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Nonlinear Hodge Theory: Applications

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INTRODUCTION

Let M be an oriented finite Riemannian manifold of dimension n with Lipschitz boundary (Morrey [6]). The analogues for forms of the Dirichlet and Neumann problems can be formulated in the following way (Conner [2], Duff and Spencer [3], Friedrichs [4], and Morrey [6]).

If ω is a p-form on M, the tangential boundary value of ω is its tangential component $T\omega$, and its normal boundary value is its normal component $N\omega$, where $T\omega$ and $N\omega$ are forms defined in ∂M (Duff and Spencer [3]). We remark that $T^* = {}^*N$, where $*$ is the Hodge star operator.

A p-form τ defined in the boundary will be called *admissible* if τ is closed and $\int_{C_p} \tau = 0$ over every p-cycle in the boundary which bounds in M.

A prescribed tangential boundary value will always be required to be admissible, and a prescribed normal boundary value will be required to be the star of an admissible form in ∂M.

Solutions are then obtained of the problem of showing the existence of a unique harmonic p-form satisfying either

DIRICHLET DATA: prescribed relative periods and tangential boundary value.

or

NEUMANN DATA: prescribed absolute periods and normal boundary value.

In this paper, we will show the existence of solutions of a *nonlinear* elliptic equation satisfying Dirichlet or Neumann boundary conditions. Our methods are based on previous work on compact manifolds [9]. However, complete results are obtained only for $p = 1$; see the Remark at the end of Section 1.

In Section 1, the theorems are stated. In Sections 2, 3, and 4 we give applications to gas dynamics, minimal surfaces and surfaces with prescribed mean curvature. For a survey of these topics, we refer to Bers [1] and Serrin [7]. In the last section, the theorems are proved.

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1. Statement of Theorems

The p-forms at $x \in M$ form a finite-dimensional inner product space. Let $\langle\ ,\ \rangle_x$ denote the inner product and $|\ |_x$ the associated norm. Let $(\ ,\ ) = \int_M \langle\ ,\ \rangle_x\, dV$ be the global inner product of two p-forms on M and $\|\ \|$ the associated norm.

For a given smooth form ω, $Q(\omega) = |\omega|_x^2$ is a C^∞ function on M. Following the usual assumptions made in gas dynamics, a function $\theta(x, t)$: $M \times R_+^1 \to R_+^1$ is called *regular* if for all (x, t),

(a) $0 < k < \theta < 1/k < \infty$,

(b) $0 < c < (\partial/\partial t)(t\theta^2(x, t)) < 1/c < \infty$.

$\theta(x, t)$ is called *admissible* if (a) holds and if there exists t_θ, $0 < t_\theta < \infty$, such that for all x and $t < t_\theta$,

(b′) $0 < (\partial/\partial t)(t\theta^2(x, t) < \infty$.

The *sonic speed* associated with θ is $Q_\theta = \text{lub}\ t_\theta$ over all t_θ for which (a) and (b′) hold, and a p-form ω is said to be *subsonic* if $\max_{x \in M} Q(\omega) < Q_\theta$.

Now suppose $\rho(x, t)$ is a given admissible density function on M which is of class $C^{2+\mu}$ in x and $C^{1+\mu}$ in t. Let d be exterior differentiation on forms and δ, its adjoint. For a p-form ω, $\rho(x, Q(\omega))\omega$ is also a p-form. We say that ω is *ρ-harmonic* if $d\omega = \delta\rho\omega = 0$.

Let $C^{1+\mu}(\overline{M})$ denote the space of p-forms whose components are in $C^{1+\mu}$. All spaces carrying the boundary and homology conditions are to be regarded as subspaces of $C^{1+\mu}(\overline{M})$.

Inhomogeneous Dirichlet data are given by an element of the space

$$\mathscr{D} = \mathscr{D}_1 \oplus \mathscr{D}_2,$$

where $\mathscr{D}_1 = \text{Ker}\, d$ and $\mathscr{D}_2 = C^{1+\mu}(\overline{M})$.

Since we can always find a closed form with prescribed relative periods and admissible tangential data, the first function space represents the Dirichlet conditions as defined in the introduction. The second factor gives the inhomogeneous term in the equation.

Inhomogeneous Neumann data belong to the space

$$\mathscr{N} = \mathscr{N}_1 \oplus \mathscr{N}_2,$$

where

$$\mathscr{N}_1 = \text{Ker}\, d \qquad \text{and} \qquad \begin{aligned} \mathscr{N}_2 &= \text{Ker}\, \delta && \text{if } n \leqslant 3 \\ &= 0 && \text{if } n > 3. \end{aligned}$$

The restriction to homogeneous data for $n > 3$ is due to a gap in the differentiability results. In fact, for $n = 3$, a completely different estimate of Ladyzhenskaya and Ural'tseva [5] is needed for the nonlinear boundary condition and we assume more smoothness of the boundary; namely, ∂M must be of class C^2.

