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Dispersive Smoothing Effects for
KdV Type Equations

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In this paper we study the smoothness properties of solutions of some nonlinear
equations of Korteweg-de Vries (KdV) type, which are of the form
\[ \partial_t u = a(x, t) u_3 + f(u_{x}, u_{t}, u, x, t), \]
where \( x \in \mathbb{R} \), \( u_{j} = \partial_{x}^{j} u \), and \( k \) and \( j \) are nonnegative integers. Our principal condition
is that \( a(x, t) \) be positive and bounded, so that the dispersion is dominant. It is
shown under certain additional conditions on \( a \) and \( f \) that \( C^s \) solutions \( u(x, t) \) are
obtained for \( t > 0 \) if the initial data \( u(x, 0) \) decays faster than it does polynomially
on \( \mathbb{R} \) and has certain initial Sobolev regularity.

A quantitative relationship between the rate of decay and the amount of gain of
smoothness is given. Let \( s_{0} \) be the Sobolev index. If
\[ \int_{\mathbb{R}} u^{2}(x, 0)(1 + |x|^{m}) \, dx < \infty, \]
for an integer \( m \geq 0 \) and the solution obeys \( |u|_{L^{\infty}(\mathbb{R})} < \infty \) for an existence time
\( 0 < t < T \), then \( u(x, t) \in H^{m}_{x}(\mathbb{R}) \) for all \( 0 < t < T \), and \( u(x, t) \in L^{1}(\mathbb{R}; H^{m+1}(\mathbb{R})) \).
Our method can also be extended to address the fully nonlinear
dispersive equations related to (1).

1. INTRODUCTION

In this paper we study the following initial value problem (IVP) for a
nonlinear PED of KdV type:
\[ \partial_t u = a(x, t) u_3 + f(u_{x}, u_{t}, u, x, t), \quad u(x, 0) = \varphi(x), \quad (x, t) \in \mathbb{R} \times [0, T], \quad 0 < T < \infty, \]

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with some conditions on \( a(x, t) \), decay for the initial data, and nonlinearity \( f \) which are stated later.

Our main object is to show that the solutions of IVP of (1) are locally smoothing due to the dispersivity of the equations. Along with this analysis we will also prove the well posedness result for these equations, basing our work on the work done on energy method. The notation of well posedness means existence, uniqueness, persistence, and continuous dependence of initial data.

There have been many articles and results in or close to this topic. For instance, T. Kato [1] showed that solutions of the IVP for the Korteweg-de Vries (KdV) equation possess a smoothing effect which is inherent to the dispersive character of the equation. Kruzhkov and Faminski [2] found a relation between the order of vanishing at infinity of the initial data and the gain of smoothness of the solution for \( t > 0 \). Corresponding work for some special nonlinear Schrödinger equations was done by Hayashi et al. [3] and Ponce [4]. Kenig et al. [11] proved that Kato’s local smoothing effect in Section 6 of [1] is a global identity in an appropriate space; they [12] also studied inhomogeneous version of this effect via the contraction principle. Constantin & Saut [5], Sjölin [6], Vega [10], Ginibre & Velo [7], and others gave the results of local gain of finite regularity for various other linear and nonlinear dispersive equations. We call the coefficient of the highest order \( u_3 \) in the KdV type equation the leading coefficient. It is interesting to remark that, as we know, all KdV and its generalized forms studied in the past were considered “constant leading coefficient” until a paper of Craig and Goodman [8] appeared. Since the only main tool for “variable leading coefficient” equations is energy estimate theory, ([8, 9] and references therein) this “variable leading coefficient” is in general more difficult to handle.

Continuing in the spirit in [8] Craig, Kappeler and Strauss [9] studied a fully nonlinear equation of KdV type in one dimension. Namely, they considered the equation

\[
  u_t + f(u_3, u_2, u, x, t) = 0 \quad (-\infty < x < +\infty),
\]

where \( f \) is and

\[
  \partial f/\partial u_3 > c > 0, \quad \text{and} \quad \partial f/\partial u_2 < 0.
\]

They proved that there is an infinite gain of regularity within the existence interval for solutions with sufficient decay at infinity without any growth conditions on the nonlinear terms. To measure decay they use weight class \( w_{\infty, r}(x) \) which decay exponentially for large negative \( x \).
Our main results extend the work in [9] as follows:

1. Weaken the condition (3) considerably in order to handle the non-linear examples such as \( u_t + u_3 - u_3 u^2 = 0 \) and \( u_t + u_3 + u_3^2 = 0 \) without smallness of initial data or growth condition.

2. Use a better one-sided weight function class. The class based on old one in [9] allows for the solution \( u(x, t) \) to have an exponential growth as \( x \to +\infty \). The spaces based on our new one show that in fact \( u(x, t) \) will have controlled polynomial growth as \( x \to +\infty \).

3. Generalize the classes of nonlinear terms in [9], and obtain the corresponding lower starting Sobolev regularity index for the IVP, which we denote \( s_r \). That is, \( s_r \) is the order of the Sobolev regularity that is required of the data \( \varphi \) to guarantee the existence, uniqueness, and gain of regularity of the solution.

We will make the assumptions on the coefficients \( a(x, t) \) and the non-linearity \( f \) in equation (1) as follows:

\[
a: R \times [0, T] \to R \quad \text{is} \quad C^\infty \quad \text{in} \quad x,
\]

and \( f: R^4 \times [0, T] \to R \) is \( C^\infty \) in all its variables. Furthermore:

(A1). There is a constant \( A \geq 1 \) such that

\[
A \geq a(x, t) \geq A^{-1}, \quad \text{for all} \quad (x, t) \in R \times [0, T].
\]

We will use the notation that \( \partial_n f = \partial_x f \).

(A2). All \( \partial_{i_1} \partial_{i_2} f \in L^2 \), provided that \( u \in H^s(R) \).

\[
\partial_{i_2} f = P_2(u_{i_2}, ..., u, x, t) + L_2(u_{i_2}, ..., u, x, t)
\]

where \( P_2(u_{i_2}, ..., u, x, t) \geq 0 \) for all \( (x, t) \in R \times [0, T] \), and

\[
\left| \int_0^x L_2(u_{i_2}, ..., u, x', t) \, dx' \right| \leq L_2^2 < \infty,
\]

on \( R \times [0, T] \), provided \( u(x, t) \in H^s(R) \) with \( s \geq 3 \).

(A3). If \( f \) depends explicitly on \( x \), then we ask that for \( i' > 0 \), for multi-indices \( (i', i_2, i_3, ..., i_0) \), and for \( u(x, t) \in H^s(R) \) with \( s \geq 3 \),

\[
\int_R \left( (\partial_{i_1}^{i'} \partial_{i_2}^{i_2} \cdot \partial_{i_0}^n f)(u_{i_2}, ..., u, x, t))^2 \, dx < \infty
\]

(A4). \( x^N \partial_{i_1}^j f(x, t) \) is bounded for all \( N \geq 0, j \geq 0 \) and for all \( (x, t) \in R \times [0, T] \). For the special interest, this paper will concern five particular forms of nonlinearity:
\[ f = f_1 = f_2(u_2, u_1, u, x, t), \quad \text{for instance, } f = -u_2^2u^n \sin(t + x), \quad n \in \mathbb{Z}^+. \]
\[ f = f_2 = \partial_x(f_2(u_1, u)), \quad \text{for instance, } f = -u_2^2u^n, \quad n \in \mathbb{Z}^+. \]
\[ f = f_3 = f_3(u_1, u, x, t), \quad \text{for instance } f = -u_2^2u^n \sin(t + x), \quad n \geq 0. \quad (8) \]
\[ f = f_4 = \partial_x(f_4(u)), \quad \text{for instance} \quad f = -u_1^2u^n, \quad n \in \mathbb{Z}^+. \]
\[ f = f_5 = f_5(u), \quad \text{for instance} \quad f = -u^n, \quad n \in \mathbb{Z}^+. \quad (9) \]

We will introduce or use notation:

\[ \text{Weight Class } V_{\nu m l}. \]

\[ \xi(x, t) \in V_{\nu m l} \text{ with real numbers } i, m, l, \text{ if } \xi(x, t) \text{ is a positive } C^\infty \text{ function on } R \times [0, T] \text{ and there exist constants } c_1, c_2, \text{ and } c_3 \text{ such that} \]
\[ 0 < c_1 \leq |t - i| \left| \frac{x}{m} \right| \leq c_2, \quad \text{for } x > 1, \ 0 < t < T \quad (10) \]
\[ 0 < c_1 \leq |t - i| \left| \frac{x}{m} \right| \leq c_2, \quad \text{for } x < -1, \ 0 < t < T \quad (11) \]
\[ (t \partial_x \xi + \partial_x \xi) \leq c_3 \xi, \quad \text{for } (x, t) \in R \times [0, T], \ j \in \mathbb{Z}^+. \quad (12) \]

\[ \text{Weighted Sobolev Spaces.} \]
\[ H'(V_{\nu m l}) = \left\{ v(x) \mid \|v\|_2 = \sum_{j=0}^{s} \int_{R} |v(x)|^2 \xi(x, \cdot) \ dx < \infty, \ \xi \in V_{\nu m l} \right\} \quad (13) \]

We remark here that the traditional Sobolev Space is \( H' = H'(V_{\nu oo}) \), without a weight.

\[ \text{Weighted Vector - Valued Spaces.} \]
\[ L^2(H'(V_{\nu m l})) = \left\{ v(x, t) \mid \|v\|_2 = \int_{0}^{T} \|v(\cdot, t)\|^2 \ dt < \infty, \ \xi \in V_{\nu m l} \right\} \quad (14) \]
\[ L^\infty(H'(V_{\nu m l})) = \left\{ v(x, t) \mid \|v\|_\infty = \sup_{t} \|v(\cdot, t)\| < \infty, \ \xi \in V_{\nu m l} \right\} \quad (15) \]

