JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **160**, 176–189 (1991)

Range Inclusion and the UUP-Property

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Submitted by C. Foias

Received September 11, 1989

Let A and B be positive operators on a complex Hilbert space H. The *parallel sum* $A:B$ of A and B is the strong operator limit (as $\varepsilon \downarrow 0$) of the net $(A+\varepsilon)(A+B+2\varepsilon)^{-1}(B+\varepsilon)$. If this net converges in norm, then (following Green and Morley), we say that the pair (A, B) is of *class* UUP. It is known that if Range $A \subseteq \text{Range}(A+B)$, then (A, B) is a UUP-pair. In this work we establish that $\text{Range}(A) \subseteq \text{Range}(A+B)$ if and only $\text{Range}(B) \subseteq \text{Range}(A+B)$. We also show that if $AB+BA+B^2 \geqslant 0$, then Range $A \subseteq \text{Range}(A+\alpha B)$ for all real α with $\alpha \geqslant 1$ and that if $AB+BA \geqslant 0$, then Range $(A) \subseteq \text{Range}(A+\alpha B)$ for all $\alpha \geqslant 0$. This implies several previously known results, e.g., that if $AB=BA$, then (A, B) is a UUP-pair. Finally we establish a number of other results which relate the class UUP to range inclusion. © 1991 Academic Press, Inc.

1. INTRODUCTION

If A and B are invertible positive operators, their parallel sum $A:B$ is defined by $A:B=(A^{-1}+B^{-1})^{-1}$; this definition is motivated by the parallel connection of resistors in electrical networks (see [2, 14]). Anderson and Duffin have studied this operation for operators on a finite-dimensional space [2]; some extensions of the theory to infinite dimensional Hilbert spaces have been given by Anderson and Schreiber [3] and Fillmore and Williams [8]. In each of these three papers some form of generaized inverse has been used for noninvertible operators. Anderson and Trapp base their definition of $A:B$ on the shorted operator (see [1] or [4]). In practice one must usually compute the parallel sum $A:B$ of A and B by some limiting or iterative technique as, for example,

$$A:B=\lim_{\varepsilon \downarrow 0}(A+\varepsilon)(A+B+2\varepsilon)^{-1}(B+\varepsilon)=\lim_{\varepsilon \downarrow 0}(A+\varepsilon):(B+\varepsilon),$$

where the limit is taken in the strong operator topology [4]. If the net $(A+\varepsilon):(B+\varepsilon)$ does converge in norm, then the pair (A, B) is said to

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be *uniformly and universally parallelizable*, or *of class* UUP [9]. This definition is due to Green and Morley, who studied the class UUP and showed [9] that if $AB = BA$, then (A, B) is of class UUP, that if A and B have closed ranges, then (A, B) is of class UUP if and only if $A + B$ has closed range, and that (A, B) is of class UUP if and only if the process of computing the parallel sum commutes with every representation of the C^*-algebra $B(H)$.

Bunce [6] and Morley [12] have independently established that if $\mathrm{Range}(A) \subseteq \mathrm{Range}(A+B)$, then (A, B) is of class UUP, but the converse does not hold [10]. We do have, however, a principle of symmetry for range inclusion, namely that $\mathrm{Range}(A) \subseteq \mathrm{Range}(A+B)$ if and only if $\mathrm{Range}(B) \subseteq \mathrm{Range}(A+B)$.

In Section 3 below, we establish this principle of symmetry together with some corollaries, and we obtain conditions on the (positive) operators A and B which will guarantee that $(A, \alpha B)$ is of class UUP for all real $\alpha \geqslant 1$. We show in particular that if $AB + BA + B^2 \geqslant 0$, then $\mathrm{Range}\, A \subseteq \mathrm{Range}(A + \alpha B)$ for all such α, and that if $AB + BA \geqslant 0$, then $\mathrm{Range}(A) \subseteq \mathrm{Range}(A + \alpha B)$ for all $\alpha > 0$. These results imply several previously known results, e.g., that if $AB = BA$, then (A, B) is of class UUP [9], and they may be of use in the study of the shorted operator, since the short of A to the range of a projection P can be expressed as the norm limit of the sequence $\{A : nP\}_{n=1}^{\infty}$ [4]. Section 5 contains an alternative treatment of the parallel sum and some further characterizations of range inclusions. In Particular, we establih that $\mathrm{Range}(B) \subseteq \mathrm{Range}(A+B)$ if and only if $B(A + B + 1/n)^{-1} (A+B)^{1/2}$ converges strong operator to $K(A+B)^{1/2}$ for some $K \in B(H)$, and that this strong operator convergence holds if and only if the same sequence is in fact norm convergent. We also show that these conditions imply that $(A+B)^{\dagger} A$ and $(A+B)^{\dagger} B$ are bounded, where $\dagger$ denotes the Moore–Penrose pseudoinverse [5].