Let $\mathscr{L}$ be a function space of forms on M. (For our purposes, $\mathscr{L}$ will either be $\mathscr{D}$ or $\mathscr{N}$). We denote by $\pi(\mathscr{L})$ the space of *paths* on $\mathscr{L}$. These will be one-parameter families $\tau(s)$, $s \in I$, satisfying:

(i) $\tau(0) = 0$,

(ii) for each $s \in I$, $\tau(s) \in \mathscr{L}$,

(iii) $\tau(s)$ depends continuously on s in the topology of $\mathscr{L}$.

The following theorems will be obtained in Section 5:

THEOREM 1. *Let ρ be admissible with sonic value Q_ρ.*

(a) *There is an open connected set $\mathcal{O} \subset \mathscr{D}$ containing the origin such that for each pair $(\gamma, \sigma) \in \mathcal{O}$, there is a unique subsonic 1-form $\omega \in C^{1+\mu'}(\overline{M})$ such that ω has the same relative periods and tangential boundary data as γ and*

$$d\omega = 0, \qquad \delta\rho\omega = \delta\sigma.$$

(b) *If $(\gamma(s), \sigma(s)) \in \pi(\mathscr{D})$ is a path, then the solution $\omega(s)$ of* (a) *depends continuously on s in the uniform norm and either*

(1) *ω is subsonic for all $s \in I$, or*

(2) *there exists a number s_ρ such that $\omega(s)$ is subsonic for $s < s_\rho$ and $\max_{x \in M} Q(\omega(s)) \to Q_\rho$ as $s \nearrow s_\rho$.*

THEOREM 2. *Let ρ be admissible with sonic value Q_ρ.*

(a) *There is an open connected set $\mathcal{O} \subset \mathscr{N}$ containing the origin such that for each pair $(\gamma, \kappa) \in \mathcal{O}$, there is a unique subsonic ρ-harmonic 1-form $\omega \in C^{1+\mu'}(\overline{M})$ having the same absolute periods as γ and the mass flow $N\rho\omega = N\kappa$ on the boundary.*

(b) *For $(\gamma(s), \kappa(s)) \in \pi(\mathscr{N})$, a path, the same conclusions hold as in part* (b) *of Theorem* 1.

Remark. If ρ is regular, weak versions of these theorems are valid for p-forms, $p > 1$. However, because of a gap in the differentiability theory for systems, one cannot use the same techniques to prove the nonregular Theorems 1 and 2 above for $p > 1$.

2. Minimal Surfaces

Theorem 1 can be applied to the problem of obtaining minimal hypersurfaces of $\mathbb{R}^{n+1}$ given in nonparametric form $x_{n+1} = \phi(x_1, ..., x_n)$. In this case the appropriate $\rho = (1 + Q)^{-1/2}$ is admissible with $Q_\rho = \infty$ (but *not* regular since $(d/dQ)(Q\rho^2) \to 0$ as $Q \to \infty$). For simplicity we discuss only the case $n = 2$.

Let Ω be a bounded, simply connected plane domain bounded by a C^2 curve and let $f\colon \partial\Omega \to \mathbb{R}$ be boundary data in $C^{2+\alpha}(\partial\Omega)$. Then one seeks a surface $z = \phi(x, y)$ of minimal area and finite gradient passing through the curve $z = f(x, y)$, $(x, y) \in \partial\Omega$. The classical results are:

(i) If Ω is convex, then for any f there exists a minimal surface $z = \phi(x, y)$ with $z = f$ on $\partial\Omega$.

(ii) If Ω is *not* convex, then for every $\epsilon > 0$, there exists an f with $\max_\Omega |f| < \epsilon$ for which there is no minimal surface given by $z = \phi(x, y)$ with $z = f$ on ∂M.

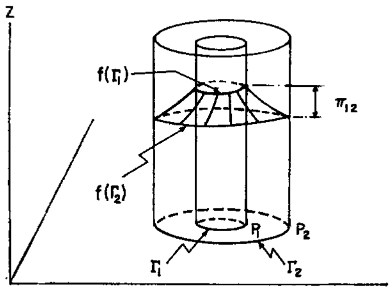
However, we obtain from Theorem 1

Example 1. For simplicity choose a family of boundary data $f(t)$ given by tf for some fixed f. (The modification for the more general example of an arbitrary family $f(t)$ is immediate.) Then our results imply (writing $\omega_t = d\phi_t$ in Theorem 1) for *any* Ω, and *any* $f \in C^{2+\alpha}$ there is a t such that, for every $t < t_\rho$, there exists a unique minimal surface $z_t = \phi_t(x, y)$ with $z_t = tf$ on $\partial\Omega$ (and $\max_\Omega |\operatorname{grad} \phi_t| < \infty$). If f is nonconstant, then $\max |\operatorname{grad} \phi_t| \to \infty$ as $t \nearrow t_\rho$. (By the maximum principle this occurs at a boundary point.)

