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An Investigation of Biorthogonal Polynomials Derivable from Ordinary Differential Equations of the Third Order*

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A theorem of Chaundy is exploited to prove the existence of only one ordinary differential equation of the third order of the type

$$\mathscr{A}(x)y''' + \mathscr{B}(x)y'' + \mathscr{C}(x)y' = \lambda y$$

from which biorthogonal polynomials are derivable. The method of proof permits us to discuss differential equations of the first and second order as well and shows that no first order ordinary differential equation and only five second order ordinary differential equations of the above type will generate biorthogonal polynomials.

I. INTRODUCTION

The object of this paper is to examine the differential equation (d.e.),

$$\mathscr{A}(x)y''' + \mathscr{B}(x)y'' + \mathscr{C}(x)y' = \lambda y,$$

in order to determine all differential equations of the above type which give rise to biorthogonal polynomials.

After defining the concept of biorthogonality to be used, it will be proved that no first order d.e., and only five second order d.e. of the above type can generate biorthogonal polynomials. The polynomials thus derived are the Hermite (and the trivially found pseudo-Hermite), Laguerre, Jacobi, pseudo-Jacobi, and the Bessel polynomials. The pseudo-Jacobi polynomials may be easily related to the Jacobi polynomials, but they do not appear to be discussed in the literature. Except for the "pseudo" polynomials, the properties of the above polynomials are discussed extensively in the literature [1-4].

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The principal theorem of this paper asserts that there is only one third order d.e. of the above type from which polynomials, biorthogonal in a sense to appear, are derivable. The appropriate d.e. are derived, and their polynomial solutions discussed in Sections IV and V. These polynomials were first discovered by Spencer and Fano [5], and were used in calculations involving the penetration of gamma rays through matter.

II. General Theory

In this section we develop the general theory to be used in the later sections.

Definition 1. Two polynomials $z_l(x)$, $y_n(x^m)$ will be called *biorthogonal* with respect to a weight function $\rho(x)$, if they satisfy a scalar product of the form,

$$(\rho z_l, y_n) \equiv \int_\Gamma \rho(x)\, z_l(x)\, y_n(x^m)\, dx = (y_n, \rho z_l) = N_n \delta_{nl},$$

where

$\rho(x)$ is a weight function, having as many continuous derivatives as the order of the d.e. we consider

$z_l(x)$ is a polynomial of degree l in x

$y_n(x^m)$ is a polynomial of degree n in x^m

$N_n \neq 0$, is a constant of normalization,

Γ is an arc or a closed curve in the complex plane,

$$\delta_{nl} = \begin{cases} 0 & \text{for} \quad n \neq l \\ 1 & \text{for} \quad n = l \end{cases}$$

m is an integer $\geqslant 1$.

If, when $m = 1$, $z_n(x) = y_n(x)$, then we call the polynomials "self-orthogonal" or merely "orthogonal."

Definition 2. Let

$$\mathscr{L} y_n \equiv \mathscr{A}(x) y_n''' + \mathscr{B}(x) y_n'' + \mathscr{C}(x) y_n' = \lambda_n y_n,$$

$$\mathscr{M} z_l \equiv \frac{1}{\rho(x)} [-(\rho \mathscr{A} z_l)''' + (\rho \mathscr{B} z_l)'' - (\rho \mathscr{C} z_l)'] = \lambda_l z_l,$$

$$\mathscr{N}(y_n, \rho z_l) \equiv [(\mathscr{A} \rho z_l)'' y_n - (\mathscr{A} \rho z_l)' y_n' + \mathscr{A} \rho z_l y_n'' + \mathscr{B} \rho z_l y_n' - (\mathscr{B} \rho z_l)' y_n + \mathscr{C} \rho z_l y_n]|_\Gamma,$$

where the prime denotes derivative with respect to x.

In a straightforward manner we arrive at

$$(\rho z_l, \mathscr{L} y_n) - (y_n, \rho \mathscr{M} z_l) = \mathscr{N}(y_n, \rho z_l) = (\lambda_n - \lambda_l)\,(y_n, \rho z_l).$$

We define the operator $\mathscr{M}$ to be the "*reduced adjoint operator*" to the "forward" operator $\mathscr{L}$. Its relation to the formal adjoint operator, is that we have removed the function $\rho(x)$ in such a way that the $z_l(x)$ shall be a polynomial of degree l in x. We will refer to $\mathscr{M} z_l = \lambda_l z_l$ as the "*reduced adjoint differential equation*", (r. a. d. e.) and to $\mathscr{N}(y_n, \rho z_l)$ as the "*conjunct*" or "*bilinear concomitant.*"

We now pose the following question. Given the differential equation

$$\mathscr{A}(x)y''' + \mathscr{B}(x)y'' + \mathscr{C}(x)y' = \lambda y \tag{2.1}$$

(where the prime denotes derivative with respect to x), then under what conditions will (2.1), along with its r. a. d. e., have polynomial solutions of degree n in x^m and x respectively, for m a fixed integer $\geqslant 1$, and $n = 0, 1, \cdots$, whenever $\lambda = \lambda_n$, a parameter independent of x, and $\mathscr{A}, \mathscr{B}, \mathscr{C}$ are functions of x independent of n?

THEOREM 1. (Bochner [6]). *If* $\alpha, \beta, \gamma, \delta, \xi, \eta, \zeta, \theta, \kappa$ *are arbitrary real numbers, then a necessary condition that*

$$A(u)y''' + B(u)y'' + C(u)y' = \lambda_n y, \tag{2.2}$$

(where the prime denotes derivative with respect to u) has polynomial solutions of order exactly n in u, for $n = 0, 1, 2, 3$ *is that,*

$$A(u) = \alpha u^3 + \beta u^2 + \gamma u + \delta \tag{2.3}$$

$$B(u) = \xi u^2 + \eta u + \zeta \tag{2.4}$$

$$C(u) = \theta u + \kappa. \tag{2.5}$$

PROOF. We assume our polynomials to be of the form,

$$y_n(u) = \sum_{j=0}^{n} a_{nj} u^j, \qquad a_{nn} = 1; \qquad a_{nj} \text{ real.} \tag{2.6}$$

For $n = 0$, (2.2) is satisfied only if $\lambda_0 = 0$. For $n = 1$, we see that $C(u) = \lambda_1(u + a_{10})$, and thereby satisfies (2.5). For $n = 2$, we have

$$2B(u) + C(u)\,(2u + a_{21}) = \lambda_2(u^2 + a_{21}u + a_{20}),$$

or $B(u)$ satisfies (2.4). For $n = 3$, a similar argument gives (2.3).

Note that if $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = 0$, then necessarily $C(u) = B(u) = A(u) = 0$. We have thus shown:

COROLLARY 1. *If* $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = 0$, then $C(u) = B(u) = A(u) = 0$, *or no d. e. of type (2.2), having polynomial solutions of all orders can occur.*

THEOREM 2. (Chaundy [7]). *Equation (2.2) will have a solution given by (2.6)* for $n = 0, 1, \cdots$, *when Eqs. (2.3), (2.4), (2.5) hold, provided* $\lambda_n \neq \lambda_l$ *for* $n \neq l$, *where*

$$\lambda_n = \alpha n(n-1)\cdot(n-2) + \xi n(n-1) + \theta n.$$

PROOF. The proof rests on our ability to determine $a_{n,j}$ for $j = 0, \cdots, n-1$, such that $a_{nn} = 1$. In order to determine the $a_{n,j}$ we proceed as follows:

Substitute (2.6) into (2.2). Upon multiplying both sides of the equation by u^3, we have

$$\sum_{j=0}^{n} a_{nj}\{u^{3+j}h_j + u^{2+j}g_j + u^{1+j}f_j + u^j e_j\} = \lambda_n \sum_{j=0}^{n} a_{nj}u^{3+j},$$

where

$$h_j = \alpha j(j-1)\cdot(j-2) + \xi j(j-1) + \theta j, \tag{2.7}$$

$$g_j = \beta j(j-1)\cdot(j-2) + \eta j(j-1) + \kappa j, \tag{2.8}$$

$$f_j = \gamma j(j-1)\cdot(j-2) + \zeta j(j-1), \tag{2.9}$$

$$e_j = \delta j(j-1)\cdot(j-2). \tag{2.10}$$

If we compare coefficients of like powers of u, we arrive at a set of $n+1$ equations in the unknowns $a_{n,j}$, $j = 0, \cdots, n-1$, and λ_n. Since we desire $a_{n,n} = 1$, we have,

$$\lambda_n = h_n = \alpha n(n-1)\cdot(n-2) + \xi n(n-1) + \theta n, \tag{2.11}$$

and if we replace h_j by its equivalent λ_j we have,

$$a_{n,n-1}\lambda_{n-1} + g_n = \lambda_n a_{n,n-1}, \tag{2.12}$$

$$a_{n,n-2}\lambda_{n-2} + a_{n,n-1}g_{n-1} + f_n = \lambda_n a_{n,n-2} \tag{2.13}$$

$$a_{n,j}\lambda_j + a_{n,j+1}g_{j+1} + a_{n,j+2}f_{j+2} + a_{n,j+3}e_{j+3} = \lambda_n a_{n,j}, \tag{2.14}$$

for $j = 0, 1, \cdots, n-3$.