2. Preliminaries

We shall be concerned with a complex Hilbert space H with inner product (,). A bounded linear Hermitian operator A on H wil be called *positive* if $(Ax, x) \geqslant 0$ for all $x \in H$. For positive operators B and C, we say that $B \geqslant C$ if $B - C$ is positive; we recall [11] that if a net A_n of positive operators is monotone and bounded, then the net has a limit in the strong operator topology. The range of the operator A is denoted below by $R(A)$, the null space by $N(A)$. For a positive operator A, the unique positive square root is denoted by $A^{1/2}$. Observe that for $A \geqslant 0$, $R(A) \subseteq R(A^{1/2})$, and $x \in N(A)$ if and only if $x \in N(A^{1/2})$ whence $\overline{R(A^{1/2})} = \overline{R(A)}$ (= the orlthogonal complement of $N(A)$). Some basic results concerning the ranges of linear operators are given by Douglas [7] and Fillmore and Williams [8].

LEMMA 1 (Douglas). *Let A and B be (bounded) operators on the Hilbert space H. The following statements are equivalent:*

(1) $(R(A) \subseteq R(B))$.

(2) $AA^* \leqslant \lambda^2 BB^*$ *for some* $\lambda \geqslant 0$.

(3) *There exists a bounded operator C such that $A = BC$. Moreover, if* (1), (2), *and* (3) *are valid, then there exists a unique operator so that*

(a) $\|C\|^2 = \inf\{\mu \mid AA^* \leqslant \mu BB^*\}$

(b) $N(A) = N(C)$, *and*

(c) $R(C) \subseteq \overline{R(B^*)}$.

COROLLARY 1. *Let A and B be bounded operators on H. There exists an invertible C on H such that $A = BC$ if A and B have the same range and nullity.*

THEOREM 1 (T. Crimmins quoted in [8]). *If $A, B \in B(H)$, then*

$$R(A) + R(B) = R(\sqrt{AA^* + BB^*}).$$

The algebra $B(H)$ of all bounded linear operators on H is a *C^*-algebra*, i.e., a Banach $*$-algebra with an isometric involution $T \to T^*$ satisfying $\|T^*T\| = \|T\|^2$ for all T. In particular, $B(H)$ is (via the standard norm for operators) a Banach space. The Banach space double dual $B(H)^{**}$ is also, in a natural way [9, 11], a C^*-algebra and the universal representation (see [9, 11]) of $B(H)$ allows us to regard $B(H)^{**}$ as a weak operator closed (hence norm closed) adjoint-closed subalgebra of $B(K)$, where K is a rather large Hilbert space obtained from $B(H)$ by means of the Gelfand–Naimark–Segal construction [11]. Let $A \mapsto \hat{A}$ be this universal representation of $B(H)$, so that $\hat{\ }: B(H) \to B(K)$ is an isometric $*$-preserving algebra isomorphism of $B(H)$ into $B(K)$. The Hilbert space K is called the *universal representation space* of the C^*-algebra $B(H)$.

The map $A \mapsto \hat{A}$ preserves the norm and all algebraic structure and all that is implied by algebraic structure. For instance, an operator A is positive if and only if there is a B with $A = B^*B$, in which case we also have $\hat{A} = \hat{B}^*\hat{B}$ (since $\hat{B}^* = (B^*)\hat{\ }$). We conclude that $\hat{A}$ is positive whenever A is positive. The map $A \mapsto \hat{A}$ is therefore order preserving. The map $A \mapsto \hat{A}$ does *not*, however, preserve strong or weak operator limits or certain infinite operations such as suprema and infima (see, for example, [9]).

We shall be interested in a class of pairs of positive operators which was first defined by Green and Morley, and we quote here their definition and some related results which will be used in our later discussion.

DEFINITION (Green and Morley [9]). Let A and B be two positive operators on a Hilbert space A. We say that the pair (A, B) is *uniformly and universally parallelizable* (or *of class UUP*) if the net $(A+\varepsilon):(B+\varepsilon)$ converges in norm.

THEOREM 2 (Bunce [6] and independently Morley [12]). *Suppose A and B are two bounded positive operators on H and $R(B) \subseteq R(A+B)$. Then (A, B) is a UUP pair.*

LEMMA 2 (Bunce, unpublished [6]). *Let A and B be positive operators in $B(H)$. If $\sup_{\varepsilon>0} \|B(A+B+2\varepsilon)^{-1}\| < \infty$, then $B=(A+B)C$ for some $C \in B(H)$.*

LEMMA 3 (Green and Morley [9]). *Let A and B be positive operators on a Hilbert space H. If A and B commute, then (A, B) is UUP. Moreover $R(B) \subseteq R(A+B)$.*

THEOREM 3 (Green and Morley [9]). *Let A and B be positive operators in $B(H)$, each of which has closed range. Then the following properties are mutually equivalent*:

(1) *$A+B$ has closed range*;

(2) *$(A:B)\hat{} = \hat{A}:\hat{B}$*;

(3) *(A, B) is of class UUP.*

3. RANGE INCLUSION

We begin with a principle of symmetry for range inclusion.