Remark. The classical result (i) shows Ω convex implies that, in fact, $t_\rho = \infty$.

Remark. Consider the regular tetrahedron $ABCD$ with base ABC in the xy plane and apex D. Let D_0 be the centroid of triangle ABC and D_t the point on the line segment D_0D which divides it in the ratio t. Then for $t = 0$ we have the point D_0 and $D_\infty = D$. We ask for a minimal surface $z = \phi_t(x, y)$ spanned by the (space) quadrilateral ABD_tC. For $t = 0$ the quadrilateral is planar and a solution exists. The case $t = \infty$ is the well-known Schwarz example for which no solution exists. The above example shows that there is a critical t_ρ such that solutions exist for $t < t_\rho$ but the corresponding surfaces $z = \phi_t(x, y)$ have somewhere a tangent plane which tends to vertical as $t \nearrow t_\rho$. We do not know the value of t_ρ.

It is clear that more general applications of the theorems could be given. In particular, one can consider minimal surfaces over multiply connected domains. The following three examples for doubly connected domains indicate the variety of situations which can occur.



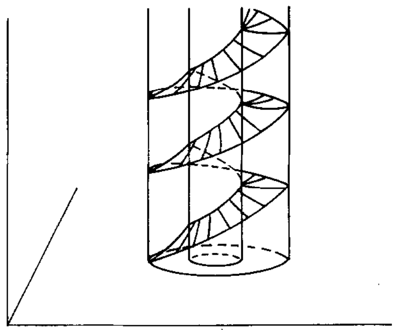
EXAMPLE 2

EXAMPLE 2 (see figure). Let Ω be a bounded doubly connected domain in $\mathbb{R}^2$. Admisslble homogeneous Dirichlet data given by a closed 1-form γ give rise to the following interpretation. Suppose first that $\int T\gamma = 0$ on *each* boundary component Γ_j of Ω. Choose a fixed point P_j on each Γ_j. Then we are given continuous functions f_j on Γ_j (where $f_j(P) = \int_{P_j}^{P} T\gamma$), each determined up to a constant. In addition, however, the difference $\pi_{12} = f_2(P_2) - f_1(P_1)$ is specified by the relative period of γ. If we draw the cylinders with bases Γ_1 and Γ_2, we can think of these data as follows: The shape of the curves $z = f_j(P)$, $P \in \Gamma_j$, are given, but the curves are allowed to slide vertically on their cylinders subject to the restriction that the difference in height between the points of the curves sitting over P_1 and P_2 is prescribed by π_{12}. The data $\{t_1 f_1, t_2 f_2, t_3 \pi_{12}\}$ are parametrized by points $t = (t_1, t_2, t_3) \in \mathbb{R}^3$. (In the notation of Theorem 1, a path in $\mathbb{R}^3$ parametrized by t induces a path in $\mathscr{D}$, each closed form $\gamma(t)$ providing the above data where $t_i = t_i(t)$, $i = 1, 2, 3$.)

Our results show that there exists an open set $\mathcal{O} \subset \mathbb{R}^3$, containing the origin, such that for $t \in \mathcal{O}$ there exists a unique minimal surface $z_t = \phi_t(x, y)$ (with finite gradient) satisfying $z_t = t_j f_j$ on Γ_j and $z(P_2) - z(P_1) = t_3 \pi_{12}$. Again one has continuous dependence, of the family, on t and $\max_\Omega |\operatorname{grad} \phi_t| \to \infty$ as t tends to the boundary of $\mathcal{O}$. As in Example 1, the last conclusion means that the family of surfaces ϕ_t "acquires" a vertical tangent (at a point on $\partial\Omega$).

Remark. If the Γ_i are concentric circles and the f_i are constant then the solutions are catenoids.

EXAMPLE 3 (see figure). If, in Example 2, we remove the restriction that the integral of $T\gamma$ be zero on *each* boundary component, the curve $f_j(\Gamma_j)$ describes a helix, lying on the cylinder over Γ_j (and again, only determined up to a vertical motion). The admissibility condition $\int T\gamma = 0$ over the *whole* boundary $\partial\Omega$ implies that the two helices have the same pitch. Since the helices are periodic in their z coordinate, the definition of π_{12} (which describes the relative period) presents no problem.



EXAMPLES 3, 4

The homogeneous Dirichlet data give the shape of the helices and their relative position. Theorem 1 gives the existence of a family $z_t = \phi_t(x, y)$ of (multivalued) minimal "ramp" surfaces having properties analogous to the surfaces of Example 2.

Remark. If the Γ_i are concentric circles and the curves $z = f_i$ describe circular helixes, then the surfaces obtained are helicoids—right helicoids if the relative period π_{12} is zero.

