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Theorems of Erdős–Ko–Rado type in polar spaces

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ABSTRACT

We consider Erdős–Ko–Rado sets of generators in classical finite polar spaces. These are sets of generators that all intersect non-trivially. We characterize the Erdős–Ko–Rado sets of generators of maximum size in all polar spaces, except for $H(4n+1,q^2)$ with $n \geqslant 2$.

1. Introduction

Finding the largest sets of pairwise non-trivially intersecting elements is one of the classical problems in extremal combinatorics. We first give the original Erdős–Ko–Rado theorem, published in [9] in 1961.

Theorem 1. *If S is a family of subsets of size k in a set Ω with $|\Omega| = n$ and $n \geqslant 2k$, such that the elements of S are pairwise not disjoint, then $|S| \leqslant \binom{n-1}{k-1}$. If $n \geqslant 2k+1$, then equality holds if and only if S is the set of all subsets of size k containing a fixed element of Ω.*

Variants of this theorem in a wide variety of contexts, including one in projective geometry, were found subsequently. The projective geometry $\mathsf{PG}(n-1,q)$ consists of the lattice of subspaces of the vector space $V(n,q)$ of dimension n over $\mathbb{F}_q$. The number of subspaces of dimension k in $V(n,q)$, with $0 \leqslant k \leqslant n$, is given by the Gaussian coefficient:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=1}^{k} \frac{q^{n+1-i}-1}{q^i-1}.$$

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(If $n < k$ or $k < 0$, then $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is zero. Note that $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q$ and $\begin{bmatrix} n \\ 1 \end{bmatrix}_q = \frac{q^n-1}{q-1}$.)

In 1975, Hsieh proved the following q-analog of Theorem 1 in [12].

Theorem 2. *If S is a set of k-dimensional subspaces in $V(n,q)$, with $n \geqslant 2k+1$, pairwise intersecting not trivially, then $|S| \leqslant \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$. Equality holds if and only if S is the set of all subspaces with dimension k, containing a fixed 1-dimensional subspace of $V(n,q)$.*

In this article, we will consider the analogous problem for generators in a polar space. Classical finite *polar spaces* are incidence structures, consisting of the subspaces of $V := V(n,q)$ totally isotropic with respect to a certain non-degenerate sesquilinear or quadratic form f. Incidence is the inclusion relation. We will only consider classical finite polar spaces, and from now on, polar spaces are implicitly assumed to be finite and classical. The *rank* of the polar space is the algebraic dimension of the maximal totally isotropic subspaces or *generators*. We explicitly list the different types of polar spaces of rank N. For the sake of clarity, we give both the notation related to Chevalley groups and the more geometric notation, based on the embedding of the polar space in a projective space.

- the hyperbolic quadric $D_N(q)$ or $Q^+(2N-1,q)$, with $V = V(2N,q)$ and f a non-degenerate quadratic form of Witt index N, with parameters $(q,1)$,
- the Hermitian variety ${}^2A_{2N-1}(q)$ or $H(2N-1,q^2)$, with $V = V(2N,q^2)$ and f a non-degenerate Hermitian form, with parameters (q^2,q),
- the parabolic quadric $B_N(q)$ or $Q(2N,q)$, with $V = V(2N+1,q)$ and f a non-degenerate quadratic form, with parameters (q,q),
- the symplectic space $C_N(q)$ or $W(2N-1,q)$, with $V = V(2N,q)$ and f a non-degenerate symplectic form, with parameters (q,q),
- the Hermitian variety ${}^2A_{2N}(q)$ or $H(2N,q^2)$, with $V = V(2N+1,q^2)$ and f a non-degenerate Hermitian form, with parameters (q^2,q^3),
- the elliptic quadric ${}^2D_{N+1}(q)$ or $Q^-(2N+1,q)$, with $V = V(2N+2,q)$ and f a non-degenerate quadratic form of Witt index N, with parameters (q,q^2).