THEOREM 4. *Let A and B be bounded operators on H. Then $R(A) \subseteq R(A+B)$ if and only if $R(B) \subseteq R(A+B)$.*

Proof. Suppose $R(A) \subseteq R(A+B)$, then by Douglas' lemma (Lemma 1 above) there exists a bounded operator C on H such that $A=(A+B)C$. Clearly $D=1-C$ is a bounded operator on H, and $(A+B)D=(A+B)(I-C)=A+B-(A+B)C=A+B-A=B$. Thus by Douglas' Lemma $R(B) \subseteq R(A+B)$. By symmetry nothing remains to be proved.

Remark 1. The calculations in the proof of Theorem 4 can be made to show the following more general result: if A and B are linear transformations on a vector space X, then $R(A) \subseteq R(A+B)$ if and only if $R(B) \subseteq R(A+B)$.

By applying (2) of Douglas' lemma, one gets a characterization of $R(A)$ due to Shmulyan [13]:

LEMMA 4 (Quoted in Fillmore and Williams [8]). *If $A \in B(H)$, then $y \in R(A)$ if and only if*

$$\sup_x \frac{|(x, y)|}{\|A^*x}\| < \infty,$$

where the supremum is taken over all x which are not elements of $N(A^*)$.

COROLLARY 2. *Let A and B be bounded operators on H. The following conditions are equivalent.*

(1) $R(A) \subseteq R(A+B)$

(2) $R(B) \subseteq R(A+B)$

(3) $\sup_x (|x, Ay)|/\|(A^* + B^*)x\|) < \infty$ *for* $y \in H$

(4) $\sup_x (|x, By)|/\|(A^* + B^*)x\|) < \infty$ *for* $y \in H$.

Proof. By Theorem 4, (1) is equivalent to (2). The equivalence of (1) and (3) (or (2) and (4)) is an immediate consequence of the Lemma 4.

LEMMA 5. *If $A, B \in B(H)$ and $R(A) \subseteq R(A+B)$, then $(A+B)^\dagger A$, $(A+B)^\dagger B$, are bounded operators on H, where $(A+B)^\dagger$ denotes the (possibly unbounded) Moore–Penrose pseudoinverse of $A+B$* [5].

Proof. Without loss of generality we may suppose that $A+B$ is one to one (see Remark 2 below). By Lemma 1 there exists a $Y \in B(H)$ such that $(A+B)Y = B$. Notice that since $A+B$ is one to one and $(A+B)(A+B)^\dagger B = B$ (because $(A+B)(A+B)^\dagger$ is the projection on $R(A+B)$), we have $(A+B)^\dagger B = Y$. The proof is completed by Theorem 4.

Remark 2. If $A+B$ is not one to one H, then we can consider

$$A+B\colon N(A+B)^\perp \to H.$$

Notice that $A+B \in B(N(A+B)^\perp, H)$ is one to one and if $R(A) \subseteq R(A+B)$, then by the previous lemma $(A+B)^\dagger A$ is bounded, i.e., $(A+B)^\dagger A \in B(N(A+B)^\perp, H)$. Now $(A+B)^\dagger A$ can be extended to a bounded operator on H.

4. Sufficient Conditions Which Imply Range Inclusion

LEMMA 6. *Let A and B be bounded positive operators on H. Then $R(A+B)$ is closed if and only if $R(A+\alpha B)$ is closed for all $\alpha>0$ if and only if $R(A+\alpha B)=R(A)+R(B)$ for all $\alpha>0$. If $R(A+B)$ is closed, then $R(A)\subseteq R(A+\alpha B)$ and $R(B)\subseteq R(A+\alpha B)$ for all $\alpha>0$.*

Proof. Closure of $R(A+B)$ implies the closure of $R(A+B)^{1/2}$. But

$$\begin{aligned} R(A+\alpha B)^{1/2} &= R(A^{1/2})+R((\alpha B)^{1/2}) \\ &= R(A^{1/2})+R(B^{1/2})=R(A+B)^{1/2} \end{aligned}$$

for all $\alpha\in\mathbb{R}^+$ by Theorem 1 (Crimmins' theorem). This implies that $R(A+\alpha B)^{1/2}$ and $R(A+\alpha B)$ are closed for all $\alpha\in\mathbb{R}^+$. Hence

$$\begin{aligned} R(A+\alpha B) = R((A+\alpha B)^{1/2}) &= R(A^{1/2})+R((\alpha B)^{1/2}) \\ &= R(A^{1/2})+R(B^{1/2}), \end{aligned}$$

which includes both $R(A)$ and $R(B)$.

LEMMA 7. *Let A and B be bounded and self-adjoint on H. If $AB+BA\geqslant 0$, then $R(A)$ and $R(B)$ are included in $R(A+\alpha B)$ for all $\alpha\in\mathbb{R}$ with $\alpha\geqslant 0$.*

Proof. Suppose $AB+BA\geqslant 0$. Since A and B are self-adjoint $A^2=A^*A$, $B^2=B^*B$ and hence $\alpha(AB+BA)+\alpha^2A^2+B^2\geqslant B^2$ for all $\alpha\geqslant 0$. This implies that $B^2\leqslant(\alpha A+B)^2$. Similarly we can conclude that $\alpha^2A^2\leqslant(\alpha A+B)^2$. Now the result follows by Lemma 1 (Douglas' lemma).

