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Homotopy analysis method for solving multi-term linear and nonlinear diffusion–wave equations of fractional order

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ABSTRACT

In this paper we have used the homotopy analysis method (HAM) to obtain solutions of multi-term linear and nonlinear diffusion–wave equations of fractional order. The fractional derivative is described in the Caputo sense. Some illustrative examples have been presented.

1. Introduction

Fractional equations, both partial and ordinary ones, have received more attention in recent years. Various phenomena in physics, like diffusion in a disordered or fractal medium, or in image analysis, or in risk management have been modeled by means of fractional equations.

There has been some attempt to solve linear problems with multiple fractional derivatives (the so-called multi-term equations) [1,2]. Not much has been done for the nonlinear problems and only a few numerical schemes have been proposed to solve the nonlinear fractional differential equations. More recently, applications have included classes of nonlinear multi-order fractional derivative equations and this motivates us to develop a numerical scheme for their solutions [3,4]. Numerical and analytical methods have included Adomian decomposition method (ADM) [5,6], variational iteration method (VIM) [7], and homotopy perturbation method (HPM) [7].

In this paper, the homotopy analysis method [8,9] is applied to solve multi-term fractional diffusion–wave equations

$$P(D)u(\bar{x},t)=\sum_{i=1}^{n}N_i\frac{\partial^2 u}{\partial x_i^2}+\varphi(\bar{x},t)u^m(\bar{x},t),$$

where

$$P(D)\equiv D_t^{s_1}-\sum_{j=2}^{r}\lambda_j D_t^{s_j},$$

$$r\geq 2,\quad r\in N,\quad 0<s_r<s_r-1<\cdots<s_2<s_1<2,\quad m=0,1,2,\ldots,N_i(\bar{x},t)\in C_\alpha.$$

$D_t^{s_j}$ are Caputo fractional derivatives.

The homotopy analysis method (HAM) [8,10] provides an effective procedure for explicit and numerical solutions of a wide and general class of differential systems representing real physical problems. Based on homotopy of topology, the

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validity of the HAM is independent of whether or not there exist small parameters in the considered equation. Therefore, the HAM can overcome the foregoing restrictions and limitations of perturbation techniques so that it provides us with a possibility to analyze strongly nonlinear problems. This method has been successfully applied to solve many types of nonlinear problems by others [11–17].

The paper has been organized as follows. Basic definitions are given in Section 2. In Section 3 the homotopy analysis method is described. Section 4 presents some illustrative example equations.

2. Basic definitions

Definition 1. A real function $f(t), t > 0$ is said to be in the space $C_\alpha, \alpha \in \mathfrak{R}$ if there exists a real number $p\,(> \alpha)$, such that $f(t) = t^p f_1(t)$ where $f_1 \in C[0, \infty]$. Clearly $C_\alpha \subset C_\beta$ if $\beta \le \alpha$.

Definition 2. A function $f(t), t > 0$ is said to be in the space C_α^m, $m \in N \bigcup \{0\}$, if $f^{(m)} \in C_\alpha$.

Definition 3. The left sided Riemann–Liouville fractional integral of order $\mu > 0$, [18,19] of a function $f \in C_\alpha, \alpha \ge -1$ is defined as:

$$I^\mu f(x,t) = \frac{1}{\Gamma(\mu)} \int_0^t \frac{f(x,\tau)}{(t-\tau)^{1-\mu}} \mathrm{d}\tau, \quad \mu > 0, t > 0,$$
$$I^0 f(x,t) = f(x,t). \tag{1}$$

Definition 4. The (left sided) Caputo fractional derivative of f, $f \in C_{-1}^m$, $m \in \mathbb{N} \bigcup \{0\}$, is defined as [1,20]:

$$D_*^\mu f(x,t) = \begin{cases} \left[I^{m-\mu} f^{(m)}(x,t)\right] & m-1 < \mu < m, \quad m \in \mathbb{N}, \\ \dfrac{\partial^m}{\partial t^m} f(x,t) & \mu = m. \end{cases} \tag{2}$$

Note that [18,19]