Equations (2.11), (2.12), (2.13), (2.14) may be solved for λ_n, a_{nj}, $j = 0, \cdots, n-1$, provided only that $\lambda_n - \lambda_j \neq 0$ for $j = 0, \cdots, n-1$, but this is assured by the hypothesis.

COROLLARY 2. *There is only one linearly independent polynomial of degree exactly n in x, which can be a solution of (2.2).*

PROOF. Suppose

$$y_n = \sum_{j=0}^{n} a_{nj} x^j \qquad \text{and} \qquad z_n = \sum_{j=0}^{n} b_{nj} x^j,$$

where $a_{nn} = b_{nn} = 1$, are two linearly independent solutions of (2.2). It follows that

$$Y_n = y_n - z_n = \sum_{j=0}^{n-1} c_{nj} x^j$$

is a polynomial of degree at most $n-1$, and must satisfy (2.2). We wish to show that $Y_n = 0$, or equivalently, $c_{nj} = 0$ for $j = 0, 1, \cdots, n-1$. If we substitute Y_n into (2.2) we find, by comparing the highest power of x on both sides that $\lambda_{n-1} c_{n,n-1} = \lambda_n c_{n,n-1}$. But $\lambda_n \neq \lambda_{n-1}$, whence $c_{n,n-1} = 0$. By a repeated application of this argument we conclude that $c_{nj} = 0$ for $j = 0, \cdots, n-1$, and all n.

If we desire Eq. (2.2) to have polynomial solutions of degree n in x^m, we merely make a change of independent variable such that $u = x^m$. This change of variable transforms Eq. (2.2) into Eq. (2.1) where,

$$\mathscr{A}(x) = \frac{x^{3-3m}}{m^3} A(x^m), \tag{2.15}$$

$$\mathscr{B}(x) = \frac{3(1-m)x^{2-3m}}{m^3} A(x^m) + \frac{x^{2-2m}}{m^2} B(x^m), \tag{2.16}$$

$$\mathscr{C}(x) = \frac{(1-m)(1-2m)}{m^3} x^{1-3m} A(x^m) + \frac{(1-m)x^{1-2m}}{m^2} B(x^m) + \frac{C(x^m)}{m} x^{1-m} \tag{2.17}$$

(Note that for $m = 1$, $\mathscr{A} = A$, $\mathscr{B} = B$, $\mathscr{C} = C$.) Equation (2.1) will have polynomial solutions of order n in x^m for $\lambda = \lambda_n$, when the conditions of Theorem 2 are satisfied.

The r.a.d.e. to Eq. (2.1), after some manipulation, may be recognized to be,

$$-\mathscr{A}z''' + \left[\mathscr{B} - \frac{3}{\rho}(\mathscr{A}\rho)'\right]z'' + \left[-\frac{3}{\rho}(\mathscr{A}\rho)'' + \frac{2}{\rho}(\mathscr{B}\rho)' - \mathscr{C}\right]z'$$

$$+ \frac{1}{\rho}[-(\mathscr{A}\rho)''' + (\mathscr{B}\rho)'' - (\mathscr{C}\rho)']z = \lambda z, \tag{2.18}$$

where the prime denotes derivative with respect to x. If we now ask that (2.18) have polynomial solutions of degree n in x when $\lambda = \lambda_n$, we find, by application of Theorem 1, that

$$-(\mathscr{A}\rho)''' + (\mathscr{B}\rho)'' - (\mathscr{C}\rho)' = 0, \tag{2.19}$$

$$-\mathscr{A} = \alpha x^3 + \epsilon x^2 + \mu x + \nu, \tag{2.20}$$

$$\mathscr{B} - \frac{3}{\rho}(\mathscr{A}\rho)' = \xi x^2 + \varphi x + \omega = \mathscr{B}^*, \tag{2.21}$$

$$-\frac{3}{\rho}(\mathscr{A}\rho)'' + \frac{2}{\rho}(\mathscr{B}\rho)' - \mathscr{C} = \theta x + \tau = \mathscr{C}^*. \tag{2.22}$$

The appearance of α, ξ, and θ is forced since Eq. (2.2) and (2.18) must have the same eigenvalue $\lambda = \lambda_n$, for $n = 0, 1, \cdots$. We will refer to Eq. (2.19)-(2.22) as our "compatibility equations." Each of these equations must be satisfied if we are to have biorthogonal polynomials, but they by themselves are *not* sufficient. We will require in addition that our polynomials satisfy a scalar product as defined previously. Hence the "conjunct" given by $\mathscr{N}(y_n, \rho z_l)$ must vanish identically for all $n, l = 0, 1, \cdots$.

By virtue of (2.19) we have,

$$(\mathscr{A}\rho)'' - (\mathscr{B}\rho)' + \mathscr{C}\rho = k, \text{ a constant.} \tag{2.23}$$

If we carry out the differentiations in $\mathscr{N}(y_n, \rho z_l)$ and substitute for (2.23) we get, with some manipulation,

$$\{kz_l y_n + \mathscr{A}\rho[z_l y_n'' - z_l' y_n' + y_n z_l''] + (\mathscr{A}\rho)'[2y_n z_l' - z_l y_n']$$

$$+ \mathscr{B}\rho[z_l y_n' - z_l' y_n]\}\,|_{\Gamma} = 0. \tag{2.24}$$

We are now in a position to apply our equations of compatibility, and our conjunct condition, in order to analyze their consequences.

III. Discussion of First and Second Order Differential Equations

Theorem 3. *There exist no biorthogonal polynomials associated with first order differential equations of type (2.1).*

Proof. This follows immediately since for first order d.e., $\mathscr{A}(x) = \mathscr{B}(x) = 0$. Hence, the forward d.e. becomes,

$$\mathscr{C}y_n' = \lambda_n y_n, \tag{3.1}$$

and the r.a.d.e. is given by,

$$-\mathscr{C}z_n' = \lambda_n z_n. \tag{3.2}$$

Therefore λ_n from (3.1) is,

$$\lambda_n = \theta n, \tag{3.3}$$

and λ_n from (3.2) must be

$$\lambda_n = -\theta n. \tag{3.4}$$

Equations (3.3) and (3.4) together imply that $\lambda_n = 0$. If we now invoke Corollary 1, Theorem 3 follows.

We now consider the possibility of there being biorthogonal polynomials associated with second order differential equations. We assume here that $\mathscr{A}(x) = 0$. For any m we have,

$$\mathscr{B}(x) = \frac{x^{2-2m}}{m^2}(\xi x^{2m} + \eta x^m + \zeta), \tag{3 5}$$

which follows from (2.16) and (2.4). On the other hand (2.21) requires that

$$\mathscr{B}(x) = \xi x^2 + \varphi x + \omega. \tag{3.6}$$

For $m \geqslant 3$, (3.5) and (3.6) will be consistent only if $\mathscr{B}(x) \equiv 0$, or the d.e. reduces to the first order differential equation previously discussed.

For $m = 2$, we will get consistency only if $\xi = \zeta = \varphi = 0$, and if,

$$\mathscr{B}(x) = \frac{\eta}{4} = \omega. \tag{3.7}$$

Equation (2.22) gives,

$$\frac{\rho'}{\rho} = \frac{1}{\eta}\left(3\theta x + 2\tau + \frac{2\kappa - \eta}{2x}\right), \tag{3.8}$$

where we have tacitly assumed that $\eta \neq 0$. (If $\eta = 0$, we would again have the first order d.e. discussed previously.)

A useful identity involving derivatives of ρ is given by,

$$\frac{\rho''}{\rho} = \left(\frac{\rho'}{\rho}\right)' + \left(\frac{\rho'}{\rho}\right)^2. \tag{3.9}$$

If we substitute (3.8) and (3.9) into (2.19), we find that the coefficient of x^2 that results is $3\theta^2/4\eta$, which must vanish identically. Thus $\theta = 0$, and we previously had $\xi = 0$, or $\lambda_n = 0$ for all n. Corollary 1 thus leads us to conclude that we can have no biorthogonal polynomials associated with second order d.e. when $m \geqslant 2$.