Defining the following extra classes of weights,
\[ V_{\nu m} = \bigcup_{j < i} V_{\nu j m l} \quad \text{and} \quad \bar{V}_{\nu m} = \bigcup_{j > m} V_{\nu j l}, \quad (16) \]

we define the extra two spaces:
\[ L^p(H'(V_{\nu m l})) = \bigcup_{j < i} L^p(H'(V_{\nu j m l})). \quad (17) \]
\[ L^p(H'(\bar{V}_{\nu m l})) = \bigcup_{j > m} L^p(H'(V_{\nu j l})). \quad (18) \]
We shall use space of $L^p(L^r)$ and $L^p(H^s(V))$ only in the cases $i = -1$ or $m = 1$, respectively. Our main theorems are as follows:

**Main Theorem 1.1. (Existence and Uniqueness).** Consider the IVP (3) with $k = 1$, and $f = f_r$, $r = 1, ..., 5$ as in (8). Assume that $a$ and $f$ satisfy (A1)-(A4). Let $N \geq 0 - r$ be an integer and $c_0 > 0$, then there exists a time $0 < T < \infty$ depending only on $c_0$ such that for all $\varphi \in H^N(R)$ with $\|\varphi\|_{H^N(R)} \leq c_0$ there exists an unique solution of the IVP such that

$$u(x, t) \in L^\infty([0, T], H^N(R)).$$

**Main Theorem 1.2. (Gain of regularity).** Let $T > 0$ and $u(x, t)$ be a solution of the initial value problem (1) in the region $R \times [0, T]$ such that

$$u \in L^\infty(H^{m-r}(V_{L+L,1})), \quad f = f_r, \quad r = 1, ..., 5,$$

for some $L \geq 1$. Then

$$u \in L^\infty(H^{m-r+1}(V_{L-1,+1,1})), \quad L^2(H^{1-r+1}(V_{L-1,1,1,1})))$$

for all $0 \leq l \leq L$, with the exception that if $l = L$ then $V_{L-1,1,1,1}$ is replaced by $V_{L-1,1,1,1}$, also $V_{L-1,1,1,0}$ is replaced by $V_{L-1,1,1,0}$ whenever it appears.

2. **THE MAIN IDENTITY**

Since the energy estimate method plays a crucial role in the study, an energy estimate identity is first presented. We will define (refer to (5) for $L_2$)

$$EX = \exp \left\{ \frac{2}{3} \int_0^T a^{-1}L_2 \, dx \right\}$$

as it is an expression which appears very often in the estimates below.

With the above notations and assumptions we can begin with

**Lemma 2.1. (Main Identity).** Let $u$ be a solution of the initial value problem (1) with enough Sobolev regularity. (for instance $u \in H^N(R)$, $N \geq \alpha + 3$.) We have the following identity (refer to (5) for $P_2$)

$$\partial_t \int \xi^2 u_2 \, dx + \int (\eta^a + 2 \xi^a P_2) u_2 \, dx = \int \theta^a u_2 \, dx + \int O^a,$$

where $\eta^a = -3(a\xi^a) + 2(a\xi^a + L_2)\xi^a.$


Solving (24) for $\xi$ we obtain that

$$\xi^n = \frac{1}{3} a^{(2e/3)-1/2} \cdot \int_x^\infty a^{-2e/3} \frac{1}{EX} \eta \, dx'$$

(25)

$$\theta^n = \partial_t \xi - (\partial_2 f) + (\partial_1 f) + (\partial_3 f) + (\partial h)$$

(26)

$$O^\xi = \left\{ \sum C_j^i a_j u_{x+a_j} + (f(u_2, ..., x, t)_x) \right\}$$

(27)

where $h$ will be given in (30) below. Furthermore, if $u(x, t) \in L^\infty([0, T], H^\infty(R))$ then there exists a constant $E > 0$, which depends on the norms of $u$ in $u(x, t) \in L^\infty(H^\infty(R))$ such that

$$\frac{1}{E} \int_x^\infty \eta \, dx' \leq \xi(x, t) \leq E \cdot \int_x^\infty \eta \, dx'.$$

(28)

The Sobolev indices necessary for (28) can be chosen to be $s_1 = 3$ and $s_2 = 2$ for $f = f_1$ and $f_2$, respectively.

Proof. Taking $x$-derivatives of (1), then integrating the identity against $2\xi u_x$, by the method of integration by parts we have

$$\int \partial_t \xi u_x^2 \, dx = \int \partial_t \xi \cdot u_x^2 \, dx$$

$$+ \int 2\xi u_x \left\{ a u_{x} + C_2 a_2 + (\partial_2 f)_1 + \partial_1 f \right\} u_{x+1}$$

$$+ \left\{ a u_1 + \partial_2 f \right\} u_{x+2} \right\} \, dx + \int O^\xi \, dx$$

(29)

Since,

$$f_x = \partial_2 f u_{x+2} + \left\{ (x + 1)(\partial_2 f)_1 + \partial_1 f \right\} u_{x+1} + h(u_x, ...)$$

(30)
Finally,

\[ \partial_t \int \zeta u_2^2 \, dx = \int \left[ 3(a_2^2 \zeta)_t - 2\zeta(xa_1 + \partial_2 f) \right] u_{2+1}^2 \, dx \]

\[ + \int \left\{ \partial_2 \zeta \left[ \left( (\partial_2 f)_1 + \partial^2 f \right)_1 + \left( \zeta [xa_1 + \partial_2 f] \right)_2 \right] \right. \]

\[ - C_2^2 \zeta a_2 - (a_2^2)_t \big|_0^x u_2^2 + \int O^* \, dx \]  

(31)

which gives (23), (24), (26), and (27). Solving (24) for \( \zeta \) we obtain (25) and (22). Since

\[ \left| \int_0^x a^{-1} L_{2} \, dx' \right| \leq \left| a^{-1} \int_0^x L_{2} \, dx \right| + \left| \int_0^x (a^{-1})_t L_{2} \, dx \right| \]

\[ \leq 2A \left| \int_0^x L_{2} \, dx' \right| + A^2 \|a_1\|_0 \left| \int_0^x L_{2} \, dx' \right| \leq E' \]  

(32)

by the assumption (A_2) and (A_3). We will now comment on the five cases more precisely. If \( f = f_r \) with \( r = 1, 2 \) then \( \int_0^x L_{2} \, dx' = \mathcal{L}_{2}(u_{1-r}, \cdot) \) and \( \|u\|_{4-r} \) bounded implies that \( \mathcal{L}_{2} \) and \( \|u\|_{0} \) are bounded, therefore, \( E' \) depends on the norms of \( u \) in \( L^\infty(H^{4-r}(R)) \). Taking \( E = \exp E' \) we prove (28), therefore this lemma. Q.E.D.

In the energy estimates, the term involved \( O^* \) is treated as a remainder term. The next result specifies the nature of estimates of \( O^* \).

**Lemma 2.2.** The expression \( O^* \) in the main identity (4) is a sum of terms which are of the form

\[ \zeta \partial_t \partial_x \partial_i \partial_j \partial_k \partial_l \partial_m \partial_n f, u_1, u_2, \ldots, u_p, u_s \quad \text{where} \quad 1 \leq v_1 \leq \cdots \leq v_p \leq \alpha \]  

(33)

In case \( r = 1, 2, 3 \)

\[ p = (1 - [r/3]) p_2 + p_4 + p_0 \geq 1 + (-1)^r, \]  

(34)

\[ v_1 + v_2 + \cdots + v_p + 0.5(1 - (-1)^r) = \alpha + p_1 + 2(1 - [r/3]) p_2 \]  

(35)

and \( p + v_{p-1} + v_p \leq \alpha + 6 - 2[r/2] \).  

(36)

The case \( r = 4, 5 \) are

\[ p = p_0 \geq 2, \]  

(37)

\[ v_1 + \cdots + v_p = \alpha + (5 - r). \]  

(38)
It is sufficient to show this lemma for the case \( r = 1 \). The term \( \mathcal{O} \) comes from differentiating the equation \( \tau \) times with respect to \( x \) and multiplying by \( \zeta \theta u \). They clearly have the form (33), where \( p \) is the total number of factors pulled out of \( f \) by differentiation, with \( p_j \) derivatives of \( f \) with respect to \( u_j (j = 2, 1, 0) \). The total number of derivatives which occur in (33) by (35). Each differentiation of \( f \) with respect to \( u_1 \) contributes an extra 2 derivatives, which leads to the 2\( p_2 \) in (35); similarly for \( u_1 \). Thus at least \( p_2 \) elements among \( \{ v_1, ..., v_p \} \) are \( \geq 2 \), at least \( p_1 \) elements among them are \( \geq 2 \) and they are all \( \geq 1 \). Therefore,

\[
v_1 + \cdots + v_{p-2} \geq 3(p_2 - 2) + 2p_1 + p_0.
\]

Combining this with (35), we obtain (36). Q.E.D.

### 3. AN IMPORTANT A PRIORI ESTIMATE

To prove the existence theorem by the fixed point principle we need to construct a mapping \( U : L^\infty (H^r) \to L^\infty (H^r) \) such that

\[
\| u^{(n)} \|_{s} = \| U^{(n)} u(0) \|_{s} \leq c_0,
\]

where \( s \) and \( c_0 > 0 \) are constants.

To guarantee this property, we will appeal to an a priori estimate which is the main object of this section. As a first step we will use the following calculation of several derivatives of the nonlinearity.