(i) $$I_t^\mu f(x,t) = \frac{1}{\Gamma(\mu)} \int_0^t \frac{f(x,s)}{(t-s)^{1-\mu}} \mathrm{d}s, \quad \mu > 0, t > 0,$$

(ii) $$D_{*t}^\mu f(x,t) = I_t^{m-\mu} \frac{\partial^m f(x,t)}{\partial t^m}, \quad m-1 < \mu < m. \tag{3}$$

(iii) $$I^\mu t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\mu+1)} t^{\gamma+\mu}, \quad \mu > 0, \gamma > -1, t > 0, \tag{4}$$

(iv) $$I^\mu D_*^\mu f(x,t) = f(x,t) - \sum_{k=0}^{m-1} \frac{\partial^k f(x,0^+)}{\partial t^k} \frac{t^k}{k!}, \quad m-1 < \mu \le m, m \in \mathbb{N}. \tag{5}$$

Definition 5. A two-parameter Mittag-Leffler function is defined by the series expansion [21]:

$$E_{\alpha,\beta}(t) = \sum_{n=0}^\infty \frac{t^n}{\Gamma(\alpha n + \beta)}, \quad \alpha, \beta > 0.$$

Then kth-derivative of Mittag-Leffler function is

$$\begin{aligned} E_{\alpha,\beta}^{(k)}(t) = \frac{\partial^k}{\partial t^k} E_{\alpha,\beta}(t) &= \frac{\partial^k}{\partial t^k} \sum_{j=0}^\infty \frac{t^j}{\Gamma(\alpha j + \beta)} = \sum_{j=0}^\infty \frac{j(j-1)\dots(j-(k-1))t^{j-k}}{\Gamma(\alpha j+\beta)} \\ &= \sum_{j=k}^\infty \frac{j(j-1)\dots(j-(k-1))t^{j-k}}{\Gamma(\alpha j+\beta)} = \sum_{j=k}^\infty \frac{j!}{(j-k)!} \frac{t^{j-k}}{\Gamma(\alpha j+\beta)} \\ &= k! \sum_{j=k}^\infty \binom{j}{k} \frac{t^{j-k}}{\Gamma(\alpha j+\beta)}. \end{aligned} \tag{6}$$

3. Analysis of the homotopy analysis method

We apply the homotopy analysis method to fractional multi-term diffusion–wave equations

$$P(D)u(\bar{x},t)=\sum_{i=1}^{n}N_i\frac{\partial^2 u}{\partial x_i^2}+\varphi(\bar{x},t)u^m(\bar{x},t), \quad (7)$$

$$u(\bar{x},0)=f(\bar{x}), \quad (8)$$

$$u_t(\bar{x},0)=g(\bar{x}), \quad (9)$$

where

$$P(D)\equiv D_t^{s_1}-\sum_{j=2}^{r}\lambda_j D_t^{s_j},$$

$$0<s_r<s_r-1<\cdots<s_2<s_1<2.$$

Let us consider the following fractional differential equation

$$\mathcal{N}[u(\bar{x},t)]=0, \quad (10)$$

where $\mathcal{N}$ is a nonlinear fractional differential operator, $\bar{x}$ and t are independent variables, $u(\bar{x},t)$ is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in a similar way. By means of generalizing the traditional homotopy method, Liao [10] constructs the so-called zero-order deformation equation

$$(1-p)\mathcal{L}[\phi(r,t;p)-u_0(\bar{x},t)]=p\,\hbar H(\bar{x},t)\mathcal{N}[\phi(r,t;p)], \quad (11)$$

where $p\in[0,1]$ is the embedding parameter, $h\neq 0$ is a non-zero auxiliary parameter, $H(\bar{x},t)\neq 0$ is non-zero auxiliary function, $\mathcal{L}=D_t^{s_1}$ $(n-1<s_1\leq n)$ is an auxiliary linear operator with the following property

$$\mathcal{L}[\phi(\bar{x},t)]=0 \quad \text{when } \phi(\bar{x},t)=0. \quad (12)$$