LEMMA 8. *Let A and B be bounded and self-adjoint on H. If $AB+BA+B^2\geqslant 0$, then $R(A)\subseteq R(A+\alpha B)$ and $R(B)\subseteq R(A+\alpha B)$ for all $\alpha\in[1,\infty)$. Moreover for each $\alpha\in[1,\infty)$, there exist a nonexpansive map C_α such that $A=(A+B)C_\alpha$.*

Proof. Let $\alpha\geqslant 1$ and $AB+BA+B^2\geqslant 0$, then $\alpha(AB+BA+B^2)+A^2\geqslant A^2$. But $(\alpha^2-\alpha)B^2\geqslant 0$, so $(A+\alpha B)^2=\alpha(AB+BA+B^2)+A^2+(\alpha^2-\alpha)B^2\geqslant A^2$. Thus $R(A)\subseteq R(A+\alpha B)$.

Now by Lemma 1 (Douglas' lemma) there is a bounded operator C_α such that $A=(A+\alpha B)C_\alpha$ and $\|C_\alpha\|^2=\inf\{\mu\mid A^2\leqslant\mu(A+B)^2\}$. Therefore $\|C_\alpha\|\leqslant 1$, because $A^2\leqslant(A+\alpha B)^2$. Also $R(B)\subset R(A+\alpha B)$ for all $\alpha\geqslant 1$ by Theorem 4.

COROLLARY 3. *Let A and B be bounded positive operators such that $AB+BA\geqslant 0$. Then $(A,\alpha B)$ and $(B,\alpha A)$ are UUP for all $\alpha\in\mathbb{R}^+$.*

Proof. By Lemma 7, $R(A) \subset R(A+\alpha B)$ and $R(B) \subseteq R(\alpha A + B)$ for all $\alpha \in \mathbb{R}^+$. Hence $(A, \alpha B)$ and $(B, \alpha A)$ are UUP for all $\alpha \geqslant 0$ by Theorem 2.

COROLLARY 4. *Let A and B be bounded positive operators such that $B + BA + B^2 \geqslant 0$. Then $(A, \alpha B)$ is UUP for all $\alpha \in \mathbb{R}^+$ with $\alpha \geqslant 1$.*

Proof. By Lemma 8, $R(A) \subseteq R(A + \alpha B)$ for all $\alpha \in [1, \infty)$. Hence $(A, \alpha B)$ is UUP for all $\alpha \in [1, \infty)$.

LEMMA 9. *Let A and B be bounded positive operators such that $A = (A+B)C$ for some C satisfying $\|C^k\| - 0$ as $k \to \infty$. Then, $R(B) = R(A+B)$ and $R(A) \subseteq R(B)$.*

Proof. It is well known that $\|C^k\| \to 0$ as $k \to \infty$ if and only if spectral radius of C is less than one, which implies that $I - C$ is invertible [11]. Now observe that $B = (A+B)(I-C)$ an $I-C$ is invertible. Notice that from Corollary 1, it follows that two positive operators have the same range if and only if they differ by an invertible factor. This implies that $R(B) = R(A+B)$ and hence that $R(A) \subseteq R(A+B) = R(B)$ by Theorem 4.

Here we give two examples. The first example shows that there exist two non-commuting positive operators A and B such that $AB + BA \geqslant 0$ (an example for Lemma 7). The second example exhibits two non-commuting positive operators such that $AB + BA + A^2 \geqslant 0$ and yet $AB + BA \ngeqslant 0$ (an example for Lemma 8).

EXAMPLE 1. Define $B_x = \left[\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $A_x = \left[\begin{smallmatrix} 100 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ $(x > 1)$. It is not difficult to verify that $A_x \geqslant 0$, $B_x \geqslant 0$,

$$B_x A_x = \begin{bmatrix} 100x & x \\ 1 & 1 \end{bmatrix}, \qquad A_x B_x = \begin{bmatrix} 100x & 1 \\ x & 1 \end{bmatrix},$$

$$A_x B_x + B_x A_x = \begin{bmatrix} 200x & x+1 \\ x+1 & 2 \end{bmatrix}, \qquad \text{and} \qquad A_x B_x \neq B_x A_x \text{ (because } x \neq 1\text{)}.$$

Notice that $A_x B_x + B_x A_x \geqslant 0$ if and only if $x \geqslant 0$ and $400x - (x+1)^2 \geqslant 0$. Thus $1 < x \leqslant 199 + \sqrt{39600}$ is sufficient for $A_x B_x + B_x A_x$ to be positive. Now let A and B be the block diagonal matrices whose ith blocks are given, respectively, by A_x and B_x with $x = 2 + 1/i$. It is evident that $A \geqslant 0$, $B \geqslant 0$, $AB + BA \geqslant 0$, and $AB \neq BA$.

It follows that $R(A) \subseteq R(A + \alpha B)$ for all $\alpha \geqslant 0$ by Lemma 7. This implies that $(A, \alpha B)$ and $(B, \alpha A)$ are UUP for all $\alpha \geqslant 0$ by Corollary 3.