**Lemma 3.1.**

\[
(f^1 u_2, ..., x, t) = u_4 \partial_2 f^1 + u_3 (\partial_1 f^1 + \partial_x d) + u_2^2 h^1 (u_3, ..., x, t) + u_4 h^0 (u_3, ..., u, x, t)
\]

\[
(\partial_x f^1 u_1, u) = u_4 \partial_1 f^2 + u_3 [\partial_0 f^2 + 3 \partial_x (\partial_1 f^2)] + g(u_3, ..., u)
\]

\[
(f^3 u_1, ..., x, t) = u_4 \partial_3 f^3 + m(u_3, ..., u, x, t)
\]

\[
(g^j (u_1, u, x, t) u_2^j)
\]

Here all \( g, g^1, h, h^1, m, \) and \( m^1 \) are smooth functions of all of their variables.
To start to construct the iteration scheme we differentiate the equation (46) for $D(r)$ times with respect to $x$, where $f \equiv f_r$, $r = 1, \ldots, 5$. By Lemma 3.1 we obtain the following differential identities based on the particular cases $r = 1, 2, \text{ and } 3$ respectively:

\begin{align}
\partial_t u_5 &= a u_7 + (4 a_1 + \partial_2 f^1) u_6 + h^0(u_3, \ldots, x, t) u_5 + h^1(u_4, \ldots, u, x, t) \\
\text{with } D(1) &= 4, & (43) \\
\partial_t u_5 &= a u_5 + (2 a_1 + \partial_1 f^2) u_4 \\
&\quad + u_3[a_2 + \partial_0 f^2 + 3(\partial_1 f^2)_1] + g(u_2, \ldots, u) \\
\text{with } D(2) &= 2, & (44) \\
\partial_t u_5 &= a u_5 + 2 a_1 u_4 + u_3[a_2 + \partial_1 f^3)_1] + m(u_2, \ldots, u, x, t) \\
\text{with } D(3) &= 2. & (45)
\end{align}

Upon substitution of $u = Av$ with $A = (I - \partial^3)^{-1}$ for (43) and $A = (I - \partial^2)^{-1}$ for (44) and (45), we have:

\begin{align}
\partial_t v &= a A v_7 + (4 a_1 + \partial_2 f^1) A v_6 + h^0(A v_3, \ldots, A v) A v_5 \\
&\quad + h^1(A v_4, \ldots, A v) & (46) \\
\partial_t v &= a A v_5 + (2 a_1 + \partial_1 f^2) A v_4 + A v_3[a_2 + \partial_0 f^2 + 3(\partial_1 f^2)_1 - a] \\
&\quad + g(A v_2, \ldots, A v) & (47) \\
\partial_t v &= a A v_5 + 2 a_1 A v_4 + A v_3[a_2 + \partial_1 f^3)_1 - a] + m(A v_2, \ldots, A v) & (48)
\end{align}

The mappings $U$ for $f_r$ are constructed by transformations of (46)-(48) through substitution of a new variable $w$ in each coefficient as follows:

\begin{align}
\partial_t v &= a A v_7 + (4 a_1 + \partial_2 f^1(A w_2, \ldots, A w)) A v_6 \\
&\quad + h^0(A w_3, \ldots, A w) A v_5 + h^4 & (49) \\
\partial_t v &= a A v_5 + (2 a_1 + \partial_1 f^2(A w_1, \ldots, A w)) A v_4 \\
&\quad + A v_3[a_2 + \partial_0 f^2(A w_1, \ldots, A w) \\
&\quad + 3(\partial_1 f^2(A w_1, \ldots, A w))_1 - a] + g(A w_2, \ldots, A w) & (50) \\
\partial_t v &= a A v_5 + 2 a_1 A v_4 + u_3[a_2 + \partial_1 f^3)_1 - a] + m^2(A v_2, \ldots, A w) & (51)
\end{align}
where
\[
\begin{align*}
  h^4(Aw_4, ..., Aw) &= \sum_{j=0}^{2} h^{4-j}(Aw_3, ..., Aw) w_j^4 \\
  g^2(Aw_2, ..., Aw) &= \sum_{j=0}^{3} g^{2-j}(Aw_1, ..., Aw) w_j^2 \\
  m^2(Aw_2, ..., Aw) &= \sum_{j=0}^{2} m^{2-j}(Aw_1, Aw) w_j^2
\end{align*}
\]

The remaining two cases \(r=4\) and \(5\) are treated similarly. In fact they are easier. If \(f = f^4(u)_1 = f_4\) or \(f = f^5(u) = f_5\) we take following forms respectively for the mapping \(U\):
\[
\begin{align*}
  \partial_t v &= av_3 + f^4(w) w_1 & \text{if } f = f^4(u)_1 = f_4 \\
  \partial_t v &= av_3 + f^5(w) w_1 & \text{if } f = f^5(u) = f_5.
\end{align*}
\]

The next step is to estimate the corresponding solutions \(v(x, t)\) to one of Eqs. (49)-(53) via the coefficients of that equation.

**Lemma 3.2. (A Priori Estimate).** Let \(v, w\) be a pair of functions in \(C^k([0, \infty], H^N(R))\) for all \(k, N\) which satisfies one of Eqs. (49)-(53). Define
\[
\xi^* = a^{1 + 2\alpha/3}(1 - [r/3]) \exp \left\{ \frac{2}{3} \int_0^\infty a^{-1} L^2 \, dx \right\}
\]
\[
\text{if } f = f_r, \quad r = 1, ..., 5.
\]

For each integer \(\alpha \geq 0\), there exist positive and nondecreasing functions \(g^{(r)}\), \(h^{(r)}\), and \(k^{(r)}\) such that for \(t \geq 0\) those \(v\) and \(w\) fit the corresponding one in following estimates
\[
\begin{align*}
  \partial_t \int \xi^* v^2 \, dx &\lesssim (g^{(r)}(\|w\|_\infty) + \|\partial_t \xi^*\|_\infty) \|v\|_\alpha^2 + h^{(r)}(\|w\|_\infty) \|w\|_\alpha^2 \\
  &\quad + k^{(r)}(\|w\|_\infty) \quad r = 1, 2, 3 \\
  \partial_t \int \xi^* v^2 \, dx &\lesssim (g^{(r)}(\|w\|_\infty) h^{(r)}(\|w\|_\infty) \quad r = 4, 5,
\end{align*}
\]
where \(\alpha = \max(1.5 + 0.5(-1)^r, \alpha)\).
Proof. It is sufficient to show this lemma for the case \( r = 1 \). Taking \( \partial_x^r \) of Eq. (49) for some \( \alpha \geq 0 \), we obtain following equation

\[
\begin{align*}
\partial_t v_x &= aA v_{x+1} + ((4 + \alpha) a_1 + \partial_2 f^3(Aw_2, ..., Aw)) A v_{x+1} + \sum_{j=5}^{\alpha + 5} h_j(Av_j) \\
&+ q_i(Aw_4, ..., Aw) Aw_{x+4} + p_i(Aw_{x+3}, ..., Aw) \quad \text{for } f = f_1,
\end{align*}
\]

where \( h_j \) is a smooth function depending on \( Aw_{x+4}, ..., Aw \) with \( i = 4 + \alpha - j \). For \( \alpha \geq 3 \), \( p_1(Aw_{x+3}, ..., Aw) \) depends at most linearly on \( Aw_{x+3} \), while for \( \alpha = 2 \), \( p_1(Aw_{x+3}, ..., Aw) \) depends at most quadratically on \( Aw_{x+3} \). Multiplying (57) by \(-2\xi u_a\) and then integrating by parts we get

\[
\begin{align*}
\partial_t v_x^2 dx &= \int \left( -\eta - 2\xi P_2(Aw_2, ..., Aw)((\xi v_{x})_1)^2 \right) dx + \int \partial_x v_x^2 dx + s_1 + s_2 \\
&- \int \left[ (\xi v_{x}^{(\alpha + 1)})_1 (Av_{x})_1 \right]^2 dx + 2 \int \left[ (\xi v_{x}^{(\alpha + 1)})_1 Av_{x+1} \right] dx \\
&+ 2 \int \xi v_{x}^{(\alpha + 1)} \cdot (Av_{x+1})_2 dx - \int 2\xi v_{x}^{(\alpha + 1)} Av_{x} Av_{x+1} dx \\
&+ \sum_{j=5}^{\alpha + 5} \int 2\xi h_j Av_{x} v_x dx - \int 2\xi (q_1 Aw_{x+4} + p_1) v_x dx
\end{align*}
\]

where

\[
\begin{align*}
S_1 &= \int \left[ -(\xi a)_3 (Av_{x+4})^2 + (2\xi a Av_{x+4})_1 Av_{x+4} \right] dx \\
S_2 &= \int (\xi e^1)_2 (Av_{x+4})^2 - (2\xi e^1 Av_{x+4})_2 Av_{x+4} dx
\end{align*}
\]

and

\[
\begin{align*}
e^1 &= (4 + \alpha) a_1 + \partial_2 f^1(Aw_2, ..., Aw) \\
\xi &= -3(\xi a)_1 + (4 + \alpha) a_1 \xi + L_2(Aw_2, ..., Aw).
\end{align*}
\]

To obtain \( \eta = 0 \) we make the choice that

\[
\xi = a^{(5 + 2\alpha)/3} \exp \left\{ \frac{1}{\alpha} \int_0^\alpha a^{-1} L_2(Aw_2, ..., Aw) dx \right\}.
\]
We need an estimate of several derivatives of the weight function
\[ \zeta \leq B^0(|w|) \]
\[
|\partial_x \zeta|_{\infty} \leq \zeta \left[ \frac{5+2x}{3} \frac{a_1}{a} \frac{a^{-1}L_2(Aw_2, ..., Aw)}{\infty} \right]
\leq \zeta \left[ \frac{5+2x}{3} A |a_1|_{\infty} + \frac{2}{3} A \cdot \bar{F}(|Aw|_{\infty}) \right]
\leq B^1(|w|).
\] (64)

Similarly,
\[
|\partial_x^j \zeta|_{\infty} \leq B^j(|w|) \quad j = 2, 3,
\] (65)
where \( F, B^j \) are positive and nondecreasing functions.

Then for the right side of (58) one gets
\[
|s_1|_{\infty} \leq g^1(|w|_1) \|v\|_a^2 \quad \text{and} \quad |s_2|_{\infty} \leq g^2(|w|_1) \|v\|_a^2
\] (66)

\[
\left| - \int \partial_x \left[ \zeta h^{4+s_n} A v_{4+s_n} \right] d\alpha \right| + 2 \int \partial_x \left[ \zeta h^{4+s_n} A v_{4+s_n} \right] A v_{4+s_n} - 2 \int \zeta h^{4+s_n} A v_{4+s_n} d\alpha \right| 
\leq g^1(|w|_1) \|v\|_a^2
\] (67)

\[
\left| \sum_{k=1}^{4+s_n} 2 \int \zeta h^{4+s_n} A v_{4+s_n} d\alpha \right| \leq H^1(|w|_{\infty}) \|v\|_a^2
\] (68)

\[
\left| 2 \int \zeta \left( q_1 A w_{4+s_n} + p_1 \right) v_{4+s_n} d\alpha \right| \leq g^4(|w|_{\infty}) \left[ \|v\|_a^2 + \|w\|_{\infty}^2 + \|w\|_a^2 \right] + \|v\|_a^2
\] + H^2(|w|_{\infty})
\] (69)

Combining (58), (66)–(69), we prove (55), therefore, and this lemma.