$u_0(\bar{x},t)$ is an initial guess of $u(\bar{x},t)$, $u(r,t;p)$ is a unknown function, respectively. It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when $p=0$ and $p=1$, it holds

$$\phi(r,t;0)=u_0(\bar{x},t), \qquad \phi(r,t;1)=u(\bar{x},t), \quad (13)$$

respectively. Thus, as p increases from 0 to 1, the solution $\phi(r,t;p)$ varies from the initial guesses $u_0(\bar{x},t)$ to the solution $u(\bar{x},t)$. Expanding $\phi(r,t;p)$ in Taylor series with respect to p, we have

$$\phi(r,t;p)=u_0(\bar{x},t)+\sum_{m=1}^{+\infty}u_m(\bar{x},t)p^m, \quad (14)$$

where

$$u_m(\bar{x},t)=\frac{1}{m!}\left.\frac{\partial^m\phi(r,t;p)}{\partial p^m}\right|_{p=0}. \quad (15)$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are so properly chosen, the series (14) converges at $p=1$, then we have

$$u_m(\bar{x},t)=u_0(\bar{x},t)+\sum_{m=1}^{+\infty}u_m(\bar{x},t). \quad (16)$$

Define the vector

$$\vec{u}_n=\{u_0(\bar{x},t),u_1(\bar{x},t),\ldots,u_n(\bar{x},t)\}.$$

Differentiating Eq. (11) m times with respect to the embedding parameter p and then setting $p=0$ and finally dividing them by $m!$, we obtain the mth-order deformation equation

$$\mathcal{L}[u_m(\bar{x},t)-\chi_m u_{m-1}(\bar{x},t)]=\hbar\,H(\bar{x},t)\,R_m(\vec{u}_{m-1},\bar{x},t), \quad (17)$$

where

$$R_m(\vec{u}_{m-1},\bar{x},t)=\frac{1}{(m-1)!}\left.\frac{\partial^{m-1}\mathcal{N}[\phi(\bar{x},t;p)]}{\partial p^{m-1}}\right|_{p=0}. \quad (18)$$

and

$$\chi_m=\begin{cases}0, & m\leqslant 1,\\ 1, & m>1.\end{cases} \quad (19)$$

Operating the Riemann–Liouville integral operator I^{s_1} on both sides of Eq. (17), we have

$$u_m(\bar{x},t) = \chi_m u_{m-1}(\bar{x},t) - \chi_m \sum_{i=0}^{n-1} u_{m-1}^{(i)}(\bar{x},0^+)\frac{t^i}{i!} + \hbar H(\bar{x},t) I^{s_1}[R_m(\vec{u}_{m-1},\bar{x},t)]. \tag{20}$$

In this way, it is easy to obtain $u_m(\bar{x},t)$ for $m \geqslant 1$, at mth-order, we have

$$u(\bar{x},t) = \sum_{m=0}^{M} u_m(\bar{x},t). \tag{21}$$

When $M \to \infty$, we get an accurate approximation of the original Eq. (10).

4. Applying HAM

In this section we apply this method for solving linear/nonlinear fractional multi-term diffusion and wave equations.

Example 1. Consider the linear multi-term fractional partial differential equation [5].

$$(D_t^{s_1} - D_t^{s_2})u = -\sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2}, \quad -\infty < x_i < \infty, t > 0, \tag{22}$$

$$u(\bar{x},0) = e^{-(x_1+x_2+x_3)}, \quad 0 < s_2 < s_1 < 1. \tag{23}$$

To solve Eq. (22) by means of homotopy analysis method, according to the initial conditions denoted in (23), it is natural to choose

$$u_0(\bar{x},t) = e^{-(x_1+x_2+x_3)}. \tag{24}$$

We choose the linear operator

$$\mathcal{L}[\phi(\bar{x},t;p)] = D_t^{s_1}[\phi(\bar{x},t;p)], \tag{25}$$

with the property $\mathcal{L}[c] = 0$. where c is constant.