EXAMPLE 2. This example gives two positive operators A and B such that $AB+BA+B^2 \geqslant 0$ and $AB+BA \not\geqslant 0$. Define

$$B_x = \begin{bmatrix} x & 1 \\ 1 & 1 \end{bmatrix}, \qquad A_x = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \text{and}$$

$$T_x = A_x B_x + B_x A_x = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \quad (x \geqslant 0).$$

It is easily verified that $A_x B_x + B_x A_x \not\geqslant 0$ and that $T_x + B_x^2 = \left[\begin{smallmatrix} x^2+1 & x+2 \\ x+2 & 4 \end{smallmatrix}\right]$. Notice that $A_x B_x + B_x A_x + B_x^2 \geqslant 0$ is and only if $x \geqslant 4/3$. Now let A and B be the block diagonal matrices $\operatorname{diag}\{A_i\}$ and $\operatorname{diag}\{B_i\}$, $i = 1, 2, \ldots$, respectively, whose ith blocks are given respectively by A_x and B_x with $x = 4/3 + 1/i$. It is evident that $A \geqslant 0$, $B \geqslant 0$, $AB + BA \not\geqslant 0$, and $AB + BA + B^2 \geqslant 0$. Therefore by Lemma 8 and Corollary 4, $R(A) \subseteq R(A + \alpha B)$ and $(A, \alpha B)$ is UUP for each $\alpha \in [1, \infty)$.

PROPOSITION 1. *Let A and B be two positive operators. Then (A, B) is UUP if and only if (UAU^*, UBU^*) is UUP for each unitary operator U on H.*

Proof. $\alpha_u : T \to U^*TU$ is a *-isomorphism of $B(H)$ onto $B(H)$. It follows that $\alpha_u(T^{-1}) = (\alpha_u(T))^{-1}$ for any invertible T, and hence that

$$\alpha_u((A+\varepsilon) : (B+\varepsilon)) = (\alpha_u(A) + \varepsilon) : (\alpha_u(B) + \varepsilon).$$

Since α_u is isometric, the result is immediate.

5. AN ALTERNATIVE APPROACH TO PARALLEL SUMS

Let H be a complex Hilbert space with inner product $(\ ,\ )$ and let C be a positive operator with $N(C) = 0$. Observe that $(Cx, x) = 0$ if and only if $C^{1/2}x = 0$ if and only if $Cx = 0$, so we may define a new inner product $\langle\ ,\ \rangle_C$ by

$$\langle x, y \rangle = (Cx, y).$$

Let H_C be the completion of H with respect to $\langle\ ,\ \rangle_C$. Thus H_C consists of (equivalence classes of) all sequences $\{x_n\}$ that are Cauchy relative to $\langle\ ,\ \rangle_C$. This means that

$$(C(x_n - x_m), (x_n - x_m)) \to 0$$

as $n, m \to \infty$. If $0 \leqslant A \leqslant C$, then given such an H_C-Cauchy sequence, we have $(A(x_n - x_m), x_n - x_m) \leqslant ((C(x_n - x_m), x_n - x_m))$ and therefore $\{A^{1/2}x_n\}_{n=1}^{\infty}$ is H-Cauchy. If $\{y_n\}$ is equivalent to $\{x_n\}$ then

$$(A(x_n - y_n), x_n - y_n) \leqslant (C(x_n - y_n), x_n - y_n) \to 0,$$

so $A^{1/2}x_n$ and $A^{1/2}y_n$ have the same limit in H. We may therefore define an operator $\hat{A}: H_C \to H$ by $\hat{A}x^* = A^{1/2}(\lim_{n\to\infty} A^{1/2}x_n)$ where x_n is any H_C-Cauchy sequence converging to x^*, with $x_n \in H$.

T. D. Morley has proposed the following conjecture:

Morley's Conjecture. (A, B) is UUP if and only if

$$\lim_{\varepsilon \downarrow 0} (A + B + \varepsilon)^{-1} B \qquad \text{(strong limit)}$$

exists as a bounded operator from H to H_{A+B}. This would mean that (A, B) is UUP if and only if $(A+B)^{1/2}(A+B+1/n)^{-1}Bx$ is H-Cauchy for each $x \in H$.

In view of this conjecture we investigate the behavior of the sequence $\{(A+B)^{1/2}(A+B+1/n)^{-1}B\}_{n=1}^{\infty}$.

THEOREM 5. *Suppose $\mathscr{A}$ is a C^*-algebra and $0 \leqslant B \leqslant A$ in $\mathscr{A}$. For $n = 1, 2, \ldots$, let $u_n = B^{1/2}(n^{-1}I + A)^{-1}A$. Then*

(i) $u_n^* u_n \leqslant A$.

(ii) $\|u_m - u_n\| \leqslant \|(n^{-1} - m^{-1})(m^{-1}I + A)^{-1}(n^{-1}I + A)^{-1}A^{3/2}\| \leqslant |n^{-1} - m^{-1}|^{1/2}$, $\|B^{1/2} - u_n\| \leqslant \|n^{-1}A^{1/2}[n^{-1}I + A]^{-1}\| \leqslant (1/n)n^{-1/2}$.