Q.E.D.

We also need another lemma to help us to set up the iteration scheme.

**Lemma 3.3.** Given initial data \( \varphi(x) \) in \( H^s = \bigcap_{N<0} H^N(R) \), there exists a unique solution for each of the equations

\[
\text{Q.E.D.}
\]
DISpersive Smoothing Effects

\[
\partial_t v = -b.v - 2[b.v + b^{(2)} A v + b^{(1)} A v + b^{(0)}] \quad (70)
\]

with \(-b^{(2)} = c_{3r}, f^* + (4-2[r/2])a_1, \quad r = 1, 2, 3.\)

\[
\partial_t v = a A v + f^*(w) A v \quad (71)
\]

\[
\partial_t v = a A v + f^*(w) \quad (72)
\]

where \(b^{(0)} = b^{(0)}(Aw_1, ..., Aw)\), \(b^{(2)} = b^{(2)}(Aw_1, ..., Aw)\) and \(b^{(1)} = b^{(1)}(Aw_1, ..., Aw)\) are smooth bounded coefficients with \(w \in H^s\). The solution is defined in any time interval in which the coefficients are defined.

The proof is standard.

4. Uniqueness Theorem

**Theorem 4.1. (Uniqueness).** Let \(0 < T < \infty\). Then for \(q \in H^s(R) (s \geq s_r, \quad r = 1, ..., 5)\), there is at most one solution \(u \in L^\infty([0, T], H^s)\) of (1) with \(u(x, 0) = \varphi\) and \(f = f_r\) with \(r = 1, ..., 5\) where \(s_r = 6 - r.\)

**Proof.** As before, it suffices to prove this theorem for \(r = 1\). Assume \(u, \quad v\) are two solutions of (1) on \(L^\infty([0, T], H^s)\) with the same data. I complete the difference

\[
f_{1}(u_2, u_1, u, x, t) - f_{1}(v_2, v_1, v, x, t) = \sum_{i=1}^{2} c_{i'}(u_i - v_i) + (u - v), \quad (73)
\]

where

\[
c_{i'} = \partial_i f_{1}(\theta_i u_2 + (1 - \theta_i)v_2, ..., \theta_i u + (1 - \theta_i)v, x, t) \quad i = 0, 1, 2.
\]

As we know that \(u\) and \(v\) satisfy

\[
\partial_{x}(u - v) = a(u - v), \quad (74)
\]

multiplying (74) by \(2\xi(u - v)\) and integrating in \(x\) over \((-\infty, \infty)\), we obtain with integration by parts that

\[
\partial_{x} \left( \int \xi(u - v)^2 dx \right) + \int (\eta + 2P_{\xi}(u - v)^2) dx = I_1, \quad (75)
\]

where \(I_1 = \int \{ \partial_{x} \xi + 2\xi c_{1}^{(0)}(\xi c_{1}^{(0)} - (\xi c_{1}^{(0)})_2 - (\xi c_{1}^{(0)})_1 - (a c_{1}^{(0)})_2)(u - v)^2\} dx. \quad (76)\)
From now on the notations of $B_i$ and $g_r$ will represent positive and increasing functions. For $I_1$ we take

$$\eta = -3(a^\xi)_{1} + 2\xi L^1_2.$$  \hfill (77)

Then we find that

$$\xi = a^{-1} \exp \left\{ \int_0^1 a^{-1} L^2_2(u_2, v_2, ..., u, v, x', t) \, dx' \right\}. \hfill (78)$$

Similar to estimates of (64), for (78) we have

$$|\partial_t \xi|_\infty \leq B_1(\|u\|_{2+2} + \|v\|_{2+2}) \xi \hfill (79)$$

and

$$|\partial_x \xi|_\infty \leq B_1(\|u\|_{5} + \|v\|_{5}) \xi \hfill (80)$$

then

$$I_1 \leq g_1(\|u\|_{2} + \|v\|_{2}) \int \xi(u - v)^2 \, dx. \hfill (81)$$

Combining (75), (76), and (80) we obtain

$$\partial_t \int \xi(u - v)^2 \, dx \leq g_1(\|u\|_{6+2} + \|v\|_{6+2}) \int \xi(u - v)^2 \, dx. \hfill (82)$$

Now using the Gronwall inequality and the fact that $\xi > E^{-1} > 0$ from Lemma 2.1 and that $u(x, 0) = v(x, 0) = \varphi(x)$ we conclude from (81) that $u \equiv v$.

Q.E.D.

5. EXISTENCE THEOREM

As we stated in Section 3, we can now construct the mapping $U: L^\infty(H^r) \to L^\infty(H^r)$ by defining that $u^{(0)} = \varphi(x)$ and $u^{(n)} = U u^{(n-1)}$, $n \geq 1$, where $u^{(n-1)}$ and $u^{(n)}$ are in the positions of $w$ and $v$ in one of Eqs. (2.59)-(2.64), respectively.

Due to Lemma 3.3, $u^{(n)}$ exists and is unique in $C(0, \infty), H^N(R)$. A choice of $C_0$ such that $\|\varphi\|_\infty \leq C_0$ and the use of the a priori estimate in Section 4 shows that $\|U^n \varphi\|_\infty \leq C_0$. So we grasp the main idea of the proof of:

Theorem 5.1. (Existence). Given IVP (1) let $N \geq 6 - r$ be an integer, and $C_0 > 0$. Then there exists a time $0 < T < \infty$ depending only on $C_0$ such
that for all \( \varphi \in N^N(R) \), with \( \| \varphi \|_{\sigma r} \leq C_0 \), there exists a solution of (1) with \( f = f_r \), \( r = 1, \ldots, 5 \) and \( u(x, t) \in L^w([0, T], H^N(R)) \).

**Proof.** With an approximation procedure given in the last part of this proof, it is suffice to prove that for \( \varphi \in H^\infty \), there exists a solution \( u \) in \( L^w([0, T], H^{6+\varepsilon}) \) of (1). As before we just work on cases \( r = 1 \). We define a sequence of approximations to (2.48) and (1) with \( f = f_1 \) as

\[
\begin{align*}
\partial_t v^{(n)} &= a A v^{(n)} + (4a_1 + \partial_x f^1(A v^{(n-1)} - A v^{(n-1)})) A v^{(n)} \\
&+ O^k_0(A v^{(n-1)} - A v^{(n-1)}) A v^{(n)} \\
&+ O^k_1(A v^{(n-1)} - A v^{(n-1)}),
\end{align*}
\]

where both the initial condition the first approximation are given by

\[
v^{(0)}(x, 0) = \varphi(x) - \partial^4 \varphi(x).
\]

Equation (82) is a linear equation at each iteration which can be solved in any interval of time in which the coefficients are defined. This is shown in Lemma (3.3). By Lemma (3.2) we obtain the corresponding estimates for \( r = 1 \)

\[
\partial_x (v^{(n)})^2 dx \leq (g^m(\|v^{(n-1)}\|_w) + \|\partial_x v^{(n)}\|_w) \|v^{(n)}\|_w^2
\]

\[
+ h^m(\|v^{(n-1)}\|_w) \|v^{(n-1)}\|_w^2 + k^m(\|v^{(n-1)}\|_w),
\]

where \( \alpha^* = \max(1, \alpha) \). First choose \( \alpha = 1 \) for \( r = 1 \) and let

\[
c_0 \geq \|\varphi - \partial^4 \varphi\|_1 \geq \|\varphi\|_{\varepsilon}.
\]

Observe that for each iterate \( v^{(n)} \), \( \|v^{(n)}(\cdot, t)\| \) is continuous in \( t \in [0, t] \) and \( \|v^{(0)}(\cdot, 0)\| \leq c_0 \). Recall that from (54)

\[
\varphi^{(n-1)}(x, 0) = a^{1+2n/3} \exp \left\{ \frac{2}{3} \left[ \int_0^t -L_2^1(A v^{(n-1)} - A v^{(n-1)} dx' \right] \right\}.
\]

We estimate

\[
\begin{align*}
L_2^1(A v^{(n-1)}(x, t), ..., A v^{(n-1)}(x, t)) \\
= L_2^1(A v^{(n-1)}(x, 0), ..., A v^{(n-1)}(x, 0)) \\
+ t(\partial_t L_2^1(A v^{(n-1)}(x, \theta, t), ..., A v^{(n-1)}(x, \theta, t)))
\end{align*}
\]
by the mean value theorem with $0 < \theta_i$, $i = 1, 2$. Define

$$I^1 = I^1(x, t) = \int_0^t a^{-1} \partial_1 L_2^1(x', t) \, dx' \quad \text{and} \quad I_0^1 = I^1(x, 0)$$

we further estimate

$$|I^1 - I_0^1| = t \left| \int_0^t \left( \partial_1 \left[ a^{-1} L_2^1 \right] \right)(x', \theta_1 t) \, dx' \right|$$

$$\leq t \left| \int_0^t \left( \partial_1 (a^{-1}) L_2^1 + a^{-1} (\partial_2 L_2^1 \partial_1 A v_2^{(n-1)}) \right. \right.$$

$$\left. + \partial_1 L_2^1 \partial_1 A v_2^{(n-1)} + \partial_2 L_2^1 \partial_1 A v_2^{(n-1)} \right) \, dx' \right|$$

$$\leq t \left( \| \partial_1 a^{-1} \|_\infty \| L_2^1 \|_2 + A \| \partial_2 L_2^1 \| \cdot \| \partial_1 A v_2^{(n-1)} \| \right.$$\n
$$\left. + \| \partial_1 L_2^1 \| \cdot \| \partial_1 A v_2^{(n-1)} \| + \| \partial_2 L_2^1 \| \cdot \| \partial_1 A v_2^{(n-1)} \| \right), \quad (88)$$