We now define a nonlinear operator as

$$\mathcal{N}[\phi] = (D_t^{s_1} - D_t^{s_2})\phi + \sum_{i=1}^{3} \frac{\partial^2 \phi}{\partial x_i^2}. \tag{26}$$

Using the above definition, with assumption $H(\bar{x},t) = 1$ we construct the zeroth-order deformation equation

$$(1-p)\mathcal{L}[\phi(\bar{x},t;p) - u_0(\bar{x},t)] = p\hbar\mathcal{N}[\phi(\bar{x},t;p)]. \tag{27}$$

Obviously, when $p = 0$ and $p = 1$,

$$\phi(\bar{x},t;0) = u_0(\bar{x},t), \qquad \phi(\bar{x},t;1) = u(\bar{x},t). \tag{28}$$

Thus, we obtain the mth-order deformation equations

$$\mathcal{L}[u_m(\bar{x},t) - \chi_m u_{m-1}(\bar{x},t)] = \hbar R_m(\vec{u}_{m-1}), \tag{29}$$

where

$$R_m(\vec{u}_{m-1}) = (D_t^{s_1} - D_t^{s_2})u_{m-1} + \sum_{i=1}^{3} \frac{\partial^2 u_{m-1}}{\partial x_i^2}.$$

Now the solution of the mth-order deformation equations (29)

$$u_m(\bar{x},t) = (\chi_m + \hbar)(u_{m-1}(\bar{x},t) - u_{m-1}(\bar{x},0)) + \hbar\left(-I^{s_1-s_2}u_{m-1} + I^{s_1}\sum_{i=1}^{3} \frac{\partial^2 u_{m-1}}{\partial x_i^2}\right). \tag{30}$$

From (24) and (30) and subject to initial condition

$$u_{m-1}(\bar{x},0) = 0, \quad m \geq 1$$

we obtain

$$u_0(\bar{x},t) = \mathrm{e}^{-(x_1+x_2+x_3)}, \qquad u_1(\bar{x},t) = \mathrm{e}^{-x_1-x_2-x_3}\left(\frac{3ht^{s_1}}{\Gamma(s_1+1)} - \frac{ht^{s_1-s_2}}{\Gamma(s_1-s_2+1)}\right),$$

$$\begin{aligned} u_2(\bar{x},t) &= \mathrm{e}^{-x_1-x_2-x_3}\left(\frac{3h^2t^{s_1}}{\Gamma(s_1+1)} + \frac{6ht^{s_1}}{\Gamma(s_1+1)} + \frac{9h^2t^{2s_1}}{\Gamma(2s_1+1)} - \frac{h^2t^{s_1-s_2}}{\Gamma(s_1-s_2+1)}\right. \\ &\quad \left. - \frac{ht^{s_1-s_2}}{\Gamma(s_1-s_2+1)} + \frac{h^2t^{2(s_1-s_2)}}{\Gamma(2(s_1-s_2)+1)} - \frac{6h^2t^{2s_1-s_2}}{\Gamma(2s_1-s_2+1)}\right) \\ &\ \ \vdots \end{aligned}$$

Hence

$$u(\bar{x},t) = u_0(\bar{x},t) + u_1(\bar{x},t) + u_2(\bar{x},t) + \cdots.$$

When $\hbar = -1$, we have

$$u_m(\bar{x},t) = \mathrm{e}^{-(x_1+x_2+x_3)}\sum_{k=0}^{m}(-3)^k\binom{m}{k}\frac{t^{m(s_1-s_2)+ks_2}}{\Gamma(m(s_1-s_2)+ks_2+1)}, \quad m = 0,1,2,\ldots,$$

$$\begin{aligned} u(\bar{x},t) &= \sum_{m=0}^{\infty}u_m(\bar{x},t) = \mathrm{e}^{-(x_1+x_2+x_3)}\sum_{m=0}^{\infty}\sum_{k=0}^{m}(-3)^k\binom{m}{k}\frac{t^{m(s_1-s_2)+ks_2}}{\Gamma(m(s_1-s_2)+ks_2+1)} \\ &= \mathrm{e}^{-(x_1+x_2+x_3)}\sum_{k=0}^{\infty}\sum_{m=k}^{\infty}(-3)^k\binom{m}{k}\frac{t^{m(s_1-s_2)+ks_2}}{\Gamma(m(s_1-s_2)+ks_2+1)}. \end{aligned}$$

Thus in view of (6) we have

$$u(\bar{x},t) = \mathrm{e}^{-(x_1+x_2+x_3)}\sum_{k=0}^{\infty}(-3)^k\frac{t^{ks_1}}{k!}E^{(k)}_{s_1-s_2,ks_2+1}(t^{s_1-s_2}),$$

which is in agreement with the given solution using ADM [5].