We now come to the case of $m = 1$. It will become convenient to categorize our r.a.d.e. according to the behavior of the quadratic coefficient $\mathscr{B}(x) = B(x)$. Following Bochner [6] we convert $B(x)$, via a real linear transformation of the form $t = rx + s$, $r \neq 0$, to their so called "normal form." This amounts to placing the zeros of $B(x)$ at convenient places on either the real or imaginary axis. We will assume that the transformation has been carried out, leaving only the five following possibilities.

1. $B(x) = 1 = \zeta$, $\xi = \eta = 0$,
2. $B(x) = x$, $\xi = \zeta = 0$, $\eta = 1$,
3. $B(x) = 1 - x^2$, $\zeta = 1 = -\xi$, $\eta = 0$,
4. $B(x) = 1 + x^2$, $\zeta = 1 = \xi$, $\eta = 0$,
5. $B(x) = x^2$, $\zeta = \eta = 0$, $\xi = 1$.

(In place of 3 and 4, Bochner chose $B(x) = x(1 - x)$, that is, both zeros on the real axis. By considering the possibility of zeros on the imaginary axis, we have found a set of "pseudo-Jacobi" polynomials not previously discussed elsewhere.)

In order to facilitate matters, we detour for a short while and analyze our conjunct condition. Equation (2.24) when specialized to second order d.e. for $m = 1$ becomes,

$$[kz_l y_n + B(x)\,\rho(z_l y_n' - z_l' y_n)]\,|_\Gamma = 0. \tag{3.10}$$

We have previously assumed that

$$y_n = \sum_{j=0}^{n} a_{nj} x^j, \qquad a_{nn} = 1, \tag{3.11}$$

and we now assume,

$$z_n = \sum_{j=0}^{n} b_{nj} x^j, \qquad b_{nn} = 1. \tag{3.12}$$

Consider (3.10) when $z_0 = 1$, $y_1 = x + a_{10}$, then

$$[k(x + a_{10}) + B(x)\rho] \,|_\Gamma = 0. \tag{3.13}$$

Similarly for $z_1 = x + b_{10}$, $y_0 = 1$, we have,

$$[k(x + b_{10}) - B(x)\rho] \,|_\Gamma = 0. \tag{3.14}$$

For Γ an arc in the complex plane, with distinct end points (a, b), we find by adding (3.13), (3.14) that $k \equiv 0$. Consequently, from (3.14) or (3.13) we have,

$$B(b)\rho(b) = B(a)\rho(a). \tag{3.15}$$

For $y_2 = x^2 + a_{21}x + a_{20}$, $z_0 = 1$, we have, since $k = 0$,

$$[B(x)\rho(x)\,(2x + a_{21})] \,|_\Gamma = 0, \tag{3.16}$$

or

$$bB(b)\rho(b) - aB(a)\rho(a) = 0. \tag{3.17}$$

But (3.17) and (3.15) imply $B(b)\rho(b) = B(a)\rho(a) = 0$, since $b \neq a$. For $k = 0$, it follows from (2.19) and (2.22), that $\tau = \kappa$, and hence for second order differential equations with $m = 1$, the polynomials are "*self-orthogonal*," i.e., $y_n = z_n$†. In this case the forward d.e. and the r.a.d.e. are identical! Two interesting facts have emerged from this development, namely, from (2.23),

$$\frac{\rho'}{\rho} = \frac{C - B'}{B}, \tag{3.18}$$

where $C - B'$ is linear in x; and $B(x)\rho(x)$ must vanish at the end points of Γ if Γ is not a closed curve. Both of these facts are assumed by Jackson [3, p. 162] in the derivation of the classical orthogonal polygomials.

We are now prepared to discuss Cases 1-5. Our intention is to display the biorthogonal polynomials derivable from the appropriate d.e. when Γ and the associated parameters are properly chosen. Case 1 leads to the Hermite and pseudo-Hermite polynomials, Case 2 leads to the generalized Laguerre polynomials, Cases 3 and 4 lead to the Jacobi and pseudo-Jacobi polynomials, and finally Case 5 leads to the generalized Bessel polynomials.

Case 1. $B(x) = 1 = \zeta$, $\xi = \eta = 0$.

Equation (2.22), with no loss of generality, may be reduced to

$$\frac{\rho'}{\rho} = \theta x, \tag{3.19}$$

$$\rho = \rho_0 e^{\theta x^2/2} \tag{3.20}$$

† Lemma 1 of section IV gives an alternative method of deducing $y_n = z_n$.

(where ρ_0 may be chosen as unity†), and the appropriate differential equation is given by,

$$y_n'' + \theta x y_n' = n\theta y_n. \tag{3.21}$$

Since we wish $\rho(x)B(x)$ to vanish on (a, b) we must choose $\theta < 0$ and the interval $(-\infty, \infty)$, or $\theta > 0$ and the interval $(-i\infty, i\infty)$. If we choose $\theta = -1$ we get the Hermite polynomials as defined by Jackson, i.e.,

$$y_n(x) = H_n(x) = (-)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}. \tag{3.22}$$

If we choose $\theta = 1$ we get what we will call the pseudo-Hermite polynomials. These polynomials are given by $y_n^{\mathrm{p.H.}}(x) = (-i)^n H_n(ix)$.

Case 2. $B(x) = x$, $\xi = \zeta = 0$, $\eta = 1$.
Equation (2.22) reduces to,

$$\frac{\rho'}{\rho} = \theta + \frac{(\kappa - 1)}{x} \tag{3.23}$$

or

$$\rho = x^{\kappa-1} e^{\theta x}. \tag{3.24}$$

The d.e. one arrives at is,

$$x y_n'' + (\theta x + \kappa) y_n' = n\theta y_n. \tag{3.25}$$

For $\rho(x)B(x)$ to vanish at (a, b) we choose $\kappa > 0$, $\theta = -1$ and $(0, \infty)$ for the interval of integration. This leads to the generalized Laguerre equation whose solution is given by

$$y_n = L_n^{(\kappa-1)}(x) = (-)^n x^{1-\kappa} e^x \frac{d^n}{dx^n} (x^{\kappa-1+n} e^{-x}).$$

The ordinary Laguerre equation results if $\kappa \equiv 1$.

Case 3. $B(x) = 1 - x^2$, $\zeta = 1 = -\xi$, $\eta = 0$.
Equation (3.18) becomes,

$$\frac{\rho'}{\rho} = \frac{(\theta + 2)x + \kappa}{1 - x^2} = \frac{(\bar\beta - \bar\alpha) - (\bar\beta + \bar\alpha)x}{1 - x^2} = \frac{\bar\beta}{1 + x} - \frac{\bar\alpha}{1 - x}, \tag{3.26}$$

where $\theta + 2 = -(\bar\alpha + \bar\beta)$, and $\kappa = \bar\beta - \bar\alpha$. Hence

$$\rho = (1 + x)^{\bar\beta} (1 - x)^{\bar\alpha}. \tag{3.27}$$

† A similar choice is made throughout this paper.

In order for $\rho(x)B(x)$ to vanish at (a, b) we choose $a = -1$, $b = 1$ and require that $\bar{\beta}, \bar{\alpha} > -1$. The differential equation may easily be seen to be,

$$(1 - x^2)y_n'' + [\bar{\beta} - \bar{\alpha} - (2 + \bar{\alpha} + \bar{\beta})x]y_n' = -n(n + 1 + \bar{\alpha} + \bar{\beta})y_n. \quad (3.28)$$

The polynomial solutions of this d.e. are the Jacobi polynomials, given by

$$y_n(x) = P_n^{(\bar{\alpha},\bar{\beta})}(x) = \frac{(-)^n}{2^n n!}(1 - x)^{-\bar{\alpha}}(1 + x)^{-\bar{\beta}} \frac{d^n}{dx^n}\{(1 - x)^{\bar{\alpha}+n}(1 + x)^{\bar{\beta}+n}\}, \quad (3.29)$$

which include as special cases, the Legendre polynomials when $\bar{\alpha} = \bar{\beta} = 0$, the ultraspherical polynomials when $\bar{\alpha} = \bar{\beta}$, and the Tchebycheff polynomials when $\bar{\alpha} = \bar{\beta} = \pm\frac{1}{2}$.