where $\| \cdot \| = \| \cdot \|_L^2$. Since

$$-\partial_1 A v_2^{(n-1)} = A \left[ a v_2^{(n-1)} + (4a_4 + \partial_2 f^1) v_4^{(n-1)} + O^4 (A v_6^{(n-2)}, \ldots, \right.$$\n
$$A v_2^{(n-2)}) \n v_2^{(n-1)} + O^4 (A v_2^{(n-2)}, \ldots, A v_2^{(n-2)}) \right] \quad (89)$$

and $A v_2^{(n-1)} = (a v_4^{(n-1)})_3 - 3(a v_4^{(n-1)})_2 + 3(a v_4^{(n-1)})_1 - a v_4^{(n-1)}$ \n
$$\leq (A + 3 |a_1|_\infty + 3 |a_2|_\infty + |a_3|_\infty) \| A v_4^{(n-1)} \|$$

$$\leq 3 \sum_{j=0}^3 |a_j|_\infty \cdot \| v^{(n-1)} \| \quad (90)$$

Similarly,

$$\| A \left[ (4a_1 + \partial_2 f^1) (A v_2^{(n-2)}, \ldots, A v_2^{(n-2)}) \right] A v_6^{(n-1)} \|$$

$$\leq \| A \left[ (4a_1 + \partial_2 f^1 A v_4^{(n-1)})_3 - 2(4a_1 + \partial_2 f^1) A v_4^{(n-1)} - \right.$$\n
$$+ (4a_1 + \partial_2 f^1) A v_6^{(n-1)} \right\|$$

$$\leq \| (4a_1 + \partial_2 f^1) A v_4^{(n-1)} \| + 2 \| (4a_1 + \partial_2 f^1) A v_4^{(n-1)} \|$$

$$\leq g^1 \left( \| v^{(n-1)} \|_1 + 16 \sum_{j=0}^3 |a_j|_\infty \right) \| v^{(n-1)} \| \quad (91)$$
and
\[
\|A[O^{k,0}(A_1^{(n-2)}, \ldots, A_d^{(n-2)})v_0^{(n-1)} + O^{k,1}(A_1^{(n-2)}, \ldots, A_d^{(n-2)})]\| \\
\leq g^2(\|v^{(n-1)}\|)\|v^{(n-1)}\|,
\]
(92)

where \(g^1\) and \(g^2\) are positive and nondecreasing. Combining (88)–(92) we obtain
\[
\|\partial_t A v^{(n-1)}\| \leq B^0 \left( \|v^{(n-2)}\| + \sum_{j=0}^{3} |a_j| \right).
\]
(93)

Similarly,
\[
\|\partial_t A v^{(n-1)}\| \leq B^1 \left( \|v^{(n-2)}\| + \sum_{j=0}^{3} |a_j| \right) \|\partial_t A v^{(n-1)}\| \\
\leq B^1 \left( \|v^{(n-2)}\| + \sum_{j=0}^{3} |a_j| \right).
\]
(94)

Inserting (93)–(94) into (88), one has
\[
|I_{10} - I_{01}| \leq t \cdot L^1(\|v^{(n-2)}\|_1) \cdot M^1(\|v^{(n-2)}\|_1),
\]
(95)

where \(L^1\) and \(M^1\) are positive and nondecreasing functions. From (95) we conclude that
\[
\xi^{(n-1)}(x, t) \geq A^{(n-1) + 2k/3} \exp \left[ \left[ I_{01} - t L^1(\|v^{(n-2)}\|_1) \cdot M^1(\|v^{(n-2)}\|_1) \right] \cdot \frac{3}{2} \right],
\]
(96)

Clearly,
\[
\xi^{(n-1)}(x, 0) \leq A^{(n-1) + 2k/3} \exp [I_{01}].
\]
(97)

Since
\[
\partial_t (\log \xi^{(n-1)}) = \frac{1 + 2k}{3} \partial_n \partial_t T,
\]
(98)

With the same argument to (95)
\[
|\partial_t I_{10}| \leq \tilde{L}^1(\|v^{(n-2)}\|_1) \tilde{M}^1(\|v^{(n-2)}\|_1).
\]
(99)

It follows from (98)–(99) that
\[
|\partial_t \xi^{(n-1)}|_\infty \leq \tilde{L}^1(\|v^{(n-2)}\|_1) \tilde{M}^1(\|v^{(n-2)}\|_1)
\]
(100)

with positive and nondecreasing functions \(L^1\) and \(M^1\). Define
\[
c_1 = 1 + 3c_0 A^{+} \exp F^1(c_0),
\]
(101)

where
\[
|I_{01}| \leq F^1(\|\varphi\|_h).
\]
(102)
with $F^3$ positive and nondecreasing. Let $T^{(n)}$ be the $\|v^{(n)}(\cdot, t)\|_2 \leq c_1$, for
$0 \leq t \leq T$, $0 \leq k \leq n$. By integrating (84) over the time interval $[0, t]$, we obtain for
$0 \leq t \leq T^{(m)}$, from (96)–(97), (100) and (101), that
\[
A^{-1+2j/3} \exp\{-F(c_0) - tL^j(c_3)M^j(c_1)\} \|v^{(n)}(\cdot, t)\|^2
\leq c_0 A^{1+2j/3} \exp F(c_0) + t\left[ g'(c_3) + L^j(c_3) \right]
\cdot \left[ M^j(c_1)c_1^3 + h^{4j}(c_3)c_3^3 + k^{4j}(c_3) \right].
\tag{103}
\]
Also $j = 0, 1$. Choosing $T$ sufficiently small depending on $c_0$, $A$ but not on $n$, one concludes that for all
\[
\|v^{(n)}(\cdot, t)\|_2 \leq c_3 = 1 + 3c_0 A^2 \exp F(c_0),
\tag{104}
\]
where $s_1 = 1$ for $r = 1$.

This shows that $T^{(n)} \geq T$. Estimate (104) implies that there exists a subsequence, still denoted by $v^{(n)}$, such that $v^{(n)} \to v$ weak* in $L^\infty([0, T]; H^s(R))$, $r = 1$.

We claim that $u = Av$ is the solution we are looking for. By the Eqs. (82) and (89), $\Delta_v v^{(n)}$ is a sum of terms each of which is the product of a coefficient, bounded uniformly in $n$, and a function in $L^2([0, T]; H^{s-\ell}(R))$ bounded uniformly in $n$, so that sequence $\Delta_v v^{(n)}$ is bounded in $L^2([0, T]; H^{s-\ell}(R))$. By Aubin’s compactness theorem, there is a subsequence such that $v^{(n)} \to v$ strongly in $L^2([0, T], H^s_{\text{loc}}(R))$. Hence, for a subsequence $v^{(n)}$, there is a $v$ such that $v^{(n)} \to v$ a.e. in $x$ and $t$. It follows that for (82), $aAv^{(n)} \to aAv$ in $L^2([0, T]; L^2_{\text{loc}}(R))$ as $Av^{(n)} \to Av$ weakly in $L^2([0, T]; H^{-2})$ and $a \to a$ strongly in $L^2([0, T], H^s_{\text{loc}}(R))$. Similarly, all other terms in (82) converge to their correct limits, implying $\partial_t v^{(n)} \to \partial_t v$ in $L^2([0, T], L^2_{\text{loc}})$ and
\[
\partial_t v + (I - \partial_4^4)f^j(Av_2, ... + (I - \partial_4^4)aAv = 0.
\]

Applying $A = (I - \partial_4^4)^{-1}$ to both sides of this equation we find that (1) with $f = f_1$ is satisfied by $u = Av$.

As a second step we prove that there exists a solution $u$ in $L^\infty([a, T]; H^s(R))$ of (1) with $N \geq 5$ for $f = f_1$, where $T$ depends only on $\|\varphi\|_s$. We already know that there is a solution $u$ in $L^\infty([0, T]; H^s(R))$. It suffices to prove that the approximating sequence $v^{(n)}$ is bounded in $L^\infty([0, T]; H^{s-\ell}(R))$. Take $\alpha = N - S_1$ and consider the (84) for $\alpha \geq s_1 + 1$, where $s_1 = 6 - 1 - s_1 = 0$. By the same argument as for $\alpha = s_1 = 5$ we conclude that there exists $T^{(n)} > 0$, depending on $\|\varphi\|_{s+\alpha}$ and $A$, but independent of $n$ such that
\[
\|v^{(n)}(\cdot, t)\|_2 \leq 1 + 3\alpha \frac{1 + 2\alpha}{3} \|\varphi\|_2 \exp\{F(\|\varphi\|_{s+\alpha})\}.
\]
for $0 \leq t \leq T^{(s)}$. Thus $v$ is in $L^\infty([0, t^{(s)}]; H^s(R))$. Now denote by $0 \leq T^{(s)} \leq \infty$ the maximal number such that for all $0 < T \leq T^{(s)}$, $u = Av$ is in $L^\infty([0, T]; H^N(R))$. We claim that $T^{(s)} \leq T^{(s)}$ for all $s > s_0 + 1$. Thus a time of existence $T$ can be chosen depending only on $\|\varphi\|_{s_0+1}$. We now approximate $\varphi$ by a sequence $\varphi_j \in C_0^\infty$ such that $\|\varphi_j - \varphi\|_N \to 0$ as $j \to \infty$. Let $u_j$ be the solution to (1) with $u_j(0) = \varphi_j$. According to the above argument, there exists $T$ which is independent of $n$ but depending on $\sup_j \|\varphi_j\|_{s_0+1}$ such that $u_j$ exists on $[0, T]$ and a subsequence $U_j \to u$ in $L^\infty([0, T], H^N(R))$. Q.E.D.

As a consequence of Theorems 5.1 and 4.1 and their proofs one gets:

**Corollary 5.1.** Let $\varphi \in H^N(R)$ with $N \geq 6 - r$ and $\varphi^{(n)}$ be a sequence converging to $\varphi$ in $H^N(R)$. Let $u$ and $u^{(n)}$ be the corresponding unique solutions given by Theorems 1.2 and 1.1 in $L^\infty([0, T]; H^N(R))$ with $T$ depending only on $\sup \|\varphi^{(n)}\|_{H^N}$. Then $u^{(n)} \to u$ weak* in $L^\infty([0, T]; H^N(R))$ and strongly in $L^2([0, T]; H^{N+1}(R))$.

