\ \ g^f_{m,a}\circ f^{int}_{l,U,V}=f^{int}_{l,U,V}\circ g^f_{m,a}=\begin{cases} g^f_{m,a} & \text{if } m<l, m=l,\ a\notin U \text{ or } l=m-1,\ a\notin V,\\ g^f_{m-1,K,K,m} & \text{if } m=l,\ a\in U,\\ g^f_{m-1,U,K,m} & \text{if } l=m-1,\ a\in V,\\ g^f_{l,U,V,m} & \text{if } l<m-1;\end{cases}$$

$$\text{(L3)}\ \ \text{(a)}\ \ g^f_{n,S,T,m}\circ f^f_{l,U,V,k}=\begin{cases} g^f_{n,S,T,m} & \text{if } n+1<l, m\leqslant k,\\ g^f_{n,S,T\cup U,m} & \text{if } n+1=l<m\leqslant k,\\ g^f_{n,S\cup U,T\cup V,m} & \text{if } n=l\leqslant m-1\leqslant k-1,\\ \text{not a semiprime operation} & \text{if } k<m;\end{cases}$$

$$\text{(b)}\ \ f^f_{l,U,V,k}\circ g^f_{n,S,T,m}=\begin{cases} g^f_{n,S,T,m} & \text{if } n+1<l, m\leqslant k,\\ g^f_{n,S,T\cup U,m} & \text{if } n+1=l<m\leqslant k,\\ g^f_{n,S,T\cup U,k} & \text{if } n+1=l\leqslant k<m,\\ g^f_{n,S\cup U,T\cup V,m} & \text{if } n=l\leqslant m-1\leqslant k-1,\\ g^f_{n,S\cup U,T\cup V,k} & \text{if } n=l<k\leqslant m,\\ g^f_{l,U,V\cup S,m} & \text{if } l+1=n<m\leqslant k,\\ g^f_{l,U,V\cup S,k} & \text{if } l+1=n\leqslant k<m,\\ g^f_{l,U,V,k} & \text{if } l+1<n, k\leqslant m;\end{cases}$$

$$\text{(L4)}\ \ \text{(a)}\ \ g^f_{n,a}\circ f^f_{m,b}=\begin{cases} g^f_{n,a} & \text{if } n+1<m,\\ \text{not a semiprime operation} & \text{if } m\leqslant n+1;\end{cases}$$

$$\text{(b)}\ \ f^f_{m,b} \circ g^f_{n,a} = \begin{cases} g^f_{n,a} & \text{if } n+1 < m, \\ g^f_{n-1,K,K,n} & \text{if } m \leqslant n \leqslant m+1, \\ g^f_{m-1,K,K,m} & \text{if } n+1 = m, \\ g^f_{m,b} & \text{if } m+1 < n; \end{cases}$$

$$\text{(L5)}\ \ \text{(a)}\ \ g^f_{n,S,T,m} \circ f^f_{l,a} = \begin{cases} g^f_{n,S,T,m} & \text{if } m \leqslant l, \\ \text{not a semiprime operation} & \text{if } l < m; \end{cases}$$

$$\text{(b)}\ \ f^f_{l,a} \circ g^f_{n,S,T,m} = \begin{cases} g^f_{n,S,T,m} & \text{if } m \leqslant l, \\ g^f_{n,S,T,l} & \text{if } n+1 < l \leqslant m, \\ g^f_{n,K,T\cup a^C,l} & \text{if } n+1 = l \leqslant m, \\ g^f_{n-1,K,K,n} & \text{if } n = l \leqslant m-1, \\ g^f_{l,a} & \text{if } l < n; \end{cases}$$

$$\text{(L6)}\ \ \text{(a)}\ \ g^f_{l,a} \circ f^f_{n,S,T,m} = \begin{cases} g^f_{l,a} & \text{if } l < n, \\ \text{not a semiprime operation} & \text{if } l \geqslant n; \end{cases}$$

$$\text{(b)}\ \ f^f_{n,S,T,m} \circ g^f_{l,a} = \begin{cases} g^f_{n,S,T,m} & \text{if } m \leqslant l, \\ g^f_{n,S,T,l} & \text{if } n+1 < l \leqslant m, \\ g^f_{n,K,T\cup a^C,l} & \text{if } n+1 = l \leqslant m, \\ g^f_{n-1,K,K,n} & \text{if } n = l \leqslant m-1, \\ g^f_{l,a} & \text{if } l < n. \end{cases}$$

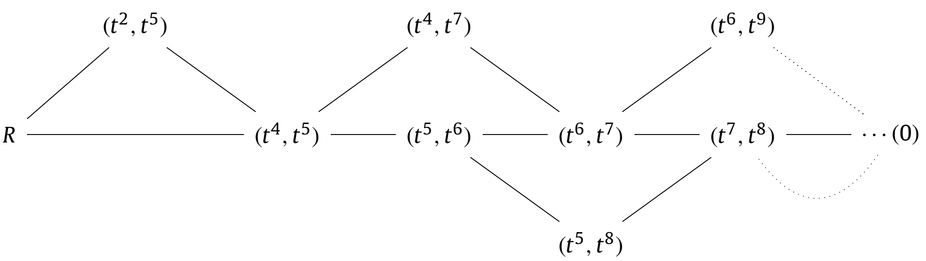
Hence M_f is a left M_0-act but not a right M_0-act. $\square$

We will now see as in the Dedekind case the only prime operation is the identity.

Theorem 4.12. *Let $R = K[[t^2, t^3]]$. Then $P_R = \{e\}$.*

Proof. Suppose f_c is one of the other semiprime operations. Then for some $i \geqslant 2$ and some $a \in K$, $f_c(P_{i,a}) = M_i$. Now since f_c is prime, $P_{i,a} = (t^i + at^{i+1}) f_c(R) = f_c(P_{i,a}) = M_i$ which is a contradiction. Hence f_c cannot be prime. Thus $P_R = \{e\}$. $\square$

To determine all the semiprime operations for other semigroup rings becomes immediately more complicated for any other semigroup. Note even for the ring $K[[t^2, t^5]]$, the diagram of two generated monomial ideals is as follows:



where each line segment in the above diagram indicates $\supseteq$. Of course, this leaves out a lot of two generated ideals in addition to all the principal ideals. But even without all these ideals we can see that there is an extra layer of difficulty that we did not have in the cuspidal cubic case. Certainly, the conductor will be involved with the classification of all semiprime operations. I believe that the non-exceptional semiprime operations over a one-dimensional semigroup ring R will decompose into the union of two submonoids of the monoid $(M_{\mathfrak{I}}, \circ)$ of maps from the set of ideals of R to itself, one being a left but not a right act of the other.

Certainly, if f_c is a prime operation over any commutative ring, then f_c is the identity on the set of principal ideals of R since $gR = gf_c(R) = f_c(g)$ for all $g \in R$. However, it is not known whether f_c must be the identity over one-dimensional domains. It may be that for one-dimensional semigroup rings, the set of prime operations will be the singleton set consisting only of the identity.

There will certainly be more prime operations if the ring is a normal domain of dimension 2 or more since the integral closure does not agree with the identity for all ideals of height 2 or more. Moreover, integral closure is a prime operation in any normal domain.

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