Case 4. $B(x) = 1 + x^2$, $\zeta = 1 = \xi$, $\eta = 0$. θ, κ real numbers.

Equation (3.18) becomes

$$\frac{\rho'}{\rho} = \frac{(\theta - 2)x + \kappa}{1 + x^2}, \quad (3.30)$$

or

$$\begin{aligned}\rho &= (1 + x^2)^{\theta/2-1} e^{\kappa \arctan x} = (1 + x^2)^{\theta/2-1} \exp\left[\frac{\kappa}{2i}\ln\left(\frac{1 + ix}{1 - ix}\right)\right] \\ &= (1 + ix)^{(\theta/2)-1-(i\kappa/2)}(1 - ix)^{(\theta/2)-1+(i\kappa/2)}. \end{aligned} \quad (3.31)$$

For $\rho(x)B(x)$ to vanish at (a, b), we choose the interval to be $(-i, i)$ and require that $\theta > 0$. For such a weight function the conjunct will vanish for all n. The differential equation we arrive at is

$$(1 + x^2)y_n'' + (\theta x + \kappa)y_n' = n(n + \theta - 1)y_n. \quad (3.32)$$

A natural question to ask at this point is the relationship between the polynomial solutions of (3.31) and the Jacobi polynomials.

If we consider the change of variable $x = -it$, then (3.32) gives,

$$(1 - t^2)y_n'' - (\kappa i + \theta t)y_n' = -n(n + \theta - 1)y_n, \quad (3.33)$$

where the prime denotes derivative with respect to t. For comparison we write the Jacobi d.e. as,

$$(1 - x^2)y_n'' + (\kappa + \theta x)y_n' = -n(n - 1 - \theta)y_n \quad (3.28)$$

where we have replaced the combinations of $\bar{\alpha}$, $\bar{\beta}$ by κ, θ, and where the prime denotes derivative with respect to x. The similarity is now quite clear. If the pseudo-Jacobi (p.J.) polynomials are evaluated for a pure imaginary argument, and further, $\theta_{\overrightarrow{\text{p.J.}}} - \theta$; $\kappa_{\text{p.J.}} \rightarrow i\kappa$, we then arrive at the ordinary Jacobi polynomials. Specifically we have,

$$\bar{\beta} + \bar{\alpha} = \theta_{\text{p.J.}} - 2 \qquad \text{and} \qquad \bar{\beta} - \bar{\alpha} = -i\kappa_{\text{p.J.}}.$$

This requires $\bar{a}$, $\bar{\beta}$ to be complex conjugates of each other. If we pursue this we see that for x pure imaginary,

$$y_n(x) = P_n^{(\bar{\alpha},\bar{\beta})}(x) = \frac{(-)^n}{2^n n!}(1 + x^2)^{-(\bar{\alpha}+\bar{\beta})/2} \exp[-i(\bar{\beta} - \bar{\alpha}) \arctan x].$$

$$\frac{d^n}{dx^n}\{(1 + x^2)^{[(\bar{\alpha}+\bar{\beta})/2]+n} \exp[i(\bar{\beta} - \bar{\alpha}) \arctan x]\}.$$

The fact that the Jacobi polynomials may be interpreted for complex values $\bar{a}$, $\bar{\beta}$ is stated by Szego [2, p. 62]. However, Szego does not list the orthogonality of these polynomials as one of their properties, nor does he state that $\bar{a}$, $\bar{\beta}$ must be complex conjugates if $P_{(n)x}^{(\bar{\alpha},\bar{\beta})}$ is to be a polynomial with real coefficients.

Case 5. $B(x) = x^2$, $\eta = \zeta = 0$, $\xi = 1$.

Equation (2.22) reduces to,

$$\frac{\rho'}{\rho} = \frac{\theta - 2}{x} + \frac{\kappa}{x^2}, \tag{3.34}$$

or

$$\rho = x^{\theta-2}e^{-\kappa/x}. \tag{3.35}$$

The restriction that $\rho(x)B(x)$ vanish at (a, b) makes it necessary to choose (a, b) to be $(0, \infty)$ for $\theta < 0$. On the other hand, for sufficiently large values of n, the scalar product of definition 1 will *not* exist, and hence we discard this possibility. There exists however a hitherto unexplored possibility, namely, a closed curve in the complex plane. In particular we choose with Krall and Frink [4], the perimeter of the unit circle. (This choice is not unique—see [4].) For this contour the conjunct will surely vanish whenever $|\theta|$ is an integer but two points remain unclear: k need not be zero, and we cannot as yet be sure that such polynomials may be normalized. We proceed as follows: If $k \neq 0$ then (2.22) gives,

$$\frac{\rho'}{\rho} = \frac{(\theta - 2)}{x} + \frac{\kappa + \tau}{2x^2} \equiv \frac{\sigma}{x} + \frac{\Omega}{x^2}. \tag{3.36}$$

If we substitute (3.36) into (2.19) we get, after some manipulation,

$$x^2\left(-\frac{\sigma}{x^2}-\frac{2\Omega}{x^3}+\frac{\sigma^2}{x^2}+\frac{2\Omega\sigma}{x^3}+\frac{\Omega^2}{x^4}\right)$$
$$+(4x-\theta x-\kappa)\left(\frac{\sigma}{x}+\frac{\Omega}{x^2}\right)+2-\theta=0. \tag{3.37}$$

The above leads to,

$$\Omega(\Omega-\kappa)=0, \tag{3.38}$$

$$2\Omega\sigma-2\Omega+(4-\theta)\Omega-\kappa\sigma=0, \tag{3.39}$$

$$\sigma^2-\sigma+(4-\theta)\sigma+2-\theta=0. \tag{3.40}$$

From (3.38) either $\Omega = 0$ or $\Omega = \kappa$. Suppose $\Omega = 0$, then $\kappa\sigma = 0$. If $\sigma \equiv 0$ then ρ would be a constant and there would be no hope of normalizing our polynomials. We therefore assume that $\kappa \equiv 0$, and since $\sigma \equiv \theta - 2$ equation (3.40) is identically satisfied.

If we solve (3.36), we get $\rho = x^{\theta-2}$. For $\theta \geqslant 2$, there is no hope of normalizing our polynomials—and for $\theta < 2$, there will exist a number $\{2-\theta\}$ such that for an index n greater than $2-\theta$ no normalization can exist. This follows since the solution of the forward equation is easily recognized to be $x^n + b_n x^r$, for $0 \leqslant r \leqslant n-1$ such that $n+r+\theta-1=0$, where $b_n \equiv 0$ if $r > n-1$ or $r < 0$. All we need show is that if $n > 2-\theta$, then $(x^n + b_n x^r)^2 x^{\theta-2}$ can have no term in x^{-1}. This is sufficient since any other integral power of x around this contour vanishes. We consider $x^{2n+\theta-2}$, $x^{n+r+\theta-2}$, $x^{2r+\theta-2}$ clearly for $n > 2-\theta$, r would be negative, thereby excluding the last two terms. The minimum power of the first term, in this case, becomes $x^{2-\theta}$, and since $2-\theta$ must be positive our conclusion follows. Consequently, for $\kappa = 0$ no polynomials of interest can arise.

We now treat the case where $\Omega \neq 0$, or (3.38) requires that $\Omega = \kappa$. In that case, we have $\kappa = \tau = \Omega$ as well, or the forward and r.a.d.e. are identical. If we assume $\kappa = 0$, we are back in the previous case and hence, we choose $\kappa \neq 0$. Equations (3.39) and (3.40) are satisfied identically in either case. The weight function becomes

$$\rho = x^{\theta-2}e^{-\kappa/x},$$

and for this weight function one may easily show that $k \equiv 0$. The appropriate d.e. is given by,

$$x^2 y_n'' + (\theta x + \kappa) y_n = n(n+\theta-1) y_n. \tag{3.41}$$

The generalized Bessel polynomials may be given by,

$$y_n(x,\theta,\kappa) = \sum_{j=0}^{n} \binom{n}{j} \frac{(n+j+\theta-2)!}{(n+\theta-2)!} \left(\frac{x}{\kappa}\right)^j. \tag{3.42}$$

The Bessel polynomials proper are given by (3.42) when $\theta = \kappa = 2$. The demonstration of the normalization of the Bessel polynomials is proved in [4], along with many other properties of interest. For θ an integer, the generalized Bessel polynomials are capable of a similar treatment as the Bessel polynomials, however for nonintegral values of θ, Krall and Frink introduce a weight function other than the one considered here.

With the conclusion of Case 5 we now summarize the results of this section in the following theorems.