### 6. PERSISTENCE THEOREM

As a starting point for the a priori dispersive smoothing results that will be discussed in the next section, I need to develop some estimates for solutions of Eq. (1) in weighted Sobolev norms. The existence of these weighted estimates is often called the “persistence” of a property of the initial data $\varphi$. We show that if $\varphi \in H^L(V_{i0}) \cap H^{b-\tau}(R)$ for $L \geq 0$, $i \geq 0$ and $f = f_\tau$, then the solution $u(-\cdot, t)$ evolves in $H^L(V_{i0})$ for $t \in [0, T]$. The time interval of such persistence is at least as long as the interval guaranteed by the existence Theorem 1.2, thus it depends only upon $\|\varphi\|_{b-\tau}$. We state:

**Theorem 6.1.** (Persistence). Let $i$ and $L$ be nonnegative integers and $0 < T < \infty$. Assume that $u$ is the solution to (1) in $L^\infty([0, T]; H^{b-\tau}(R))$ with initial data $\varphi(x) = u(x, 0)$ in $H^{b-\tau}(R)$. If, in addition, $\varphi$ in $H^L(V_{i0})$ then

$$u \in L^\infty([0, T]; H^{b-\tau} \cap H^L(V_{i0}))$$

and

$$\int_0^T \int |\partial_x^{L-1} u(x, t)|^2 \eta \, dx \, dt < \infty$$

where $\eta \in V_{i-1, j, 0}$ for $i \geq 1$ and $j > 1$ and in $V_{i-1, 0}$ for $i = 0, j > 1$. 

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Proof. As before we just work on cases \( r = 1 \). We prove this theorem by induction on \( \alpha \), for \( u \in L^\infty([0, T]; H^2 \cap H^r(V_{\infty})) \) for \( 0 \leq \alpha \leq L \). We first derive formally some a priori estimates for the solution where the bounds involve only the norms of \( u \) in \( L^\infty([0, T]; H^2 \cap (V_{\infty})) \) and of \( \varphi \) in \( H^{2-r} \). One must justify these estimates; we do it by approximating \( u(x, t) \) by smooth solutions and by approximating weight functions in \( V_{\infty} \) by smooth bounded functions. According to the existence result, \( u(x, t) \) evolves in \( L^\infty([0, T]; H^{2-r}(R)) \) with \( N = \max(L, 6 - r) \). In particular, \( u_j(x, t) \) is in \( L^\infty([0, T]; H^{2-r}(R \times [0, T])) \) for \( 0 \leq j \leq N - 1 \). To obtain estimates (105) and (106) we use the following procedure. In the rigorous derivation of these estimates there are two approximations performed; we approximate general solutions by smooth solutions, and do general weight functions by bounded weight functions. The first of these procedures has already been discussed in Section 5, we will focus on the second. Given a smooth weight function \( \eta(x) \in W_{-1, 1, 6} \) with \( j > 1 \), we take a sequence \( \eta_j(x) \) of smooth bounded weight functions approximating \( \eta(x) \) from below, uniformly on any half line \((c, +\infty)\). Define the weight functions for the \( x \)th induction step by

\[
\zeta^{nd} = 4a^{(2n/3) - 1} \left( 1 + \int_0^\infty a^{-2x/3} E X^{-1} \eta^{(x')} d x' \right) \cdot E X. \tag{107}
\]

Then \( \zeta^{nd} \) are bounded weight functions which approximate a desired weight function \( \zeta \in W_{000} \) from below, uniformly on compact sets. These weight functions are designed to satisfy the usual relations

\[
0 < \zeta^{nd} \leq \left[ 3(\zeta^{nd})_1 - 2a(\zeta^{nd})_2 + 2(\zeta^{nd})_2 \right] \quad \text{for} \quad x \geq 0. \tag{108}
\]

To prove the \( x \)th induction step we start from the main identity (23) with \( \eta \) and \( \zeta \) given by \( \eta^{nd} \) and \( \zeta^{nd} = \zeta^{nd} \), and \( R = O^\ast \)

\[
\partial_t \int \zeta^{nd} u^2 \, dx + \int (\eta^{nd} + 2\zeta^{nd} P_2) u^2 \, dx = \int (\partial_t u^2 + R) \, dx, \tag{109}
\]

where \( P_2 \geq 0 \), the second term in (109) is nonnegative as is known. From (2.7)

\[
\theta/\zeta^{nd} = 1/\zeta^{nd} \left\{ \partial_t \zeta^{nd} - (a_1 \zeta^{nd})_2 + \alpha(a_1 \zeta^{nd})_2 - \frac{\alpha}{2} \left( a_2 \zeta^{nd} \right) \right. \\
- \gamma \left( \left( \partial_2 f \right)_1 + \partial_1 f \right) + \left( \zeta^{nd} \partial_2 f \right)_2 \right\}.
\]
Since
\[ \frac{\partial_x \zeta^\delta}{\zeta^\delta} = \frac{1}{\zeta^\delta} \left\{ \frac{1}{3} (a^{(2 \alpha \lambda)} - 1) \left( \frac{1}{3} \right) + \frac{1}{3} (a^{-1} \eta^\delta) \right\} \]
\[ = \partial_x \left\{ \frac{1}{3} (a^{(2 \alpha \lambda)} - 1) \log(1/a^\delta) \right\} + \frac{1}{\zeta^\delta} \frac{1}{3} (a^{-1} \eta^\delta), \]
and estimates
\[ |\partial_x \zeta^\delta| \cdot (\zeta^\delta)^{-1} \leq g_i'(\|u\|_{L^\infty}) \quad r = 1 \]
\[ \left| \int_0^T \partial_x [a^{-1} L_2] dx' \right| \leq B_1(\|u\|_{L^\infty}) \]
\[ \left| \int_0^T \partial_x (a^{(2 \alpha \lambda)} - 1) \left( \frac{1}{3} \right) + \frac{1}{3} (a^{-1} \eta^\delta) \right| \leq B_2(\|u\|_{L^\infty}) \left( \int_x^\infty \eta^\delta dx' \right) \]
then
\[ |\partial_x \zeta^\delta| \cdot (\zeta^\delta)^{-1} \leq g_i'(\|u\|_{L^\infty}) \]
where \( g_i \) are positive and nondecreasing functions with \( i = 1, ..., 4 \) and \( r = 1 \).
Combining (111)-(113) it follows that \( \theta(\zeta^\delta)^{-1} \) is a function bounded in \( x \) and \( t \) by a constant \( C \), which depends only on \( \|\varphi\|_{L^\infty} \) and \( T \) which is in particular independent of \( \delta \). Thus, (107) yields
\[ \partial_x \int \zeta^\delta u_2^2 - \int \eta^\delta u_1^2 + \int R \leq C \int \zeta^\delta u_2^2 \]
To use Gronwall's argument to conclude the induction step, it thus suffices to prove
\[ \int |R| \leq C \left( \int \zeta^\delta u_2^2 + 1 \right), \]
where the constant \( C \) depends upon \( T \), \( \|\varphi\|_{L^\infty} \), and the norm of \( \varphi \) in \( H^\delta \left( V_{(0)} \right) \) only. The first induction step is to obtain a weighted estimate for \( \alpha = 0 \). By applying the mean value theorem to the difference \( f(u_2, u_1, u, x, t) - f(0, 0, 0, x, t) \), we have
\[ f(u_2, u_1, u, x, t) = \sum_{j=0}^2 u_j g_j + h, \]
where \( h = f(0, 0, 0, x, t), g_j, f \).
We now start with the main identity with $\alpha = 0$ and the above form

$$
\partial_t \int \xi^{0,\alpha} u^2 \, dx = \partial_t \int \xi^{0,\alpha} u^2 \, dx + \int 2\xi^{0,\alpha} u u_\alpha \, dx + \int 2\xi^{0,\alpha} u \left( \sum_{j=0}^2 u_j g_j + f(0, 0, 0, x, t) \right) \, dx \tag{117}
$$

we define

$$
\xi^{0,\alpha} = \frac{1}{4} a^{-1} \left( 1 + \int_{x}^{+\infty} \text{Ex}^{-1} \eta'(x') \, dx' \right) \cdot \text{Ex} \tag{118}
$$

with the property that

$$
0 < \eta' \leq -\left[ 3(\xi^{0,\alpha})_1 - 2a \xi^{0,\alpha}_1 - 2\xi^{0,\alpha} L_2 \right], \quad \text{for} \quad \alpha = 0. \tag{119}
$$

We know that (117) can be rewritten as

$$
\partial_t \int \xi^{0,\alpha} u^2 \, dx \leq C \int \xi^{0,\alpha} u^2 \, dx + \int |R| \, dx, \tag{120}
$$

and it suffices to prove

$$
\int |R| \, dx \leq C \left( \int \xi^{0,\alpha} u^2 \, dx + 1 \right).
$$

Since $R = h \cdot 2\xi^0 u$, so

$$
\int h \cdot \xi^0 u \, dx \geq -c' \left( \int h^2 \xi^0 \, dx \right)^{1/2} - c
$$

or

$$
\int |R| \, dx \leq c' \left( \int h^2 \xi^0 \, dx \right)^{1/2} \leq c. \tag{121}
$$

Combining (120)–(121) yields

$$
\partial_t \int \xi^0 u^2 \, dx \leq C \int \xi^0 u^2 \, dx + C.
$$

One applies Gronwall's lemma to conclude

$$
\int \xi^0 u^2 \, dx \leq C \int \xi^0 \phi^2 \, dx + C
$$

for $0 \leq t \leq T$. As $C$ does not depend on $\delta > 0$, the weighted estimate (6) remains true in the limit for $\delta \to 0$. 
To estimate \( \int R \, dx \) for \( \alpha > 0 \) we use Lemma 3.2. If \( v_p \leq \alpha - 2 \) we perform once by integration by parts, and then use the induction hypothesis to bound the resulting terms by the quantity

\[
C \left[ \left( \int \xi^\alpha u''_{n+1} \, dx \right)^{1/2} + \left( \int \xi^\alpha u''_n \, dx \right)^{1/2} \right] \left( \int \xi^\alpha u^2_{n-1} \, dx \right)^{1/2}. \tag{122}
\]

The expression (122) is bounded in the previous induction steps. Suppose that \( \alpha - 1 = v_p = v_{p-1} \) \( > v_p = v_{p-1} \), then the term (33) has the estimate

\[
\left| \int b \xi^\alpha u'_{n-1} \, dx \right| \leq \|bu'_{n-1}\|_{\infty} \left( \int \xi^\alpha u^2_{n} \, dx \right)^{1/2} \left( \int \xi^\alpha u^2_{n-1} \, dx \right)^{1/2}.
\]

From (35) and (36), the multiplicity \( l \geq 2 \) occurs only when \( \alpha \leq 7 - r \) for \( f = f_r \) with \( r = 1, 4 \). Thus for \( \alpha > 7 - r \) a differential inequality of the necessary form is obtained.