The parameters (q,q^e) listed above are such that each totally isotropic 2-space (or *line*) is incident with exactly $q+1$ totally isotropic 1-spaces (or *points*), and every totally isotropic $(N-1)$-space (or *dual line*) is incident with exactly q^e+1 maximal totally isotropic subspaces (or *generators*). We will also refer to totally isotropic 3-spaces as *planes*. The *projective dimension* of a subspace is its dimension minus one. Polar spaces of rank two are also known as classical *generalized quadrangles*.

Lemma 3. *(See [3], Lemma 9.4.1.) Let $\mathcal{P}$ be a polar space of rank N with parameters (q,q^e). The number of totally isotropic subspaces with dimension m is given by $\begin{bmatrix} N \\ m \end{bmatrix}_q \prod_{i=0}^{m-1}(q^{N-i-1+e}+1)$. In particular, the number of generators is $(q^e+1)\cdots(q^{N-1+e}+1)$.*

The previous lemma also allows us to compute the number of generators through a fixed point in a polar space of rank N, as it is the number of generators in the residual polar space of rank $N-1$ and of the same type.

If two totally isotropic subspaces in a polar space intersect trivially, or hence if they have no point in common, then we say that they are *disjoint*. It is our goal to study sets of generators in a polar space pairwise not disjoint. We will refer to such sets as *EKR sets of generators*. We will say that such a set is a *maximal EKR set of generators* if it is not a proper subset of another EKR set of generators. A simple example of an EKR set of generators is the *point-pencil* construction, consisting of all generators through a fixed point. We will prove in this paper that in many polar spaces, these are the unique EKR sets of generators of maximum size.

In the case of (classical) generalized quadrangles, the generators are lines. It is a trivial observation that the maximal sets of lines pairwise intersecting in a generalized quadrangle are the sets of lines through a point, so we will only focus on polar spaces of rank at least three.

This article is structured as follows. We will give some preliminary observations on maximal EKR sets of generators in Section 2. In Section 3, we will approach the problem in a graph-theoretic way and give bounds on the size of EKR sets of generators already found by Stanton in [14]. We will give an alternative proof in Section 4 for the characterization in those polar spaces discussed by Tanaka in [15]. Sections 5, 6 and 7 are devoted to a complete characterization in all but one remaining case. Finally, we prove in Section 8 that in one particular type of Hermitian variety, the bound from [14] cannot be reached, and we give a characterization for small rank.

An overview of our results is given in Section 9.

2. General observations on maximal EKR sets of generators

We will first obtain some results by use of purely geometric results, which already hold for EKR sets of generators when only assuming maximality.

Lemma 4. *Let π_a, π_b and π_c be pairwise non-disjoint generators in a polar space. The intersections $\pi_a \cap \pi_b$ and $\pi_a \cap \pi_c$ cannot be complementary subspaces of π_a.*

Proof. Suppose $\pi_a \cap \pi_b$ and $\pi_a \cap \pi_c$ are complementary subspaces of π_a. As π_b and π_c are assumed to meet non-trivially, they must have a point p in common, not in π_a. This point would be collinear with all points in $\pi_a \cap \pi_b$ and with all points in $\pi_a \cap \pi_c$, and hence with all points in $\langle \pi_a \cap \pi_b, \pi_a \cap \pi_c \rangle = \pi_a$, which would contradict the assumption that π_a is a maximal totally isotropic subspace. □

Lemma 5. *Let S be a maximal EKR set of generators. If a dual line is incident with at least two elements of S, then all generators through it are in S.*

Proof. Let μ be a dual line, incident with two distinct elements π_a and π_b of S. Suppose a third generator π' through μ is not in S. As S is assumed to be maximal, there must be a generator $\pi_c \in S$ disjoint from π' and hence also from μ, so it intersects π_a in a point not on μ. The generators π_a, π_b and π_c contradict Lemma 4. □

The previous lemma motivates us to introduce the following terminology. We say that a dual line in a polar space is *secant*, *tangent* or *external* with respect to a maximal EKR set of generators S if all, one or none of the generators through it are in S, respectively.