THEOREM 4. *The only second order d.e., of type (2.1), apart from real linear transformations, which give rise to biorthogonal polynomials, are those discussed in Cases 1-5. Moreover, the polynomials derivable from these d.e. are orthogonal.*

THEOREM 5. *The only second order d.e., of type (2.1), apart from real linear transformations, which give rise to orthogonal polynomials whose scalar product is defined on the real line, are those discussed in cases 1-3. The polynomials so derivable are the Hermite, Laguerre and Jacobi polynomials.*

IV. Discussion of Third Order Differential Equations

This section is devoted to the principal theorem of this paper, namely:

THEOREM 6. *There exists only one ordinary d.e. of type (2.1) of the third order, apart from real linear transformations, from which biorthogonal polynomials are derivable.*

The proof of this theorem depends on our ability to show that no biorthogonal polynomials are possible for $m \geqslant 3$, and for $m = 1$. The proof for $m \geqslant 3$ will follow from a direct application of our compatibility equations. Although this direct application when applied to $m = 1$ will also work, it is immensely tedious. Instead we appeal to an indirect argument to show that no biorthogonal polynomials are possible. Finally, a discussion for $m = 2$ will lead to the differential equations for the only biorthogonal polynomials possible.

PROOF. An examination of Eqs. (2.15) and (2.20) shows that,

$$-\frac{x^{3-3m}}{m^3}(\alpha x^{3m} + \beta x^{2m} + \gamma x^m + \delta) = \alpha x^3 + \epsilon x^2 + \mu x + \nu. \qquad (4.1)$$

For any $m \geqslant 1$, we see that $\alpha \equiv 0$. Moreover, for $m \geqslant 4$ it is quite clear that α, β, γ, δ, ϵ, μ, ν are all identically zero. Hence it follows that $\mathscr{A} \equiv 0$, or no third order differential equation leading to biorthogonal polynomials is possible when $m \geqslant 4$. For $m = 3$ we have α, γ, δ, ϵ, μ are zero, and $-\beta/27 = \nu$. If $\beta = 0$, the same conclusion as above follows. For $\beta \neq 0$, we find that Eq. (2.21) gives,

$$\begin{aligned} \frac{\rho'}{\rho} &= \frac{9}{\beta}\left[(\xi x^2 + \varphi x + \omega) + \frac{2}{9}\frac{\beta}{x} - \frac{1}{9}\left(\xi x^2 + \frac{\eta}{x} + \frac{\zeta}{x^4}\right)\right], \\ &= \frac{8\xi}{\beta}x^2 + \frac{9\varphi}{\beta}x + \frac{9\omega}{\beta} + \frac{2\beta - \eta}{\beta x} - \frac{\zeta}{\beta x^4}. \end{aligned} \tag{4.2}$$

If we now turn our attention to (2.19) we see that it may be equivalently stated to be,

$$\begin{aligned} -\mathscr{A}\frac{\rho'''}{\rho} + (\mathscr{B} - 3\mathscr{A}')\frac{\rho''}{\rho} + (2\mathscr{B}' - 3\mathscr{A}'' - \mathscr{C})\frac{\rho'}{\rho} \\ -(\mathscr{A}''' - \mathscr{B}'' + \mathscr{C}') = 0. \end{aligned} \tag{4.3}$$

By virtue of some obvious identities such as (3.9) and

$$\frac{\rho'''}{\rho} = \left(\frac{\rho'}{\rho}\right)'' + \frac{3\rho'}{\rho}\left(\frac{\rho'}{\rho}\right)' + \left(\frac{\rho'}{\rho}\right)^3, \tag{4.4}$$

we may evaluate (4.3) for the coefficient of the highest coefficient in x, which must be identically zero. If we carry this program out, we find that necessarily

$$\frac{11(8)^2}{27}\frac{\xi^3}{\beta^2} = 0. \tag{4.5}$$

Since $\beta \neq 0$ we conclude that $\xi \equiv 0$. If we apply the same procedure again we find,

$$\frac{9^2\varphi^2}{\beta^2} = 0 \tag{4.6}$$

or $\varphi \equiv 0$.

We now apply a similar program to Equation (2.22), or

$$-3\mathscr{A}\frac{\rho''}{\rho} + (2\mathscr{B} - 6\mathscr{A}')\frac{\rho'}{\rho} - 3\mathscr{A}'' + 2\mathscr{B}' - \mathscr{C} = \theta x + \tau. \tag{4.7}$$

The coefficient of the highest term in x when $\zeta = \varphi = 0$, becomes $4\theta/3 = 0$, or $\theta = 0$ too! But $\alpha = \xi = \theta = 0$ implies that $\lambda_n \equiv 0$ and if we apply Corollary 1 we see that hence no biorthogonal polynomials are possible when $m = 3$.

We now consider the case of $m = 1$. The proof of this case depends on the following two lemmas.

LEMMA 1. *If two sequences $y_n(x)$, $z_l(x)$, are both polynomials of orders n, l in x respectively, such that they satisfy a scalar product of the form*

$$\int_\Gamma \rho(x) z_l(x) y_n(x) dx = N_n \delta_{nl},$$

then necessarily $z_n(x) \equiv y_n(x)$, where $n, l = 0, 1, \cdots$.

PROOF. As before, we assume that the leading coefficient of our polynomials is equal to one. Hence for $n = l = 0$, $y_0 = z_0$. Suppose now that $z_n(x) = y_n(x)$ for $n = 0, 1, \cdots, M-1$. Then we wish to conclude that $z_M(x) = y_M(x)$. Since z_M and y_M are both polynomials, there exists some linear combination of the y_n such that

$$z_M = \sum_{j=0}^{M} h_j y_j(x).$$

Since the coefficient of the highest power of x on both sides is unity, we see that $h_M = 1$. Our scalar product holds for $n = 0, \cdots, M-1$, and in particular we see that,

$$\int_\Gamma z_M y_n \rho dx = \sum_{j=0}^{M} h_j \int_\Gamma y_j y_n \rho dx = \sum_{j=0}^{M} h_j N_n \delta_{nj} = 0,$$

for each value of n. The last term implies, however, that $h_n \equiv 0$ for $n = 0, \cdots M-1$, since $N_n \neq 0$ or $z_M = y_M$.

Lemma 1 implies that if $m = 1$, so that the forward and r.a.d.e. both have polynomial solutions satisfying a scalar product of the form we consider, then independently of the fact that they satisfy a d.e. of some order, the polynomials of the same index must be identical. We now state and prove a second Lemma.

LEMMA 2. *If the polynomial solutions of the forward and r.a.d.e. of type (2.1) are $y_n(x)$, then $\mathscr{A}(x) \equiv 0$.*

PROOF. The forward d.e. is given by,

$$\mathscr{A}(x) y_n''' + \mathscr{B}(x) y_n'' + \mathscr{C}(x) y_n' = \lambda_n y_n,$$

and the r.a.d.e. is given by,

$$-\mathscr{A}(x) y_n''' + \mathscr{B}^*(x) y_n'' + \mathscr{C}^*(x) y_n' = \lambda_n y_n.$$

Since $m = 1$, $\mathscr{A}$ is at most cubic, $\mathscr{B}, \mathscr{B}^*$ are at most quadratic polynomials, and finally $\mathscr{C}, \mathscr{C}^*$ are at most linear polynomials. If we subtract the second from the first we get,

$$2\mathscr{A}y_n''' + (\mathscr{B} - \mathscr{B}^*)y_n'' + (\mathscr{C} - \mathscr{C}^*)y_n' = 0.$$

This must be true for $n = 0, 1, \cdots$, and is in particular, true for $n = 1, 2, 3$. For $n = 1$ we see that it can only be true if $\mathscr{C} = \mathscr{C}^*$. For $n = 2$, it can be true only if $\mathscr{B} = \mathscr{B}^*$. For $n = 3$ it will be true only if $\mathscr{A} \equiv 0$, and the Lemma is proved.

We now return to the case where $m = 2$, this being the only case to be considered in our discussion for third order d.e.