Furthermore, for \( \alpha \leq 6 - r \), \( \|u_{n-1}\|_{\infty} \) is bounded by hypothesis, and again the estimate is complete. Finally, \( \alpha = 7 - r \) we use the Sobolev lemma

\[
\left| \int b \xi^\alpha u''_{n-1}u_{n-2} \, dx \right| \leq \|bu''_{n-1}\|_{\infty} \|u_{n-1}\|_1 \left( \int \xi^\alpha u^2_{n-1} \, dx \right)^{1/2} \leq \|b\|_{\infty} \|u_{n-1}\|_1 \left( \int \xi^\alpha u^2_{n-1} \, dx \right). \tag{123}
\]

Using estimates (122) and (6)–(123) and by applying the Gronwall's argument, one obtains for any \( 0 \leq t \leq T \)

\[
\int \xi^\delta(t) u_x^2(t) \, dx + \int_0^t \eta^\delta u''_{n+1}(t) \, dx \, dt \leq c_1 \exp\{c_2 t\} \left( \int \varphi^\delta_x \xi^\delta \, dx + 1 \right). \tag{124}
\]

The constants \( c_i \) are independent of \( \delta \) so that letting \( \delta \to 0 \) the desired estimate (105) and (106) are obtained.

7. Gain of Regularity

In this section we state and prove our main theorem, which tells us that if the initial data \( u(x, 0) \) decays faster than polynomially on \( \mathbb{R}^+ \), and possesses certain minimal Sobolev regularity, then the solution \( u(x, t) \in C^\infty \) for \( t > 0 \).
The main theorem also quantifies the relationship between the rate of decay and the amount of gain regularity as follows:

One has a gain of \( \mathcal{L} \) derivative, at the expense of \( \mathcal{L} \) powers of \( x \) in the weight functions as \( x^{\mathcal{L}^2} \), which means that the requirement is that \( \|x\|^{\mathcal{L}^2} u(x, t) \in L^2(R^+) \). In the main theorem we take \( 7 - r \leq \alpha = l + 6 - r \leq L + 6 - r \). The weight functions in the cases \( r = 1, \ldots, 5 \) are chosen dependent upon the relationship between \( \alpha \) and \( L \).

(1). If \( \alpha \leq L + 6 - r \) we take any \( \eta \in V_{L - \alpha + 5 - r, \alpha - 6 - r} \) to get

\[ \zeta \in V_{L - \alpha + 6 - r, \alpha - 6 - r} \]  \hspace{1cm} (125)

(2). In case \( \alpha = L + 6 - r \) we take any \( \eta \in V_{-1, -1, -1} \) to get

\[ \zeta \in V_{0, L, L}. \] \hspace{1cm} (126)

To prove the main theorem, a crucial step is to estimate the error terms in the main identities. Therefore, we begin with

**Lemma 7.1. (Estimate of Error Terms).** If \( 7 - r \leq \alpha = l + 6 - r \leq L + 6 - r \) and the weight functions are chosen as in (125)-(126), then

\[ \left| \int_0^T (\theta u^2 + R) \, dt \right| \leq c, \] \hspace{1cm} (127)

where \( c \) depends only on the norms of \( u \) in

\[ L^\infty(H^\beta(V_{L - \beta + 6 - r, \beta - 6 + r} \cap L^2(H^\beta + 1(V_{L - \beta + 5 - r, \beta - 5 + r} \cap L^2))) \] \hspace{1cm} (128)

for \( 6 - r \leq \beta \leq \alpha - 1 \), and on the norms of \( u \) in \( L^\infty(H^\beta(V_{L, 0, 0})). \)

**Proof.** As before we work just on cases \( r = 1 \). Recall from Lemma 2.2 that

\[ R = \tilde{R} + R, \quad \text{where} \quad \tilde{R} = \sum_{j=3}^n \frac{1}{j!} a_j \zeta_{3, \ldots, j} \cdot 2\zeta \] \hspace{1cm} (129)

and \( R \) is the sum of terms of the form

\[ \zeta \partial_x \partial_x' \cdots \partial_y \cdot f u_{x_1} u_{x_2} \cdots u_{y_1} u_{y_2} \] \hspace{1cm} (130)

with other conditions.
We must estimate both $R$ and $\theta$. We begin with an estimate for $R$.

(1) *Estimate for $R$. Assuming $v_{p-1} \leq \alpha - 2(p \geq 2)$ or else $p = 1$. It suffices to do it for a term of $R$ of the form (130). By the induction hypothesis, $u$ is bounded in

$$L^\alpha(H^\beta(V_L^{-(\beta - 6 + r)^+}, (\beta - 6 + r)^+))$$

for $0 \leq \beta \leq \alpha - 1$. It follows from Lemma 7.3, which will be given later, that

$$\sup_x \sup_t \xi u_\beta^2 < \infty \quad \text{for} \quad 0 \leq \beta \leq \alpha - 2$$

(131)

and $\xi \in V_L^{-(\beta - 5 + r)^+}, (\beta - 5 + r)^+,(\beta - 5 + r)^- \in$ the (130). We estimate $u_\alpha$ and $u_\beta$ using the weight $L^2$ bounds

$$\int_0^T \int \xi u_\beta^2 \, dx \, dt \quad \text{for} \quad \xi \in V_L^{-(\beta - 6 + r)^+}, (\beta - 6 + r)^+,(\beta - 7 + r)^+,$$

(132)

and the same with $v_{p}$ replaced by $\alpha$. It is sufficient to check the powers of $t$ and the powers of $|x|$ as $x \to \pm \infty$, respectively.

In the term (130) the factor $\xi$ contributes the power $t^{\alpha - 6 + r}$, according to (125). Each factor $u_\alpha$ uses up the power $t^{(\beta - 5 + r)^+}$ in the estimate (131) for $j = 1, 2, ..., p - 1$. Factors $u_\alpha$ and $u_\beta$ use up the powers $t^{(\beta - 7 + r)^+}$ and $t^{(\beta - 7 + r)^-}$ in estimate (132). Thus, the difference in the exponent of $t$ in expression (130) is

$$M = (\alpha - 6 + r) - \frac{1}{2} \sum_{j=1}^{p-1} (v_j - 5 + r)^+$$

$$- \frac{1}{2} (v_{p-1} - 7 + r)^+ - \frac{1}{2} (\alpha - 7 + r)^+. \quad (133)$$

I claim that $M \geq 0$, so that the extra power of $t$ has a positive exponent and thus the term is bounded by a constant. To prove that claim, I argue as follows. Clearly

$$2M = 2(\alpha - 2(6 - r)) - \sum_{j=1}^{p-1} (v_j - 5 + r)^+$$

$$- (v_{p-1} - 7 + r)^+ - (\alpha - 7 + r)^+. \quad (134)$$

Let $q$ be the largest index less than or equal to $p - 1$ such that

$$v_q \leq 5 - r, \quad q \in \{1, 2, ..., p\}, \quad q = q_2 + q_1 + q_0,$$
where $q_2$ of the derivatives come from $u_1$, $q_1$ from $u_1$, $q_0$ from $u_0$. Then
$v_1 + \cdots + v_p \geq 3q_2 + 2q_1 + q_0$. Now by (2.20)
$$
\begin{align*}
\alpha &= v_1 + v_2 + \cdots + r' - 2p_2 - p_1 \\
&\geq 3q_2 + 2q_1 + q_0 + v_{q+1} + \cdots + v_p - 2p_2 - p_1.
\end{align*}
$$
(135)

Hence, if $q < p$
$$
2M = 3q_2 + 2q_1 + q_0 - 2p_2 - p_1 + (v_p - 5 + v)
+ (5 - r)(p - 1 - q) - (v_p - 7 + r)^+ \\
\geq p - (4 - r) + (2 - r)(p - q) + (p_1 + p_0 - q_1 - q_0)
+ (p_0 - q_0).
$$
(136)

It follows from (136) that
$$
2M \geq p - (4 - r) + (2 - r) = p - 2 \geq 0,
$$
for $r = 1$, if $p \geq 2$. If $p = 1$ when $q < p$ then
$$
2M \geq \alpha - (5 - r) - (v_p - 7 + r)^+ \geq 0 \text{ because } v_p \leq \alpha + 2
$$
(137)

Finally, if $v_p \leq 5 - r$, then $p = q$ and $2M = \alpha - (5 - v) > 0$.

Similarly, the difference in the powers of $|x|$ as $x \to -\infty$ is
$$
N = (L - \alpha + 6 - r) - \frac{1}{2} \sum_{j=1}^{p-1} (L - v_j + (5 - r)^+) \\
- \frac{1}{2}(L - \alpha + 6 - r) \leq 0,
$$
(138)
so that the extra power of $|x|$ has a negative exponent and thus the term is bounded by a constant.