Example 2. Consider the two-dimensional fractional two-term wave equation

$$(D_t^{s_1} - \lambda D_t^{s_2})u = 2\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right), \quad -\infty < x_i < \infty, t > 0, \tag{31}$$

$$u(\bar{x},0) = \sin x_1.\sin x_2, \tag{32}$$

$$u_t(\bar{x},0) = 0, \quad 1 < s_2 < s_2 < 2. \tag{33}$$

To solve Eq. (31) by means of HAM, according to (32), it is natural to choose

$$u_0(\bar{x},t) = \sin x_1.\sin x_2. \tag{34}$$

We choose $\mathcal{L} = D_t^{s_1}$ with the property $\mathcal{L}[c] = 0$, where c is constant.

We define a nonlinear operator as

$$\mathcal{N}[\phi] = (D_t^{s_1} - \lambda D_t^{s_2})\phi - 2\left(\frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2}\right). \tag{35}$$

Assume that $H(\bar{x},t) = 1$, we construct the zeroth-order deformation equation

$$(1-p)\mathcal{L}[\phi(\bar{x},t;p) - u_0(\bar{x},t)] = p\hbar\mathcal{N}[\phi(\bar{x},t;p)]. \tag{36}$$

According to Eqs. (17)–(19), we obtain the mth-order deformation equations

$$\mathcal{L}[u_m(\bar{x},t) - \chi_m u_{m-1}(\bar{x},t)] = \hbar R_m(\vec{u}_{m-1}), \tag{37}$$

where

$$R_m(\vec{u}_{m-1}) = (D_t^{s_1} - \lambda D_t^{s_2})u_{m-1} - 2\left(\frac{\partial^2 u_{m-1}}{\partial x_1^2} + \frac{\partial^2 u_{m-1}}{\partial x_2^2}\right).$$

Now, the solution of Eq. (37) for $m \geq 1$ becomes

$$u_m(\bar{x},t) = (\chi_m + \hbar)(u_{m-1}(\bar{x},t) - \acute{u}_{m-1}(\bar{x},0)t - u_{m-1}(\bar{x},0)) - \hbar\left(\lambda I^{s_1-s_2}u_{m-1} + 2I^{s_1}\left(\frac{\partial^2 u_{m-1}}{\partial x_1^2} + \frac{\partial^2 u_{m-1}}{\partial x_2^2}\right)\right). \tag{38}$$

From (34) and (38) and subject to initial condition

$$u_{m-1}(\bar{x},0)=0,\qquad \acute{u}_{m-1}(x,0)=0\quad m\geq 1$$

we now successively obtain

$$u_0(\bar{x},t)=\sin x_1.\sin x_2,\qquad u_1(\bar{x},t)=\sin(x_1)\sin(x_2)\left(\frac{4ht^{s_1}}{\Gamma(s_1+1)}-\frac{ht^{s_1-s_2}\lambda}{\Gamma(s_1-s_2+1)}\right),$$

$$\begin{aligned}u_2(\bar{x},t)&=\sin(x_1)\sin(x_2)\left(\frac{4h^2t^{s_1}}{\Gamma(s_1+1)}+\frac{4ht^{s_1}}{\Gamma(s_1+1)}+\frac{16h^2t^{2s_1}}{\Gamma(2s_1+1)}-\frac{h^2\lambda t^{s_1-s_2}}{\Gamma(s_1-s_2+1)}\right.\\&\quad\left.-\frac{h\lambda t^{s_1-s_2}}{\Gamma(s_1-s_2+1)}+\frac{h^2\lambda^2t^{2(s_1-s_2)}}{\Gamma(2(s_1-s_2)+1)}-\frac{8h^2\lambda t^{2s_1-s_2}}{\Gamma(2s_1-s_2+1)}\right),\\&\quad\vdots\end{aligned}$$