(iii) *$\{u_n^* u_n\}_{n=1}^{\infty}$ converges to B.*

Proof. These assertions are an immediate consequence of [11, 4.6.21, p. 290] if we consider $\mathscr{B} = \mathscr{A}$ and ϕ to be I, the identity map on the C^*-algebra $\mathscr{A}$.

THEOREM 6 (Anderson and Trapp [4], Fillmore and Williams [8]). *Let A and B be positive operators on H. Then there are unique operators C and D which satisfy*

$$A^{1/2} = (A+B)^{1/2}C, \qquad N(C^*) \supseteq N((A+B)^{1/2}),$$

$$B^{1/2} = (A+B)^{1/2}D, \qquad N(D^*) \supseteq N((A+B)^{1/2}).$$

Remarks 3. Let A, B, C, and D be as above. In fact, one has $C = \sqrt{A+B}^{\dagger}\sqrt{A}$ and $D = \sqrt{A+B}^{\dagger}\sqrt{B}$, even though $\sqrt{A+B}^{\dagger}$ may be unbounded. Also since $R(A) \subseteq R(A^{1/2}) \subseteq R(A+B)^{1/2}$, there are operators E and F given by Lemma 1 such that $A = (A+B)^{1/2}E$, $B = (A+B)^{1/2}F$, $R(E) \subseteq \overline{R(A+B)}$, and $R(F) \subseteq \overline{R(A+B)}$. By the uniqueness condition of

Lemma 1 (Douglas' lemma), $E = CA^{1/2}$, $F = DB^{1/2}$, and $E + F = (A+B)^{1/2}$ (because $A + B = (A+B)^{1/2}(A+B)^{1/2}$ and $(A+B) = (A+B)^{1/2}(E+F)$).

LEMMA 10. *Let A and B be positive operators, and $R(B) \subseteq R(A+B)$. Then $\{(A+B)^{1/2}(A+B+1/n)^{-1}B\}_{n=1}^{\infty}$ is a norm convergent sequence in H and its limit is F (recall $B = (A+B)^{1/2}F$).*

Proof. First we show that the given sequence is Cauchy. If we replace both A and B in Theorem 5 by $A+B$, it follows that

$$u_n = (A+B)^{1/2}(n^{-1}I + A + B)(A+B) \qquad \text{and}$$
$$\|u_m - u_n\| \leqslant |n^{-1} - m^{-1}|^{1/2}.$$

Furthermore, from Lemma 1, there exist a $\lambda \geqslant 0$ such that $B^2 \leqslant \lambda^2(A+B)^2$. Hence

$$\begin{aligned}
&\left\| (A+B)^{1/2}\left[\left(A+B+\frac{1}{n}\right)^{-1} - \left(A+B+\frac{1}{m}\right)^{-1}\right]B\right\| \\
&\quad\leqslant \left\| (A+B)^{1/2}\left[\left(A+B+\frac{1}{n}\right)^{-1} - \left(A+B+\frac{1}{m}\right)^{-1}\right]\lambda(A+B)\right\| \\
&\quad\lambda \left\| (A+B)^{1/2}\left[\left(A+B+\frac{1}{n}\right)^{-1} - \left(A+B+\frac{1}{m}\right)^{-1}\right](A+B)\right\| \\
&\quad\leqslant \lambda\, |n^{-1} - m^{-1}|^{1/2}.
\end{aligned}$$

This implies that $\{(A+B)^{1/2}(A+B+1/n)^{-1}B\}_{n=1}^{\infty}$ is a Cauchy sequence and hence it converges to a limit T^*. From this it follows that $\{B(A+B+1/n)^{-1}(A+B)^{1/2}\}_{n=1}^{\infty}$ converges to T in norm. Again from Theorem 5 applied to the sequence $\{B^{1/2}(A+B+1/n)^{-1}(A+B)\}_{n=1}^{\infty}$ we have $B^{1/2}(A+B+1/n)^{-1}(A+B) \to B^{1/2}$ (in norm). Hence $B(A+B+1/n)^{-1}(A+B) \to B$ (in norm). On the other hand $B(A+B+1/n)^{-1}(A+B) \to T(A+B)^{1/2}$. Thus $T(A+B)^{1/2} = B$ and hence $B = (A+B)^{1/2}T^*$. Clearly, $N(T) \supseteq N((A+B)^{1/2})$, so by the uniqueness of F, $T^* = F$ and nothing remains to be proved.

Remark 4. If the conditions of Lemma 10 hold, then we have also that $(A+B)^{1/2}(A+B+1/n)^{-1}A \to E$ in norm (because $R(A) \subseteq R(A+B)$ by Theorem 4).

LEMMA 11. *Let T be a positive operator and let E_T be the range projection of T, i.e., the (self-adjoint) projection of H onto $\overline{R(T)}$. Then the sequence $\{(T+1/n)^{-1}T\}_{n=1}^{\infty}$ converges in the strong operator topology monotonically up to E_T.*

Proof. This follows easily from spectral theory and the monotone convergence theorem.