For $m = 2$, Eq. (4.1) gives. $\alpha = \gamma = \delta = \epsilon = \nu = 0$, and $-\beta/8 = \mu$. We now assume, with no loss of generality, that $\mu = 8 = m^3$, or $\mathscr{A} = x$. Equation (2.21) gives,

$$\frac{\rho'}{\rho} = -\frac{\xi x}{4} - \frac{\varphi}{3} + \frac{\eta - 24 - 4\omega}{12x} + \frac{\zeta}{12x^3}. \tag{4.8}$$

If we examine (4.7) for the highest power of x we find that, $\xi^2/8 = 0$ or $\xi \equiv 0$. Similarly, if we examine (4.7) for the lowest power of x, we find that $\zeta^2/48 = 0$, or $\zeta \equiv 0$. Hence (4.8) becomes,

$$\frac{\rho'}{\rho} = \sigma + \frac{\Omega}{x}, \tag{4.9}$$

where

$$\sigma = -\frac{\varphi}{3}; \qquad \Omega = \frac{\eta - 24 - 4\omega}{12}.$$

For the sake of compactness we state the following formulas found so far for $m = 2$.

$$\mathscr{A}(x) = x, \tag{4.10}$$

$$\mathscr{B}(x) = \frac{\eta - 12}{4}, \tag{4.11}$$

$$\mathscr{C}(x) = \frac{12 - \eta + 2\kappa}{4x} + \frac{\theta x}{2}, \tag{4.12}$$

$$\mathscr{B}^* = \varphi x + \omega, \tag{4.13}$$

$$\mathscr{C}^* = \theta x + \tau. \tag{4.14}$$

If we substitute Eqs. (4.9)-(4.14) into Eqs. (4.3) and (4.7) and equate the coefficients of the powers of x on both sides of the equation we find, after some manipulation,

$$x: \qquad \sigma\left(\sigma^2 + \frac{\theta}{2}\right) = 0, \tag{4.15}$$

$$\text{const:} \qquad -3\sigma^2\Omega + \sigma^2\left(\frac{\eta}{4} - 6\right) - \frac{\theta\Omega}{2} - \frac{\theta}{2} = 0, \tag{4.16}$$

$$x^{-1}: \qquad 3\Omega\sigma - 3\sigma\Omega^2 + 2\sigma\Omega\left(\frac{\eta}{4} - 6\right) - \frac{(12 - \eta + 2\kappa)}{4}\sigma = 0, \tag{4.17}$$

$$x^{-2}: \qquad -2\Omega + 3\Omega^2 - \Omega^3 + (\Omega^2 - \Omega)\left(\frac{\eta}{4} - 6\right)$$

$$-\frac{(12 - \eta + 2\kappa)}{4}(\Omega - 1) = 0, \tag{4.18}$$

from Equation (4.3). Equation (4.7) gives

$$x: \qquad \sigma^2 + \frac{\theta}{2} = 0 \tag{4.19}$$

$$\text{constant:} \qquad -6\sigma\Omega + \left(\frac{\eta}{2} - 12\right)\sigma = \tau \tag{4.20}$$

$$x^{-1}: \qquad -3(\Omega^2 - \Omega) + \left(\frac{\eta}{2} - 12\right)\Omega - \frac{(12 - \eta + 2\kappa)}{4} = 0. \tag{4.21}$$

We previously had that $\alpha = \xi = 0$, and if λ_n is to be different from zero, we require that $\theta \neq 0$. For $\theta \neq 0$, then we may use (4.19) and (4.16) to show that

$$\eta = 20 + 8\Omega. \tag{4.22}$$

Equation (4.22), together with (4.20), give

$$\tau = -2(1 + \Omega)\sigma. \tag{4.23}$$

Equations (4.22) and (4.21) give

$$\kappa = 2(\Omega + 1)(\Omega + 2). \tag{4.24}$$

Equations (4.17) and (4.18) are identically satisfied when we substitute (4.22) and (4.24) into them. Finally from our definitions of σ and Ω we have,

$$\varphi = -3\sigma, \tag{4.25}$$

$$\omega = -(1 + \Omega). \tag{4.26}$$

We previously had $\mu = -\beta/8 = -1$. We are now in a position to write the forward d.e.,

$$xy_n''' + 2(1+\Omega)y_n'' + \left[\frac{\theta}{2}x + \frac{\Omega(1+\Omega)}{x}\right]y_n' = \lambda_n y_n, \tag{4.27}$$

and the r.a.d.e. is given by,

$$-xz_l''' - (1+\Omega+3\sigma x)z_l'' + [\theta x - 2(1+\Omega)\sigma]z_l' = \lambda_l z_l. \tag{4.28}$$

The appropriate weight function, except for an arbitrary multiplicative constant, taken as unity, is,

$$\rho = x^{\Omega}e^{\sigma x}. \tag{4.29}$$

Equation (4.27) has polynomial solutions of order n in x^2, and (4.28) has polynomial solutions of order l in x, when $\lambda_n = n\theta$, $n = 0, 1, \cdots$. The constant k in Eq. (2.23) is easily evaluated by considering $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ in terms of Ω, σ, θ giving,

$$(x^{1+\Omega}e^{\sigma x})'' - 2(1+\Omega)(x^{\Omega}e^{\sigma x})' + \left[\frac{\theta x}{2} + \frac{\Omega(1+\Omega)}{x}\right]x^{\Omega}e^{\sigma x} = k. \tag{4.30}$$

By virtue of (4.19), k is easily shown to be zero.

We now turn our discussion toward the evaluation of the conjunct. We require that the conjunct vanish for $n, l = 0, 1, \cdots$, so that for $n = 0$, $l = 1$, we have $y_0 = 1$, $z_1 = x + b_{10}$, or necessarily,

$$[2(x^{1+\Omega}e^{\sigma x})' - 2(1+\Omega)x^{\Omega}e^{\sigma x}]\Big|_{\Gamma} = 0, \tag{4.31}$$

or

$$x^{1+\Omega}e^{\sigma x}\Big|_{\Gamma} = 0. \tag{4.32}$$

For Γ we select a segment, one of whose end points is zero and the other $+\infty$, and choose $\Omega > -1$ when $\sigma < 0$. (If the second end point is at $-\infty$ we would require $\sigma > 0$, but this case is easily reduceable to the above.) σ may now be chosen to be -1 with no loss of generality. Under these assumptions it is easily shown that the conjunct vanishes identically. Note that

$$x^{1+\Omega}e^{-x}(z_l y_n'' - z_l' y_n' + y_n z_l'')\Big|_0^{\infty} = 0,$$

$$[2(x^{1+\Omega}e^{-x})' - 2(1+\Omega)x^{\Omega}e^{-x}]y_n z_l'\Big|_0^{\infty} = 0,$$

and

$$[2(1+\Omega)x^{\Omega}e^{-x} - (x^{1+\Omega}e^{-x})']z_l y_n'\Big|_0^{\infty} = 0.$$

The first two statements follow in an obvious fashion, but the third depends on the fact that y'_n is expressible as x times a polynomial of degree $n - 1$ in x^2. This completes Theorem 6, cited at the beginning of this section.

V. Discussion of the Polynomials of Spencer and Fano

In this section we solve the differential equations leading to the Spencer and Fano polynomials and exhibit some of their properties. We let $\sigma = -1$ or $\theta = -2$, and rewrite (4.28), (4.27) as follows,

$$xz_l''' + (1 + \Omega - 3x)z_l'' + 2(x - 1 - \Omega)z_l' = 2lz_l, \tag{5.1}$$

$$xy_n''' + 2(1 + \Omega)y_n'' + \left[\frac{\Omega(1 + \Omega)}{x} - x\right] y_n' = -2ny_n. \tag{5.2}$$

Equation (5.1) has for its coefficients linear functions of x. Such equations are always solvable by the techniques of the Laplace transformation [8]. Suppose,

$$z_l(x) = \int_c e^{xt}\varphi(t)dt, \tag{5.3}$$

is a representation of $z_l(x)$ for some contour c and some function $\varphi(t)$, both of which are to be determined. Then replacing $z_l(x)$ by this integral in (5.1) gives,

$$\int_c [xt^3 + (1 + \Omega - 3x)t^2 + 2(x - 1 - \Omega)t - 2l]e^{xt}\varphi(t)dt = 0. \tag{5.4}$$

If we integrate (5.4) by parts, we find that a necessary and sufficient condition that the integral (5.3) be a solution of (5.1) is that,

$$\int_c \{[(1 + \Omega)t(t - 2) - 2l]\varphi(t) - \frac{\partial}{\partial t}[t(t - 1)(t - 2)\varphi(t)]\}e^{xt}dt$$
$$+ e^{xt}t(t - 1)(t - 2)\varphi(t)\Big|_c = 0. \tag{5.5}$$

Let us choose $\varphi(t)$ so that,

$$[(1 + \Omega)t(t - 2) - 2l]\varphi(t) = \frac{\partial}{\partial t}[t(t - 1)(t - 2)\varphi(t)]. \tag{5.6}$$