Hence $u(\bar{x},t)=u_0(\bar{x},t)+u_1(\bar{x},t)+u_2(\bar{x},t)+\cdots$. When $\hbar=-1$, we obtain

$$u_0(\bar{x},t)=\sin x_1.\sin x_2,$$

$$u_m(\bar{x},t)=\sin x_1.\sin x_2\sum_{k=0}^{m}(-4)^k\binom{m}{k}\lambda^{m-k}\frac{t^{ms_1-(m-k)s_2}}{\Gamma(ms_1-(m-k)s_2)+1},\quad m=0,1,2,\ldots.$$

Thus, the solution u is

$$u(\bar{x},t)=\sin x_1.\sin x_2\sum_{k=0}^{\infty}(-4)^k\frac{t^{ks_1}}{k!}E^{(k)}_{s_1-s_2,ks_2+1}(\lambda t^{s_1-s_2}),$$

which is in agreement with the obtained solution using ADM [5].

Example 3. Consider the nonlinear fractional equation

$$(D_t^{\frac{3}{2}}-D_t^{\frac{1}{2}})u+u_{xx}+u^2=0,\quad -\infty<x<\infty,t>0,\tag{39}$$

$$u(x,0)=x,\qquad u_t(x,0)=\sin x.\tag{40}$$

According to (11), the zeroth-order deformation can be given by

$$(1-p)\mathcal{L}[\phi(x,t;p)-u_0(x,t)]=p\hbar H(x,t)\mathcal{N}[\phi(x,t;p)].\tag{41}$$

According to the initial conditions (40), we can choose the initial guess as follows:

$$u_0(x,t)=x,\qquad \acute{u}_0(x,0)=\sin x.\tag{42}$$

and we choose the auxiliary linear operator

$$\mathcal{L}=D_t^{\frac{3}{2}}.\tag{43}$$

We also define a nonlinear operator as

$$\mathcal{N}[\phi(x,t;p)]=(D_t^{\frac{3}{2}}-D_t^{\frac{1}{2}})\phi+\phi_{xx}+\phi^2.\tag{44}$$

Using the above definition,with assumption $H(x,t)=1$ we obtain the mth-order deformation equations

$$D_t^{\frac{3}{2}}[u_m(x,t)-\chi_m u_{m-1}(\bar{x},t)]=\hbar R_m(\vec{u}_{m-1}),\tag{45}$$

where

$$R_m(\vec{u}_{m-1})=(D_t^{\frac{3}{2}}-D_t^{\frac{1}{2}})u_{m-1}+(u_{m-1})_{xx}+\sum_{i=0}^{m-1}u_iu_{m-1-i}.$$

The solution of the mth-order deformation equations (45):

$$\begin{aligned}u_m(x,t)&=(\chi_m+\hbar)(u_{m-1}(x,t)-\acute{u}_{m-1}(x,0)\,t-u_{m-1}(x,0))\\&\quad+\hbar\left(-I^{-1}u_{m-1}+I^{\frac{3}{2}}(u_{m-1})_{xx}+I^{\frac{3}{2}}\sum_{i=0}^{m-1}u_iu_{m-1-i}\right).\end{aligned}\tag{46}$$

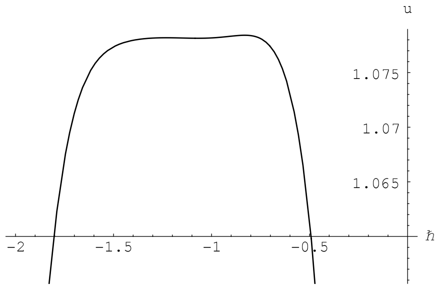

Fig. 1. ħ-curve of u(0.5,0.5).

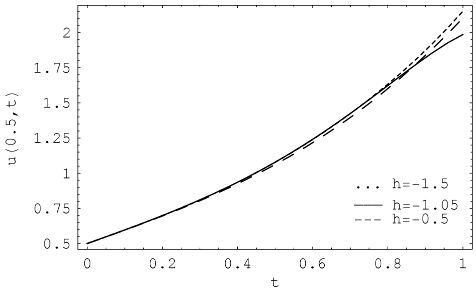

Fig. 2. $u(0.5, t)$.