The difference in the powers of $|x|^{-1}$ as $x \to +\infty$ is
$$
2\lambda = \alpha - (6 - r) - \frac{1}{2} \sum_{j=1}^{p-1} (v_j - 5 + r)^+ \\
- \frac{1}{2}(v_p - 6 + r)^+ - \frac{1}{2}(\alpha - 6 + r).
$$
(139)

We claim that $\lambda \geq 0$. At first we assume that $v_p \geq 6 - r$, and then we write
$$
2\lambda = \alpha - v_p - \sum_{j=1}^{p-1} (v_j - 5 + r)^+
$$
when $q < p,
$$
2\lambda \geq (2 - r)(p_2 - q_2) + (3 - r)(p_1 - q_1)
+ (4 - r)(p_0 - q_0) + p - (5 - r).
$$
(141)
It follows from (141) that $2\lambda \geq (2-1)(p-q) + p - 4 \geq p - 2 \geq 0$, for $r = 1$ and $p - 2 = q$. If $p - 1 = q$, then $2\lambda = \pi - v_p \geq 0$.

Finally, if $v_p < 5 - r$, then $2\lambda = \pi - (6 - r) \geq 1 > 0$.

(2) **Estimate for $R$.** Similarly,

\[ 2M \geq \pi - 1 - (v_p - 7 + r)^+ \geq 0; \]

\[ -2N \geq \pi - (6 - r) - (v_p - 6 + r)^+ \geq 0; \]

(142)

\[ 2\lambda \geq \pi - (6 - r) - (v_p - 6 + r)^+ \geq 0. \]

(3) **Estimate for the term of $R$ for $p = 0$.** This is the term of $\xi u \partial_x f$, where all the differentiation acts on the explicit $x$ variable of $f$.

By (116)

\[ \int \int \xi u \partial_x^2 f \, dx \, dt = \int \int \xi u \left( \sum_{j=0}^{2} u_j \partial_x^j g_j + \partial_x^2 h \right) \, dx \, dt \]

(143)

and it is also known that each $\partial_x^j g_j$ is bounded. Hence

\[ \int \int \xi u \partial_x^2 f \, dx \, dt \leq c \left( \int \int \xi u \, dx \, dt \right)^{1/2} \times \left( \sum_{j=0}^{2} \int \int \xi u_j \, dx \, dt \right)^{1/2} \left( \int \int (\partial_x^2 h)^2 \, dx \, dt \right)^{1/2} \]

(144)

is finite because of $\text{(A_4)}$.

(4). The analysis of all the terms of $R$ will be completed with the case of $v_{p-1} \leq \pi - 1$ with $p \leq 2$. In the case I get from Lemma 2 that $p + v_p + v_{p-1} \leq \pi + 6$, which implies

\[ 2(\pi - 1) \leq \pi + 5 - r. \]

(145)

Then the corresponding values for $\pi$ and $p$ are determined as follows:

$\pi = 6$ and $p = 2$. Thus the only term of this form is $\xi \partial_x f \partial_x u_0$. The term $\xi \partial_x f \partial_x u_0$ is integrated by parts once, leading to a term of the form

$\{ c_1 \xi \partial_x (\partial_x) + c_2 \xi \partial_x (\partial_x) \} u_{6-r} , \pi \in V_{L-1,1,1} , \text{ and } u \text{ is bounded in } L^\infty (H^{b-\pi} (V_{L,0})) \cap L^2 (H^{\pi - \pi} (V_{L-1,1,0})).$

For $\pi < 1$ we estimate from the interpolation inequality of $|v|_3 \leq |v|_2^{1/2} |v'|_2^{1/2}$ that
\[
\int_0^T \int_{-\infty}^1 t |x|^{L-1} |u_{6-1}|^3 \, dx \, dt \\
\leq T (\sup_{t, x} |u_{5-1}|) \left( \int_1^\infty (|x|^{L-1} |u_{5-1}|^2 \, dx \, dt \right)^{1/4} \\
\times \left( \int_1^\infty (|x|^{L-1} |u_{7-1}|^2 \, dx \, dt \right)^{3/4}, (146)
\]

which is bounded.

For \( x > 1 \) according to Lemma 0.7 which will be given later,
\[
\int_2^\infty t |x|^{-\beta} |u_{6-1}|^3 \, dx \, dt \leq C \left( \int_1^\infty |x|^{-\beta-1} |u_{7-1}|^2 \, dx \right)^{1/4} \\
+ \left( \int_1^\infty |x|^{-\beta-3} |u_{6-1}|^2 \, dx \right)^{1/4} \\
\times \left( \int_1^\infty |x|^{-3\beta+1/5} |u_{6-1}|^2 \, dx \right)^{5/4}, (147)
\]

where \( \beta \leq 1 \). Multiplying by time \( t \) and integrating over \([0, T]\) we obtain
\[
\int_0^T \int_1^\infty |x|^{-\beta} |u_{6-1}|^2 \, dx \, dt \\
\leq \varepsilon \int_0^T \int_1^\infty |x|^{-\beta-1} (u_{5-1}^2 + u_{7-1}^2) \, dx \, dt \\
+ c(\varepsilon) \int_0^T \int_1^\infty t |x|^{-3\beta+1/5} u_{6-1}^2 \, dx \right)^{5/3}, (149)
\]

which is also bounded since \( u \) is bounded in \( L^\infty(H^{\beta-\epsilon}(V_{L0})) \cap L^2(H^{1+\epsilon}(V_{2, -1, 0})) \). Therefore, we complete the estimate of \( R \).

(5) Estimate for \( \theta u_{5-1}^2 \). One can show that \( \theta \) involves derivatives of \( u \) only up to order 4 and is bounded in \( L^\infty(H^1(R)) \) hence that \( \theta u_{5-1}^2 \) is a sum of the same type we have already encountered in \( R \), so that its integral can be bounded in the same manner. Indeed, combining (26), (111)–(113) we obtain \( |\theta| \leq \delta(|\| | \|) \), for some positive and nondecreasing functions \( \delta \).

Now by the estimates (1)–(5) we complete the proof of this lemma.

The next lemma has been used to estimate the term \( \xi F \xi^2 u_{6-1}^2 \) in the proof of Lemma 7.1.
**Lemma 7.2.** Let $\beta \leq 0$ and $\delta \leq 0$. Whenever $(4\beta - \delta)/5 \leq 0$, there exists a constant $c > 0$ such that for $u \in H^1(R)$

$$
\int_2^\infty t \left| x \right|^{-\beta} \left| u \right|^3 \, dx \, dt \leq C \left( \left( \int_1^\infty \left| x \right|^{3} \left| \tilde{u} \right|^2 \, dx \right)^{1/4} 
+ \left( \int_1^\infty \left| x \right|^{-\delta - 2} \left| u \right|^2 \, dx \right)^{1/4} 
\times \left( \int_1^\infty \left| x \right|^2 \left| u \right|^2 \, dx \right)^{5/4} \right).
$$

(150)

With the help of the above lemmas we will come to our main theorem:

**Main Theorem 1.2.** See Section 4 for the content.

**Proof.** We will use induction on $\alpha$, begin with $\alpha = 6 - r$, the estimate will be applied to smooth approximations of the solution. Let $u$ be a solution satisfying (20). The equation itself implies that $\tilde{\eta}, u \in L^\infty(H^{3 - \gamma}(V_{L0}))$. Hence $u$ is a weakly continuous function of $t$ with values in $H^{6 - \gamma}(V_{L0})$. In particular, $u(\cdot, t) \in H^6 - \gamma(V_{L0})$ for every $t$. Let $t_0 \in (0, T)$ and let $\{u^n(\cdot)\}$ be a sequence of functions in $C^\infty_c(R)$ which converges to $u(\cdot, t_0)$ strongly in $H^6 - \gamma(V_{L0})$. Let $u^n(x, t)$ be the unique solution of (36) with the correct initial data at $t = t_0$.

By the existence Theorem 5.1, it is guaranteed to exist in a time interval $[t_0, t_0 + \delta]$, where $\delta > 0$ does not depend on $n$. According to the persistence Theorem 6.1 we have

$$
u^n \in \right\{ H^\infty(V_{L0}) \right\} \cap (\tilde{V}^\infty_{L - 1, 1, 0})
$$

(151)

with a bound that depends only on the norm of $\varphi^n$ in $H^6 - \gamma(V_{L0})$. Furthermore, Theorem 6.1 guarantees the nonuniform bounds

$$
\sup_{[t_0, t_0 + \delta]} \sup_{x \in \mathbb{R}} (1 + \left| x \right|)^{k'} \left| \partial_x^\nu u^n(x, t) \right| < \infty
$$

(152)

for each $n$, $k'$ and $\nu$.

The main identity (23) and the estimate (127) are therefore valid for each $u^n$ in the interval $[t_0, t_0 + \delta]$. The multiplier $\eta$ may be chosen arbitrarily in its weight class (125) or (126) and then $\zeta = \zeta^n$ is defined by (22) and (25) and depends on $n$. However, the constants $c_1$ and $c_2$ in (10)–(11) are independent of $n$. From the main identity (23) and the estimate (127) we have

$$
\sup_{[t_0, t_0 + \delta]} \int \zeta^n(u^n)^2 \, dx + \int_0^{t_0 + \delta} \int \eta(u^n)^2 \, dx \, dt \leq \infty
$$

(153)
by (127), here $c$ is independent of $n$. This estimate (153) is proved by induction for $\alpha = 7 - r, 8 - r, \ldots$.

Thus $u^{(n)}$ is also bounded in

$$L^\infty(H^\alpha(V_{L-\alpha + (4 - r), \alpha - (6 - r), \alpha - (4 - r)})$$

$$\cap L^2(H^{\alpha + (4 - r), \alpha - (5 - r), \alpha - (6 - r)})$$

(154)

for $\alpha \geq 7 - r$. Since $u^{(n)} \to u$ in $L^\infty(H^{\alpha - \eta})$, by Corollary 5.1, it follows that $u$ belongs to the space (154). Since $\delta$ is fixed, this result is valid over the whole interval $[0, T]$. In the last step of the induction we have $\alpha = L + 6 - r, \xi \in V_{AL}$ and $\eta \in V_{-1, L+1, L}$, this completes the proof the main theorem.

Q.E.D.

**Lemma 7.3.** For $\xi$ in $V_{m0}$ and $i \geq 0, m \geq 0$, there exists a constant $c$ such that for $u$ in $H^i(V_{m0})$

$$\sup_{x \in R} |\xi u^2| \leq c \int R |\xi|^2 + |\partial u|^2 \, dx.$$  (155)

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**REFERENCES**