This is a first order d.e. whose solution is easily verified to be

$$\varphi(t) = \frac{\varphi_0}{2\pi i}\frac{(t - 1)^{2l+\Omega}}{[t(t - 2)]^{l+1}}, \tag{5.7}$$

where $\varphi_0/2\pi i$ is an arbitrary multiplicative constant. We must now choose c such that,

$$e^{xt}t(t-1)(t-2)\varphi(t)\Big|_c = \frac{\varphi_0}{2\pi i}\frac{e^{xt}(t-1)^{1+\Omega+2l}}{[t(t-2)]^l}\Big|_c = 0. \qquad (5.8)$$

Equation (5.8) will surely hold if c is a closed contour enclosing $t = 0$, but excluding $t = 1, 2$, as long as l is an integer. (This particular choice of contour does not restrict Ω in any way, however it will become clear in the treatment of (5.2) that we require Ω to be integral. We will continue with that assumption, i.e., Ω is integral.) Hence, we have for this contour,

$$z_l(x) = \frac{\varphi_0}{2\pi i}\int_c \frac{e^{xt}(t-1)^{\Omega+2l}}{[t(t-2)]^{l+1}}\,dt. \qquad (5.9)$$

This may easily be evaluated by Cauchy's integral formula, giving,

$$z_l(x) = \frac{\varphi_0}{l!}\frac{\partial^l}{\partial t^l}\left[\frac{e^{xt}(t-1)^{\Omega+2l}}{(t-2)^{l+1}}\right]\Big|_{t=0}. \qquad (5.10)$$

If we place $z_0(x) = 1$, we get $\varphi_0 = 2(-1)^{1+\Omega}$, or

$$z_l(x) = \frac{(-)^\Omega}{2^l l!}\frac{\partial^l}{\partial t^l}\left[\frac{e^{xt}(1-t)^{\Omega+2l}}{(1-t/2)^{l+1}}\right]\Big|_{t=0}, \qquad (5.11)$$

and if

$$z_l(x) = \sum_{j=0}^{l} b_{lj}x^j, \qquad (5.12)$$

then

$$b_{lj} = \frac{(-)^l}{2^{2l}}\frac{(2l+\Omega)!}{l!}\frac{2^j}{j!}\sum_{k=0}^{l-j}\frac{(-)^k(2l-j-k)!2^k}{(\Omega+2l-k)!k!(l-j-k)!} \qquad (5.13)$$

where

$$b_{ll} = \frac{(-)^l}{2^l l!}.$$

Instead of discussing (5.2) directly, it will be more expedient to discuss the formal adjoint differential equation to (5.1) for a function

$$Y_n(x) = x^\Omega e^{-x}y_n(x),$$

where $Y_n(x)$ satisfies,

$$-xY_n''' + (\Omega - 2 - 3x)Y_n'' + 2(\Omega - 2 - x)Y_n' = (2n+2)Y_n. \qquad (5.14)$$

If we suppose that

$$Y_n = \int_{c'} e^{xt}\psi(t)dt \tag{5.15}$$

represents a solution for some contour c' and some function $\psi(t)$, then as before,

$$\int_{c'} [-xt(t+1)(t+2) + (\Omega-2)t^2 + 2(\Omega-2)t - 2(n+1)]e^{xt}\psi(t)dt = 0. \tag{5.16}$$

Integrating by parts we get,

$$\int_{c'} [(\Omega-2)t^2 + 2(\Omega-2)t - 2(n+1) + (t+1)(t+2) + t(t+2) + t(t+1)]e^{xt}\psi(t)dt + \int_{c'} t(t+1)(t+2)\psi'(t)e^{xt}dt - t(t+1)(t+2)e^{xt}\psi(t)\Big|_{c'} = 0. \tag{5.17}$$

We shall choose $\psi(t)$ such that

$$\frac{\psi'(t)}{\psi(t)} = \frac{n}{t} - \frac{2n+1+\Omega}{t+1} + \frac{n}{t+2}, \tag{5.18}$$

then the solution to (5.18) is given by,

$$\psi(t) = \frac{\psi_0}{2\pi i}\frac{[t(t+2)]^n}{(t+1)^{2n+1+\Omega}}, \tag{5.19}$$

for $\psi_0/2\pi i$ an arbitrary constant. Suppose we choose c' such that,

$$-t(t+1)(t+2)e^{xt}\psi(t)\Big|_{c'} = -\frac{\psi_0}{2\pi i}e^{xt}\frac{[t(t+2)]^{n+1}}{(t+1)^{2n+\Omega}}\Big|_{c'} = 0, \tag{5.20}$$

then it is clear that if we choose a contour encircling $t = -1$ such that n, Ω are integral, (5.20) will hold. We therefore arrive at

$$Y_n(x) = \frac{\psi_0}{2\pi i}\int_{c'} \frac{e^{xt}[t(t+2)]^n}{(t+1)^{2n+1+\Omega}}dt. \tag{5.21}$$

We may simplify (5.21) by moving our contour such that $t + 1 = s$, or

$$e^x Y_n(x) = \frac{\psi_0}{2\pi i}\int_{c''} e^{xs}\frac{(s^2-1)^n}{s^{2n+1+\Omega}}ds, \tag{5.22}$$

where c'' is the new contour. Equation (5.20) gives,

$$-\frac{\psi_0}{2\pi i}\, e^{x(s-1)}\, \frac{(s^2-1)^{n+1}}{s^{2n+\Omega}}\Big|_{c''} = 0. \tag{5.23}$$

The application of Cauchy's integral formula on (5.22) gives,

$$e^x Y_n(x) = \frac{\psi_0}{(2n+\Omega)!}\, \frac{\partial^{2n+\Omega}}{\partial s^{2n+\Omega}}\, [e^{xt}(s^2-1)^n]\Big|_{t=0}. \tag{5.24}$$

Note that $e^x Y_n(x)$ will be a polynomial of degree $2n+\Omega$ for a contour c'' encircling the point $s=0$, when n, Ω are integers. If we choose $\psi_0 = 1$, and perform the necessary differentiations we get,

$$e^x Y_n(x) = e^x e^{-x} x^\Omega y_n(x) = \sum_{j=0}^{n} (-)^j \binom{n}{j} \frac{x^{2j+\Omega}}{(2j+\Omega)!}, \tag{5.25}$$

or

$$y_n(x) = \sum_{j=0}^{n} (-)^j \binom{n}{j} \frac{x^{2j}}{(2j+\Omega)!}, \tag{5.26}$$

which is easily verified to be a solution of (5.2). We are now in a position to determine a convenient normalization of these polynomials. Their biorthogonal character for $n \neq l$ followed directly from the fact that the conjunct vanished identically for $\lambda_n \neq \lambda_l$. We now establish their normalization. We evaluate N_n, given by (5.27) in direct fashion

$$\int_0^\infty z_n(x) y_n(x^2) x^\Omega e^{-x} dx = N_n, \tag{5.27}$$

or

$$\int_0^\infty \sum_{j=0}^{n} b_{nj} x^j \sum_{j=0}^{n} (-)^j \binom{n}{j} \frac{x^{2j+\Omega}}{(2j+\Omega)!}\, e^{-x} dx = N_n. \tag{5.28}$$

Since x^j for j less than n is some linear combination of z_m for $m = 0, \cdots, j$, and $y_n(x^2)$ is orthogonal to each of these, it suffices to consider just the term $b_{nn} x^n$. Therefore

$$b_{nn} \int_0^\infty x^n \sum_{y=0}^{n} (-)^j \binom{n}{j} \frac{x^{2j+\Omega}}{(2j+\Omega)!}\, e^{-x} dx = N_n, \tag{5.29}$$

or

$$b_{nn} \sum_{j=0}^{n} (-)^j \binom{n}{j} \frac{(2j+n+\Omega)!}{(2j+\Omega)!} = N_n, \tag{5.30}$$

or

$$\frac{(-)^n}{2^n n!} \sum_{j=0}^{n} (-)^j \binom{n}{j} \frac{d^n}{dx^n} x^{2j+n+\Omega} \Big|_{x=1} = N_n, \tag{5.31}$$

or

$$\frac{(-)^n}{2^n n!} \frac{d^n}{dx^n} [x^{n+\Omega}(1 - x^2)^n] \Big|_{x=1} = \frac{(-)^n(-)^n 2^n n!}{2^n n!} = N_n = 1. \tag{5.32}$$