THEOREM 7. *Let A and B be bounded positive operators on H. The following statements are equivalent*:

(1) $R(B) \subseteq R(A+B)$

(2) $B(A+B+1/n)^{-1}(A+B)^{1/2} \to K(A+B)^{1/2}$ *(strong operator) for some* $K \in B(H)$

(3) $B(A+B+1/n)^{-1}(A+B)^{1/2} \to K(A+B)^{1/2}$ *(norm) for some* $K \in B(H)$

(4) $\{B(A+B+1/n)^{-1}\}_{n=1}^{\infty}$ *is a strong operator convergent sequence.*

Proof. (4)$\Rightarrow$(1). If $\{B(A+B+1/n)^{-1}\}_{n=1}^{\infty}$ is a strong operator convergent sequence, then by the uniform boundedness principle $\sup_n \|B(A+B+1/n)^{-1}\| < \infty$. Hence by Lemma 2, $R(B) \subseteq R(A+B)$.

(1)$\Rightarrow$(4). If $R(B) \subseteq R(A+B)$, then there exists C with $B = (A+B)C$, by Lemma 1 (Douglas' lemma). Therefore $B(A+B+1/n)^{-1} = C^*(A+B)(A+B+1/n)^{-1}$ and hence $\|B(A+B+1/n)^{-1}\| = \|C^*(A+B)(A+B+1/n)^{-1}\| \leqslant \|C^*\| \, \|(A+B)(A+B+1/n)^{-1}\| \leqslant \|C^*\|$.

(1)$\Rightarrow$(3). Suppose $R(B) \subseteq R(A+B)$. Then by Lemma 10, $(A+B)^{1/2}(A+B+1/n)^{-1}B \to F = DB^{1/2}$ in norm. From Lemma 5, we have that $X = (A+B)^{\dagger}B$ is a bounded operator on H.

Now we claim that $X^*(A+B)^{1/2} = B^{1/2}D^*$. Indeed, $(A+B)(A+B)^{\dagger}$ is the projection on $R(A+B)$, so $R(B) \subseteq (A+B)$ implies that $B = (A+B)(A+B)^{\dagger}B = (A+B)X$. Thus $X^*(A+B) = B = B^{1/2}B^{1/2} = B^{1/2}D^*(A+B)^{1/2}$. Therefore, $X^*(A+B)^{1/2} = B^{1/2}D^*$, since both vanish on $N((A+B)^{1/2})$. Thus $((A+B)^{1/2}(A+B+1/n)^{-1}B)^* \to X^*(A+B)^{1/2}$.

(2)$\Rightarrow$(1). Suppose that $B(A+B+1/n)^{-1}(A+B)^{1/2} \to K(A+B)^{1/2}$ (strong operator) for some $K \in B(H)$. We show that $R(B) \subseteq R(A+B)$. Since by Lemma 11, $(A+B+1/n)^{-1}(A+B) \to E_{A+B}$ (the range projection of $A+B$) in the strong operator topology by Lemma 11, we have $B(A+B+1/n)^{-1}(A+B) \to B$ (strong operator). Therefore, $B = K(A+B)$ and hence $R(B) \subseteq R(A+B)$ by Lemma 1.

COROLLARY 5. *If $R(B) \subseteq R(A+B)$, then*

$$\sup_x \frac{\|B^{1/2}D^*x\|}{\|(A+B)^{1/2}x\|} < \infty,$$

where the supremum is taken over all x which are not elements of $N((A+B)^{1/2})$. (B and D are as above.)

Proof. We showed in the previous theorem that $X^*(A+B)^{1/2}=B^{1/2}D^*$. Thus

$$\sup \frac{\|B^{1/2}D^*x\|}{\|(A+B)^{1/2}x\|}=\sup \frac{\|X^*(A+D)^{1/2}x\|}{\|(A+B)^{1/2}x\|}$$
$$\leqslant \sup \frac{\|X^*\|\ \|(A+B)^{1/2}x\|}{\|(A+B)^{1/2}x\|}=\|X^*\|.$$

COROLLARY 6. *If $R(B)\subseteq R(A+B)$, then*

$$\sup \frac{\|B(A+B+1/n)^{-1}(A+B)^{1/2}x-B^{1/2}D^*x\|}{\|(A+B)^{1/2}x\|}\leqslant 2\ \|(A+B)^{\dagger}B\|.$$

Proof. From the proof of Theorem 7, $X^*(A+B)=B$ and $X^*(A+B)^{1/2}=B^{1/2}D^*$ where $X=(A+B)^{\dagger}B$. Thus

$$\sup \frac{\|B(A+B+1/n)^{-1}(A+B)^{1/2}x-B^{1/2}D^*x\|}{\|(A+B)^{1/2}x\|}$$
$$=\sup \frac{\|X^*(A+B)(A+B+1/n)^{-1}(A+B)^{1/2}x-X^*(A+B)^{1/2}x\|}{\|(A+B)^{1/2}x\|}$$
$$=\sup \frac{\|X^*[(A+B)(A+B+1/n)^{-1}-I](A+B)^{1/2}x\|}{\|(A+B)^{1/2}x\|}$$
$$\leqslant \|X^*\|\ \left\|(A+B)\left(A+B+\frac{1}{n}\right)^{-1}-I\right\|\leqslant 2\ \|X^*\|$$
$$\left(\text{because } (A+B)\left(A+B+\frac{1}{n}\right)^{-1}\leqslant I\right).$$