EXAMPLE 4 (see figure). If, for the doubly connected domain Ω, we give Neumann boundary data we obtain, by Theorem 2, a family of "ramp" minimal surfaces as in Example 3, where now the data $t\gamma$ give the prescribed pitch of ϕ_t and the normal derivatives at the boundary surfaces—the cylinders. More precisely it gives $\rho(\partial\phi_t/\partial n)$, where $\rho = (1 + Q)^{-1/2}$ and $Q = |\operatorname{grad} \phi_t|^2$. The case $\partial\phi_t/\partial n = 0$ corresponds to a free boundary problem in the class of all ramped surfaces of prescribed pitch and with boundaries on the cylinders.

3. SURFACES WITH PRESCRIBED MEAN CURVATURE

We now consider the analogous problem for surfaces of prescribed mean curvature (and finite gradient). Such surfaces are described by an inhomogeneous system $d\omega = 0$, $\delta\rho\omega = 2\Lambda$ for a 1-form ω. Locally (globally if Ω is simply connected), we can write $\omega = d\varphi$ and obtain the equation $\delta(\rho d\phi) = 2\Lambda$ where $\Lambda \in C^1(\Omega)$ is the required curvature of the manifold $x_{n+1} = \phi(x_1, ..., x_n)$.

Again, for simplicity, we restrict outselves to surfaces in $\mathbb{R}^3$ defined over bounded simply connected domains $\Omega \subset \mathbb{R}^2$. (However, see Serrin [7] for how to make the obvious generalizations to $n > 2$.) The known results corresponding to (i) and (ii) are [7]:

(i′) If at each point of $\partial\Omega$, the inequality $2 |\Lambda| \leqslant \kappa$ holds, where κ is the curvature of $\partial\Omega$, then for any boundary data $f \in C^{2+\alpha}$ on $\partial\Omega$, there exists a surface $z = \phi(x, y)$ with $z = f$ on $\partial\Omega$ and having mean curvature Λ. (Note that the hypothesis implies that Ω is convex.)

(ii′) If the inequality $2 \mid \Lambda \mid \leqslant \kappa$ fails to hold at a single point of $\partial\Omega$, then there exists arbitrarily small data f for which there exists no solution surface.

However, Theorem 1 gives

EXAMPLE 5. Let Ω be convex, $\Lambda \in C^1(\Omega)$, and $f \in C^{2+\alpha}(\partial\Omega)$. Then there exists a family of surfaces $z_t = \phi_t(x, y)$ for $0 \leqslant t < t_\rho$ for which $z_t = f$ on $\partial\Omega$ and the mean curvature of ϕ_t is equal to $t\Lambda$. As $t \nearrow t_\rho$, $\max_\Omega \mid \operatorname{grad} \phi_t \mid \to \infty$. The convexity of Ω ensures that a solution (a minimal surface in fact) exists for $t = 0$.

Remark. The classical results (i′) and (ii′) imply for a convex domain Ω with boundary curvature κ, that for Λ a constant, $t_\rho = \min_{\partial\Omega} \kappa/2 \mid \Lambda \mid$.

Remark. As in Example 1 we could of course have allowed an additional parameter by varying f.

Remark. If Ω is *not* convex and Λ fixed, then we cannot be certain that a solution exists for $f = 0$. If, however, it is known that a solution exists for, say, f_0 then the statement of Theorem 1 may be modified to allow paths in the parameter space $\mathscr{D}$ with initial points other than the origin. The necessary changes present no difficulties.

Remark. Since surfaces of constant mean curvature arise as soap films separating regions of differing pressure, we can visualize the case of Ω not necessary convex, $f = 0$ and mean curvature $t\Lambda_0$, Λ_0 a constant, as follows. If a hole Ω is cut in the top of an enclosed chamber and the pressure within the chamber increased, then our results show the existence of solutions, i.e., bubbles of constant mean curvature $t\Lambda_0$, at least up to the point when the bubble has somewhere a vertical tangent. This phenomenon is observed when a balloon or an inner tube with a "weak spot" is inflated.

4. COMPRESSIBLE SUBSONIC FLOWS

Theorem 2 applies to the problem of showing the existence of a steady compressible flow on a manifold with boundary. The mass density $\rho = \rho(x, Q)$ is admissible with a finite sonic speed. (For example, in the special case of a polytropic flow, the sonic speed $Q_\rho = 2/(\gamma + 1)$, where γ is the adiabatic constant.) The components of the ρ-harmonic form give the velocity components of the flow.

In particular, one has the following.

EXAMPLE 6. Let S be a smooth *compact* manifold embedded in R^n. For a given mass density ρ, one can ask for a steady compressible flow *inside* S with prescribed circulations $C = \{c_i\}$ and no mass flow through the boundary (which is the manifold S). By Theorem 2, there exists a t_ρ such that, for $t < t_\rho$, there is a unique subsonic flow, described by a 1-form ω_t, having circulations tC. In fact, Theorem 2 gives more; we can obtain a mass flow $N(\rho\omega_t) = tN\kappa$ through the boundary (where the coclosed form κ carries the mass flow data).