It is not difficult to establish some contiguous relationships between the polynomials corresponding to Ω and $2 + \Omega$. In order to do so, we return to

$$z_l^\Omega(x) = \frac{\varphi_0}{2\pi i} \int_c \frac{e^{xt}(t-1)^{2l+\Omega}}{[t(t-2)]^{l+1}} dt, \tag{5.9}$$

then applying the operator

$$\left(\frac{\partial}{\partial x} - 1\right)^2$$

to both sides gives,

$$\left(\frac{\partial}{\partial x} - 1\right)^2 z_l^\Omega(x) = \frac{\varphi_0}{2\pi i} \int_c e^{xt} \frac{(t-1)^{2l+2+\Omega}}{[t(t-2)]^{l+1}} dt = z_l^{\Omega+2}(x). \tag{5.33}$$

If we apply

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} - 2\right)$$

to (5.9) we get,

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} - 2\right) z_l^\Omega(x) = \frac{\varphi_0}{2\pi i} \int_c e^{xt} \frac{(t-1)^{2l+\Omega}}{[t(t-2)]^l} = z_{l-1}^{\Omega+2}(x). \tag{5.34}$$

Subtracting (5.34) from (5.33) gives,

$$z_l^\Omega(x) = z_l^{\Omega+2}(x) - z_{l-1}^{\Omega+2}(x). \tag{5.35}$$

An analagous procedure may be carried out for the $Y_n^\Omega(x)$. Since

$$e^x Y_n^{\Omega+2}(x) = \frac{1}{2\pi i} \int_{c''} e^{xs} \frac{(s^2-1)^n ds}{s^{2n+1+\Omega+2}}, \tag{5.22}$$

then

$$\frac{\partial^2}{\partial x^2} [e^x Y_n^{\Omega+2}(x)] = \frac{1}{2\pi i} \int_{c''} \frac{e^{xs}(s^2-1)^n ds}{s^{2n+1+\Omega}} = e^x Y_n^\Omega(x), \tag{5.36}$$

$$\left(\frac{\partial^2}{\partial x^2} - 1\right) [e^x Y_n^{\Omega+2}(x)] = \frac{1}{2\pi i} \int_{c''} \frac{e^{xs}(s^2-1)^{n+1} ds}{s^{2(n+1)+1+\Omega}} = e^x Y_{n+1}^\Omega, \tag{5.37}$$

or by subtracting (5.37) form (5.36) we have,

$$e^x Y_n^{\Omega+2}(x) = e^x Y_n^{\Omega} - e^x Y_{n+1}^{\Omega}, \tag{5.38}$$

and finally,

$$x^{\Omega+2} y_n^{\Omega+2}(x) = x^{\Omega} y_n^{\Omega} - x^{\Omega} y_{n+1}^{\Omega}. \tag{5.39}$$

With some simple manipulations (5.39) is seen to be equivalent to the well known relation between the binomial coefficients,

$$\binom{n+1}{j} = \binom{n}{j} + \binom{n}{j-1}. \tag{5.40}$$

Another property of these polynomials that we choose to find is a recurrence relation among *four* neighboring polynomials. We begin with a study of the $z_n(x)$.

$$x^2 z_n(x) = \sum_{j=0}^{n+2} c_{nj} z_j, \tag{5.41}$$

where

$$c_{nj} = \int_0^\infty x^2 z_n(x) y_j(x^2) x^{\Omega} e^{-x} dx.$$

In particular, $c_{n,j} \equiv 0$ for $j < n-1$ since $x^2 y_j$ is a linear combination of $y_{j+1}, \cdots, y_0$, and z_n is orthogonal to each of these terms. By equating like terms of x on both sides of (5.41) we get

$$b_{n,k-2} = \sum_{j=k}^{n+2} c_{nj} b_{jk}, \tag{5.42}$$

where it is to be assumed that $b_{n,j} \equiv 0$ for $j > n$, or $j < 0$. Equation (5.42) may be solved explicitly for c_{nj}.

$$c_{n,n+2} = \frac{b_{n,n}}{b_{n+2,n+2}} \tag{5.43}$$

$$c_{n,n+1} = \frac{1}{b_{n+1,n+1}} (b_{n,n-1} - c_{n,n+2} b_{n+2,n+1}) \tag{5.44}$$

$$c_{n,n} = \frac{1}{b_{n,n}} (b_{n,n-2} - c_{n,n+2} b_{n+2,n} - c_{n,n+1} b_{n+1,n}) \tag{5.45}$$

$$c_{n,n-1} = \frac{1}{b_{n-1,n-1}} (b_{n,n-1} - c_{n,n+2} b_{n+2,n-1} - c_{n,n+1} b_{n+1,n-1} - c_{n,n} b_{n,n-1}) \tag{5.46}$$

$$c_{n,j} \equiv 0 \tag{5.47}$$

for $j < n - 1$, where the coefficients b_{lj} are directly evaluated from (5.13).

A similar procedure, if carried out for the $y_n(x^2)$ would be as follows:

$$x^2 y_n(x^2) = \sum_{j=0}^{n+1} d_{nj} y_j(x^2), \tag{5.48}$$

where

$$d_{nj} = \int_0^\infty x^2 y_n(x^2) z_j(x) x^\Omega e^{-x} dx, \tag{5.49}$$

such that $d_{nj} \equiv 0$ for $j < n - 2$. It then follows that,

$$a_{n,k-1} = \sum_{j=k}^{n+1} d_{nj} a_{jk}, \tag{5.50}$$

which permits us to solve for d_{nj} when $j = n + 1, n, n - 1, n - 2$.

Finally, we exhibit a generating function for the $y_n(x)$ polynomials. We notice that

$$x^\Omega y_n(x) = \sum_{j=0}^{n} (-)^j \binom{n}{j} \frac{x^{2j+\Omega}}{(2j+\Omega)!}. \tag{5.51}$$

If we multiply both sides by t^n and sum on n we get,

$$\sum_{n=0}^\infty x^\Omega y_n(x) t^n = \sum_{n=0}^\infty \sum_{j=0}^n (-)^j \binom{n}{j} \frac{x^{2j+\Omega} t^n}{(2j+\Omega)!},$$

$$= \sum_{j=0}^\infty (-)^j \frac{x^{2j+\Omega}}{(2j+\Omega)!} \sum_{n=j}^\infty \binom{n}{j} t^n$$

$$= \sum_{j=0}^\infty (-)^j \frac{x^{2j+\Omega}}{(2j+\Omega)!} \frac{t^j}{(1-t)^{j+1}}$$

$$= \frac{(1-t)^{(\Omega/2)-1}}{t^{\Omega/2}} \left\{ \cos x \sqrt{\frac{t}{1-t}} - \sum_{j=0}^{(\Omega/2)-1} \frac{(-)^j}{(2j)!} \left[x \sqrt{\frac{t}{1-t}} \right]^{2j} \right\}, \tag{5.52}$$

for Ω an even integer, and

$$= \frac{(1-t)^{(\Omega/2)-1}}{t^{\Omega/2}} \left\{ \sin x \sqrt{\frac{t}{1-t}} - \sum_{j=0}^{(\Omega-3)/2} \frac{(-)^j}{(2j+1)!} \left[x \sqrt{\frac{t}{1-t}} \right]^{2j+1} \right\},$$

for Ω an odd integer.

With this we conclude our discussion of the Spencer and Fano polynomials.

VI. Conclusion

It would be of interest to determine the extent to which our theory may be generalized to ordinary differential equations of some fixed order greater than three. Theorems 1 and 2 are easily generalizable, but the equivalent of our compatibility equations and our conjunct conditions rapidly become unwieldy. For odd order d.e. Lemmas 1 and 2 easily generalize and answer the nonexistence of self-orthogonal polynomials,* but it remains a difficult task to determine whether any biorthogonal polynomials are possible. On the other hand, for differential equations of even order we have even less information. For instance, it is not difficult to prove the existence of at least one d.e. (and hence, many) of even order having nonself-orthogonal polynomials for its solution. One arrives at such an equation by iterating the differential operators appearing in the Spencer and Fano d.e. once, giving a d.e. of sixth order having the Spencer and Fano polynomials as their solution. This same device when applied twice to the second order d.e. of Cases 1-5 also lead to sixth order d.e. having as solutions the polynomials discussed in Cases 1-5.

Another set of questions left unanswered concerns the merit of deliberately using nonself-orthogonal polynomials for the expansion of arbitrary functions. Questions of convergence of such systems are left open, along with their related moment problems which are barely hinted at in the existing literature.

Finally, the full exploration of the specific polynomials of Section V might be carried out in an attempt to broaden the field of special functions.

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* Krall in [9] proves this using a different approach.