Remark 5. If $\{(A+B)^{1/2}(A+B+1/n)^{-1}Bx\}_{n=1}^{\infty}$ is a Cauchy sequence for each $x\in H$, then by the principle of uniform boundedness

$$\sup\left\{\left\|(A+B)^{1/2}\left(A+B+\frac{1}{n}\right)^{-1}B\right\|;n\in N\right\}<\infty.$$

PROPOSITION 2. *Let A and B be two bounded positive operators and suppose that*

$$\sup_n \left\|(A+B)^{1/2}\left(A+B+\frac{1}{n}\right)^{-1}B\right\|<\infty.$$

Then

$$\left\{\left\|(A+B)^{1/2}\left(A+B+\frac{1}{n}\right)^{-1} Bx\right\|\right\}_{n=1}^{\infty}$$

is Cauchy for each $x \in H$.

Proof. It is easy to see (from the spectral theory of $A+B$) that

(1) $(A+B+1/n)^{-2} \leqslant (A+B+1/m)^{-2}$ for $n \leqslant m$.

(2) $(A+B)(A+B+1/n)^{-2} \leqslant (A+B)(A+B+1/m)^{-2}$ for $n \leqslant m$.

(3) $B(A+B)(A+B+1/n)^{-2}B = (B(A+B+1/n)^{-1}(A+B)^{1/2})(B(A+B+1/n)^{-1}(A+B)^{1/2})^* \leqslant B(A+B)(A+B+1/m)^{-2}B = (B(A+B+1/m)^{-1}(A+B)^{1/2})(B(A+B+1/m)^{-1}(A+B)^{1/2})^*$ for $n \leqslant m$.

Hence

$$\begin{aligned}\left\|(A+B)^{1/2}\left(A+B+\frac{1}{n}\right)^{-1} Bx\right\| &= \left\|\left(B\left(A+B+\frac{1}{n}\right)^{-1}(A+B)^{1/2}\right)^* x\right\| \\ &\leqslant \left\|\left(B\left(A+B+\frac{1}{m}\right)^{-1}(A+B)^{1/2}\right)^* x\right\| \\ &= \left\|(A+B)^{1/2}\left(A+B+\frac{1}{m}\right)^{-1} Bx\right\|\end{aligned}$$

for all $n \leqslant m$ and all $x \in H$. From boundedness of $\{(A+B)^{1/2}(A+B+1/n)^{-1}B\}_{n=1}^{\infty}$, it follows that $\{\|(A+B)^{1/2}(A+B+1/n)^{-1}Bx\|\}_{n=1}^{\infty}$ is a bounded monotone sequence in R for each $x \in H$. Hence $\{\|(A+B)^{1/2}(A+B+1/n)^{-1}Bx\|\}_{n=1}^{\infty}$ is a convergent sequence and is therefore Cauchy for each $x \in H$.

Acknowledgments

The author is very thankful to his thesis advisor, Professor William L. Green at School of Mathematics, Georgia Institute of Technology for his constructive comments and his thoughtful suggestions and ideas during preparation of this work.

References

1. W. N. Anderson, Shorted operators, *SIAM J. Appl. Math.* **20** (1971), 520–525.
2. W. N. Anderson and R. J. Duffin, Series and parallel addition of matrices, *J. Math. Anal. Appl.* **11** (1963), 576–594.
3. W. N. Anderson, Jr., and M. Schreiber, On the infimum of two projections, *Acta Sci. Math. (Szeged)* **23** (1972), 165–168.

4. W. N. Anderson and G. E. Trapp, Shorted Operators, II, *SIAM J. Appl. Math.* **23** (1975), 60–71.
5. A. Ben Israel and A. Chappes, Contributions to the theory of generalized inverses, *J. Indust. Appl. Math.* **11** (1963), 667–669.
6. J. Bunce, unpublished.
7. R. G. Douglas, On majorization, factorization, and range inclusion of operators on Hilbert space, *Proc. Amer. Math. Soc.* **17** (1966), 413–416.
8. P. A. Fillmore and J. P. Williams, On operator ranges, *Adv. in Math.* **7** (1971), 254–281.
9. W. L. Green and T. D. Morley, Operator means, fixed points, and the norm convergence of monotone approximants, *Math. Scand.* **60**, No. 2 (1987), 202–218.
10. W. L. Green and T. D. Morley, Parallel sums and norm convergence, submitted for publication.
11. R. V. Kadison and J. R. Ringrose, Fundamentals of the theory of operator algebras, Vol. I, *in* "Elementary Theory," Pure and Applied Math., Vol. 100, Academic Press, New York.
12. T. Morley, unpublished.
13. Yu. L. Shmulyan, Two-sided division in a ring of operators, *Math. Notes* **1** (1967), 400–403.
14. A. H. Zemanian, Infinite network of positive operators, *Circuit Theory Appl.* **2** (1974).