Remark. The 1-homology of a solid torus T in R^3 is generated by a single closed curve. If it is assumed that there is no "seepage" (i.e., no mass flow through the boundary) then it follows that there exists a unique subsonic flow in T with prescribed circulation c for $0 \leqslant c < c_\rho$. As the circulation increases to c_ρ, the speed of the flow tends somewhere in T to sonic speed. This must occur at the boundary (cf. [10]).

Remark. An idealized reaction turbine provides a concrete illustration of the preceding; the mass flow is prescribed on the boundary of the ***runner***, a solid torus. It is normal component points inward at the guide vane exit (which is the runner entrance—where the curvature is positive if T is a "standard" torus) and outward at the inner chamber (the runner exit—where the curvature is negative). If the turbine is unloaded, so that no torque is developed, then the tangential component of the flow velocity at the runner entrance determines the circulation in T.

5. PROOFS OF THE THEOREMS

Let $\mathscr{L}_2$ be the Hilbert space completion of smooth p-forms on M with respect to the inner product (,). The following orthogonal decompositions of $\mathscr{L}_2$ are valid (cf. [2, 3, 6]):

$$\mathscr{L}_2 = \mathscr{E}_T \oplus \mathscr{E}_N^* \oplus H = \mathscr{E} \oplus \mathscr{E}_N^* \oplus H_N$$

where,

$$\mathscr{E}_T = \{d\nu \mid T\nu = 0\}, \qquad \mathscr{E}_N^* = \{\delta\tau \mid N\tau = 0\}, \qquad H = \{\sigma \mid d\sigma = \delta\sigma = 0\},$$

$$\mathscr{E} = \{d\nu\}, \qquad \mathscr{E}^* = \{\delta\nu\}, \qquad \text{and} \qquad H_N = \{\sigma \mid d\sigma = \delta\sigma = 0,\ N\sigma = 0\}.$$

Let $\theta(x, t)$ be regular as defined in Section 1. Then θ induces the following

VARIATIONAL PRINCIPLE. Suppose $\mathscr{L}_2 = V \oplus V^\perp$ with V a closed subspace. Let α and β be given p-forms on M. Then, there exists a unique p-form τ such that

(1) $\tau - \alpha \in V$,

(2) $\theta(x, Q(\tau))\tau - \beta \in V^{\perp}$.

Proof. Let $E(\tau) = \frac{1}{2} \int_M (\int_0^{Q(\tau)} \theta(x, t)\, dt)\, dV$ be the energy functional defined by θ and let

$$I(\tau) = E(\tau) - (\tau, \beta).$$

The conditions on θ ensure that

$$\min_{\tau - \alpha \in V} I(\tau)$$

exists, is unique, and satisfies (2), which is just the Euler equation of the variational problem. See [9], where a similar argument is given in more detail.

COROLLARY (Continuous dependence). *Suppose $\alpha(s), \beta(s) \in \pi(\mathscr{L}_2)$ are paths. Consider the mapping of s into τ_s, where τ_s is the extremal. If this mapping is compact in the uniform topology, then τ_s is a continuous function of s.*

Proof. Suppose s_n converges to s_0. By continuity, $\alpha_n = \alpha(s_n) \to \alpha_0 = \alpha(s_0)$ in $\mathscr{L}_2$ and $\beta_n \to \beta_0$ in $\mathscr{L}_2$. By assumption, a subsequence of $\{\tau_n\}$ converges uniformly to τ_0. We claim that τ_0 is extremal which, by uniqueness, shows that the choice of a subsequence was unnecessary. For suppose $\varphi - \alpha_0 = v \in V$. Then, $\varphi_n = \alpha_n + v$ is admissible for α_n and $I(\tau_n) \leqslant I(\varphi_n)$ for every n, since τ_n is extremal. Since the integrals converge uniformly, $I(\tau_0) \leqslant I(\varphi)$, which proves the claim.

Assuming ρ is regular, we will solve the regular Dirichlet and Neumann problems and then obtain the nonregular solutions from these (see Section 1).

REGULAR THEOREM 1. *Let $(\gamma, \sigma) \in \mathscr{D}$. Then, there is a unique $\omega \in C^{1+\mu'}(\overline{M})$ which satisfies $d\omega = 0$, $\delta\rho\omega = \delta\sigma$ and $\omega - \gamma \in \mathscr{E}_T$. The solution depends continuously on γ and σ.*

REGULAR THEOREM 2. *Let $(\gamma, \kappa) \in \mathscr{N}$. Then, there is a unique $\omega \in C^{1+\mu'}(\overline{M})$ which satisfies $d\omega = \delta\rho\omega = 0$, $\omega - \gamma \in \mathscr{E}$ and $N\rho\omega = N\kappa$. The solution depends continuously on γ and κ.*

We first obtain weak solutions of these theorems for arbitrary ρ and then use Di Giorgi–Moser–Morrey techniques to show differentiability in the case $p = 1$.