Table 1

x	$\hbar = -1.05$			$\hbar = -1$		
	$t = 0.1$	$t = 0.5$	$t = 1$	$t = 0.1$	$t = 0.5$	$t = 1$
0.2	2.48×10^{-12}	0.00177	0.00227	4.06×10^{-9}	0.00503	0.00601
0.4	3.91×10^{-9}	0.00053	0.00288	5.34×10^{-9}	0.00242	0.0051
0.6	1.25×10^{-9}	0.00154	0.00066	2.13×10^{-9}	0.00030	0.00613
0.8	2.76×10^{-9}	0.00109	0.00049	2.63×10^{-9}	0.00161	0.00495
1	5.02×10^{-9}	0.00001	0.00085	6.08×10^{-9}	0.00139	0.0030

From (42) and (46) and subject to initial conditions $u_m(x, 0) = 0,\ \acute{u}_m(x, 0) = 0,\ m \geq 1$ we obtain

$$u_0(x, t) = x, \qquad u_1(x, t) = -h\, t \sin(x) - h\, t\, x + 0.752253\, h t^{3/2} x^2,$$

$$\begin{aligned} u_2(x, t) \; = \; & -h \sin(x)\, t - h^2 \sin(x)\, t - h^2\, x\, t - h\, x\, t + 0.752253\, h\, x^2\, t^{3/2} \\ & + 0.752253\, h^2\, x^2\, t^{3/2} + 0.5\, h^2 \sin(x)\, t^2 + 0.5\, h^2\, x\, t^2 \\ & - 0.601802\, h^2\, x \sin(x)\, t^{5/2} + 0.333333\, h^2\, x^3\, t^3 + 0.333333\, h^2\, t^3 \\ & - 0.902703\, h^2\, x^2\, t^{5/2} + 0.300901\, h^2 \sin(x)\, t^{5/2} - 0.601802\, h^2\, x \sin(x)\, t^{5/2} \\ & \vdots \end{aligned}$$

Hence, the solution of Eq. (39) in series from is obtained as

$$u(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + \cdots.$$

In Figs. 1 and 2 $u(x, t) = (u_0 + \cdots + u_7)$ is drawn.

The following tables show the error value for the HAM approximate solution u when $\hbar = -1, \hbar = -1.05, \hbar = -1.2$ and ADM solution $(u(x, t) = (u_0 + u_1 + u_2))$ [5]. It is clear that we obtain the better results when $\hbar = -1.05$ (see Table 1 and Table 2).

5. Conclusion

In this work, the HAM was applied to derive approximate analytical solutions of both linear and nonlinear multi-term fractional partial differential equations. The explicit series solution linear fractal partial diffusion–wave equations are obtained, which are the same as those results given by the Adomian decomposition method for $\hbar = -1$. This accords with the conclusion that the homotopy analysis method logically contains the Adomian decomposition method [10,9].

The HAM provides us with a convenient way to control the convergence of approximation series by means of the so-called ħ-curve, it is easy to determine the valid regions of ħ to gain a convergent series solution, which is a fundamental

Table 2

x	$\hbar = -1.2$			ADM [5]		
	$t = 0.1$	$t = 0.5$	$t = 1$	$t = 0.1$	$t = 0.5$	$t = 1$
0.2	0.00001	0.00086	0.0042	0.0094	0.4555	4.0243
0.4	0.00003	0.00029	0.00104	0.0095	0.3927	3.7036
0.6	0.00005	0.00131	0.00177	0.0086	0.4067	4.343
0.8	0.00007	0.0011	0.00221	0.0074	0.5323	6.052
1	0.00010	0.00048	0.000627	0.0064	0.79030	8.667

qualitative difference in analysis between HAM and other methods. The results show that HAM is a powerful mathematical tool for solving wide classes of multi-term fractional differential equations.

We point out that the corresponding analytical and numerical solutions are obtained using *Mathematica*.

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