In Regular Theorem 1, apply the variational principle with $V = \mathscr{E}_T$, $\theta = \rho$, $\alpha = \gamma$ and $\beta = \sigma$. We obtain

(1) $\tau - \gamma \in \mathscr{E}_T$,

(2) $\rho\tau - \sigma \in \mathscr{E}_T{}^{\perp}$.

Setting $\tau = \omega$, we obtain from (1), since $d\gamma = 0$, that ω is a weak solution of $d\omega = 0$, and from (2), that ω is a weak solution of $\delta\rho\omega = \delta\sigma$.

In Regular Theorem 2, we recall (cf. [9]) that if ρ is regular, there is a conjugate function ν, also regular, such that $\tau = {}^*\rho\omega$ if and only if $\omega = {}^*\nu\tau$. The correspondence is one–one and continuous in any of the function space topologies we consider.

We now apply the variational principle with $V = \mathscr{E}^{*\perp}$, $\theta = \nu$, $\alpha = {}^*\kappa$, and $\beta = {}^*\gamma$ and obtain

(1) $\tau - {}^*\kappa \in \mathscr{E}^{*\perp}$

(2) $\nu\tau - {}^*\gamma \in \mathscr{E}^*$.

Letting $\omega = {}^*\nu\tau$, we find that $\omega - \gamma \in \mathscr{E}$ and $\rho\omega - \kappa \in \mathscr{E}^\perp$. Using the second decomposition of $\mathscr{L}_2$ and $d\gamma = \delta\kappa = 0$, we see that ω is a weak solution of $d\omega = \delta\rho\omega = 0$ and $N\rho\omega = N\kappa$.

Having obtained weak solutions to these problems, we now show that they are smooth if $p = 1$.

The first step is to write down the Euler equations for both problems.

Choose local coordinates $(x_1, \dots, x_n)$ so that a coordinate patch is given locally by a semiball

$$S_r = \left\{x \,\middle|\, \sum x_i^2 \leqslant r^2 \text{ and } x_n \geqslant 0\right\}$$

with the boundary represented by $x_n = 0$. In Regular Theorem 1, in these coordinates, $\omega = \omega_i\, dx^i$ and $Q(\omega) = g^{ij}\omega_i\omega_j$. The integrand in $I(\omega)$ is

$$f(x, \omega) = \frac{g^{1/2}}{2}\left(\int_0^{Q(\omega)} \rho(x, t)\, dt - \langle\omega, \sigma\rangle\right),$$

and the Euler equation is

$$\int_{S_r} \sum_{i=1}^{n} \frac{\partial f}{\partial\omega_i}\frac{\partial\zeta}{\partial x_i}\, dx = 0 \qquad \text{for all } \zeta \in C_0^\infty(S_r), \tag{3}$$

where

$$\frac{\partial f}{\partial\omega_i} = \sum_{j=1}^{n} g^{1/2}g^{ij}(\rho(x, Q(\omega))\,\omega_j - \sigma_j)$$

and $T\omega = T\gamma$ on ∂M.

In Regular Theorem 2, we derive the Euler equation for the conjugate variational problem in two dimensions. Letting

$$\tilde{\omega} = {}^*\rho\omega = \tilde{\omega}_i\, dx^i$$

the integrand is

$$g(x, \tilde{\omega}) = \frac{g^{1/2}}{2}\left(\int_0^{Q(\tilde{\omega})} \nu(x, t)\, dt - \langle \tau, {}^*\gamma \rangle\right)$$

and the Euler equation is

$$\int_{S_r} \sum_{i=1}^{2} \frac{\partial g}{\partial \tilde{\omega}_i} \frac{\partial \zeta}{\partial x_i}\, dx = 0 \qquad \text{for all } \zeta \in C_0^\infty(S_r) \tag{4}$$

where

$$\frac{\partial g}{\partial \tilde{\omega}_i} = \sum_{j=1}^{2} g^{1/2} g^{ij}(\nu(x, Q(\tilde{\omega}))\, \tilde{\omega}_j - ({}^*\gamma)_j)$$

and $T\tilde{\omega} = T^*\kappa$ on ∂M.

We demonstrate differentiability for ω. Since the Euler equation and boundary conditions are completely analogous, the same results are true for $\tilde{\omega}$, and hence for ${}^*\nu\tilde{\omega}$, which is the solution of Regular Theorem 2, for $n = 2$.

In higher dimensions, the Euler equation for the conjugate problem gives rise to a system which cannot be handled by standard techniques. It is therefore necessary to treat the mass flow boundary value problem directly.

Let $\hat{\omega}$ be the 1-form which solves the problem in Regular Theorem 2. Then, $\rho\hat{\omega} - \kappa \in \mathscr{E}^\perp$ and therefore

$$\int_{S_r} \sum_{i,j=1}^{n} g^{1/2}(g^{ij}\rho\hat{\omega}_i - \kappa_i) \frac{\partial \zeta}{\partial x_j}\, dx = 0 \qquad \text{for every } \zeta \in H_2^{\,1}(M)$$

and $N\rho\hat{\omega} = N\kappa$ on ∂M.

If $\kappa = 0$ then $N\hat{\omega} = 0$. Differentiability for this problem is then basically the same as it is for the tangential problem and is true in any number of dimensions. If $\kappa \neq 0$, and $n = 3$, then a more delicate argument due to Ladyszenskaya and Ural'tseva [5] gives the necessary results. We first sketch the differentiability theory for ω, the solution of the tangential problem. Then, we will describe the techniques used in the inhomogeneous mass flow problem.

Since $d\omega = d\gamma = 0$, there exist functions φ and ψ in S_r such that $\omega = d\varphi$ and $\gamma = d\psi$. Let $\chi = \varphi - \psi$.

LEMMA 1. *In S_r, $\chi = 0$ for $x_n = 0$ and satisfies*

$$\int_{S_r} \sum_i \left(\sum_j A_{ij} \frac{\partial \chi}{\partial x_j} + E_{ij}\right) \frac{\partial \zeta}{\partial x_i}\, dx = 0 \qquad \textit{for all } \zeta \in C_0^\infty(S_r),$$

where

(i) $k\,|\,\xi\,|^2 \leqslant \sum_{i,j} A_{ij}\xi^i\xi^j \leqslant K\,|\,\xi\,|^2$,

(ii) $\sum_{i,j} E_{ij}^2 \leqslant M$.

Proof. Setting $A_{ij} = g^{1/2}g^{ij}\rho(x, Q(\omega))$ and $E_{ij} = g^{1/2}(A_{ij}\gamma_i - \sigma_j)$ in the Euler equation (3), conditions (i) and (ii) follow from the fact that ρ is regular and γ and σ are continuous.

Choose a unit vector e in a tangential direction, say x_k. Let

$$W = \varphi_h - \psi_h = \frac{\varphi(x + he) - \varphi(x)}{h} - \frac{\psi(x + he) - \psi(x)}{h}.$$

LEMMA 2. *In S_r, $W = 0$ for $x_n = 0$ and satisfies*

$$\int_{S_r} \sum_i \Big(\sum_j B_{ij} \frac{\partial W}{\partial x_j} + F_{ij}\Big) \frac{\partial \zeta}{\partial x_i}\, dx = 0 \qquad \textit{for all } \zeta \in C_0^\infty(S_r),$$

where

(i') $c\,|\,\xi\,|^2 \leqslant \sum_{i,j} B_{ij}\xi^i\xi^j \leqslant C\,|\,\xi\,|^2$,

(ii') $\sum |\,F_{ij}\,|^2 \leqslant N(1 + |\,\omega\,|^2)$.

Proof. Applying the difference quotient procedure in S_r, let

$$B_{ij} = \int_0^1 \frac{\partial^2 f}{\partial \omega_i\, \partial \omega_j}(x + t\, he, \omega(x) + t\, \Delta\omega)\, dt,$$

$$F_{ij} = \int_0^1 \frac{\partial^2 f}{\partial \omega_i\, \partial x_k}(x + t\, he, \omega(x) + t\, \Delta\omega)\, dt + B_{ij}\frac{\partial \psi_h}{\partial x_j},$$

where $\Delta\omega = \omega(x + he) - \omega(x)$.

Since

$$\frac{\partial^2 f}{\partial \omega_i\, \partial \omega_j} = (\rho g^{ij} + 2\rho'(g^{ik}\omega_k)(g^{jm}\omega_m))\, g^{1/2},$$

where ρ is regular, condition (i) holds.

Since $\gamma, \sigma \in C^{1+\mu}$ and

$$\frac{\partial^2 f}{\partial \omega_i\, \partial x_k} = \sum_{j=1}^{n} \left\{\frac{\partial}{\partial x_k}(g^{1/2}g^{ij})(\rho\omega_j - \sigma_j) + g^{1/2}g^{ij}\left(\frac{\partial \rho}{\partial x_k}\,\omega_j - \sigma_j\right) - g^{1/2}g^{ij}\frac{\partial \sigma_j}{\partial x_k}\right\},$$

we see that (ii') holds.

Standard arguments applied to W show

LEMMA 3. *The derivatives of the components of ω are in $\mathscr{L}_2$ and*

$$\int_{S_r} |\,\nabla\omega_j\,|^2\, dx < N', \qquad j = 1,\ldots, n,$$

with N' depending only on c, C, and N.

THEOREM (Morrey). *Let ∂M be Lipschitz. Let $u \in H^1_{20}(M)$ be a function satisfying*

$$\int \nabla\zeta(a \cdot \nabla u + e)\, dx = 0 \qquad \text{for all } \zeta \in C_0^\infty.$$

Suppose matrix a satisfies (i) *with constants b and B, and matrix e satisfies*

(ii″) $\int_{S_r} |e|^2\, dx \leqslant L r^{n-2+2\alpha}$, $0 < \alpha \leqslant 1$, *for every r, where* $|e|^2 = \sum_{i,j} |e_{ij}|^2$.

Then, there exists α_0, $0 < \alpha_0 < 1$, depending only on n, b, and B, such that $u \in C^\beta(\overline{M})$, where $\beta = \min(\alpha_0, \alpha)$. Also

(iii) $\int_{S_r} |\nabla u|^2\, dx \leqslant L' r^{n-2+2\beta}$ *where L' depends only on n, b, B, L, and α.*

To complete the proof of Regular Theorem 1, let $u = \chi$ of Lemma 1. Condition (ii″) of Morrey's theorem is satisfied with $\alpha = 1$. Therefore, χ, and hence φ, is Holder continuous and

$$\int_{S_r} |\nabla\varphi|^2\, dx = \int_{S_r} |\omega|^2 \leqslant L' r^{n-2+2\alpha_0} \qquad \text{for every } r.$$

Next, let $u = W$ in Lemma 2. Condition (ii′) and the above inequality imply (ii″). Therefore, with α_0 replaced by α_1 and $\beta = \min(\alpha_0, \alpha_1)$, we find that $W \in C^\beta$ independently of h. Letting $h \to 0$ and using the fact that γ is continuously differentiable, it follows that ω is Holder continuous with exponent $\beta = \min(\alpha_0, \alpha_1)$. Standard arguments for linear elliptic equations with Holder continuous coefficients now give $\omega \in C^{1+\mu'}(\overline{M})$ with $\mu' = \min(\alpha_0, \alpha_1, \mu)$, which means ω is a classical solution.

Next, we briefly describe the differentiability theory for the mass flow problem. The solution $\hat{\omega}$ is closed and therefore, $\hat{\omega} = d\hat{\varphi} = d\hat{\phi}$ locally, where

$$\int_{S_r} \sum_{i,j=1}^{n} g^{1/2} g^{ij} \left(\rho \frac{\partial\hat{\varphi}}{\partial x_i} - \kappa_i\right) \frac{\partial\zeta}{\partial x_j}\, dx = 0 \qquad \text{for all } \zeta \in H_2^1(M)$$

and $\rho(\partial\hat{\varphi}/\partial x_n) = \kappa_n$ on the boundary. Letting $\hat{W} = (\hat{\varphi}(x + he) - \hat{\varphi}(x))/h$, where e is a unit vector in the tangential direction, say x_k, we find, applying the difference quotient procedure, that

LEMMA 2′. *$\hat{W}$ satisfies the equation*

$$\int_{S_r} \sum_i \left(\sum_j B'_{ij} \frac{\partial\hat{W}}{\partial x_j} + F'_{ij}\right) \frac{\partial\zeta}{\partial x_i} = 0 \qquad \text{for all } \zeta \in H_2^1(M),$$

with conditions (i′) *and* (ii′) *as in Lemma* 2 *valid.*

LEMMA 3′. *The derivatives of the components of $\hat{\omega}$ are in $\mathcal{L}_2$ with norms depending only on the constants in* (i′) *and* (ii′).

This lemma is proved by letting $\zeta = \eta^2 \hat{W}$, where η is a function which vanishes outside the circular part of the boundary of S_{2r}. Note that it is not necessary for the functions ζ to vanish on the boundary of M.

If $n = 3$, then $\hat{\omega} \in H_2^1(M)$ which implies that $\hat{\varphi} \in H_2^2(S_r)$. From Sobolev's lemma, $\hat{\varphi}$ is continuous and $|\hat{\varphi}(x)| \leqslant C \|\hat{\varphi}\|_2^2$.

THEOREM (Ladyzhenskaya and Ural'tseva). *The solution* $\hat{\varphi} \in C^{1+\alpha}(S_r)$ *for some* α *and* $\|\hat{\varphi}\|_{1,\alpha}$ *is bounded by a constant depending only on* $\max |\hat{\varphi}(x)|$, *the constants in* (i') *and* (ii'), *and the boundary of* M, *where the boundary is of class* C^2.

It follows that $\hat{\omega} \in C^\alpha$, and is, in fact a classical solution.

LEMMA 4. *Suppose* (γ, σ) *and* (γ, κ) *are paths in* $\mathscr{D}$ *or* $\mathscr{N}$. *Then, the extremal* ω_s, *for each* s, *depends continuously on* s *in the uniform topology.*

Proof. If $\{s_n\}$ is a bounded sequence in I, then $\omega_n = \omega(s_n) \in C^\beta(M)$ for each n, with the same modulus of continuity. Hence, $\{\omega_n\}$ are equicontinuous. Since they are uniformly bounded in $\mathscr{L}_2$, they are uniformly bounded. By Arzela's theorem, the mapping from $s \in I$ to ω_s, the extremal, is compact. Hence, the conclusion of Lemma 4 follows from the corollary to the variational principle.

This completes the proofs of the regular theorems.

The method of Shiffman regularization (see [8] or [9]) now gives the solution of Theorems 1 and 2.

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