Let S be an EKR set of generators in a polar space with $\pi \in S$. Consider all secant dual lines with respect to S in π. We will refer to their intersection as the *nucleus* of π (with respect to S), we will denote it by π_s and s will be the projective dimension of π_s. The nuclei of the elements of S will play a crucial role in our characterization of the EKR sets of generators of maximum size. In the following lemma, we prove fundamental properties of the nuclei.

Lemma 6. *Let S be a maximal EKR set of generators in a polar space of rank N and with parameters (q, q^e). If π_s is the nucleus of the generator $\pi \in S$, then the secant dual lines in π are those through π_s, and the tangent dual lines in π are those not through π_s. The number of elements of S that intersect $\pi \in S$ in a dual line is given by $q^e \begin{bmatrix} N-s-1 \\ 1 \end{bmatrix}_q$.*

Moreover, if a generator $\pi' \in S$ meets π in just a point, then this point must be in π_s.

Proof. Suppose $\pi \in S$ has nucleus π_s with projective dimension s. If $\pi' \in S$ intersects π in a point p, then p must belong to every secant dual line μ in π by Lemma 4, hence $p \in \pi_s$.

Let μ be a dual line through π_s. By Lemma 5, μ is either secant or tangent. Suppose that μ is tangent, hence there exists a π_1 through μ such that $\pi_1 \notin S$. Since S is maximal, there must be a $\pi_2 \in S$ disjoint from π_1, but π_2 must intersect π, so this intersection would be a point not in μ and hence not in π_s either, contradicting the above. So μ is secant.

The number of dual lines in π through π_s is given by $\begin{bmatrix} N-s-1 \\ 1 \end{bmatrix}_q$, and through each such dual line there are q^e other elements of S, and hence there are exactly $q^e \begin{bmatrix} N-s-1 \\ 1 \end{bmatrix}_q$ elements of S meeting π in a dual line. □

3. Graph-theoretic approach to polar spaces

Theorems 1 and 2 can be interpreted as results regarding the Johnson and Grassmann graphs, respectively (see for instance Sections 9.1 and 9.3 in [3]). In this article, we have to consider the dual polar graph, the vertices of which are the generators of a polar space. Two generators are neighbors in the dual polar graph if they meet in a subspace of codimension one and when we say that two generators are neighbors, we will mean with respect to this relation, unless stated otherwise. First, we need some algebraic background.

For any finite set Ω, we can consider the real vector space $\mathbb{R}\Omega$, which has an orthonormal basis corresponding to the elements of Ω. For every subset $S \subseteq \Omega$, we define the column vector $\chi_S \in \mathbb{R}\Omega$ as the *characteristic vector* of S, with the entries of χ_S corresponding to an element of S equal to one, and all other entries equal to zero.

Bose and Shimamoto [2] introduced the notion of a d-class *association scheme* on a finite set Ω as a pair $(\Omega, \mathcal{R})$ with $\mathcal{R}$ a set of symmetric relations $\{R_0, R_1, \dots, R_d\}$ on Ω such that the following axioms hold:

(i) R_0 is the identity relation,
(ii) $\mathcal{R}$ is a partition of Ω^2,
(iii) there are *intersection numbers* p_{ij}^k such that for $(x, y) \in R_k$, the number of elements z in Ω for which $(x, z) \in R_i$ and $(z, y) \in R_j$ equals p_{ij}^k.

All the relations R_i are symmetric regular relations with valency p_{ii}^0, and hence define regular graphs on Ω.

With each relation R_i, we can associate the $(|\Omega| \times |\Omega|)$-matrix A_i, the rows and columns of which are indexed by the elements of Ω and with $(A_i)_{xy} = 1$ if $(x, y) \in R_i$ and $(A_i)_{xy} = 0$ if not. The axioms for an association scheme immediately imply that all A_i are symmetric, A_0 is the identity matrix, the sum of all A_i is the all-one matrix and $A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k$. Hence it follows that the vector space spanned by $\{A_0, \dots, A_d\}$ is closed under multiplication, and we refer to it as the *Bose–Mesner algebra*. We also let $\circ$ denote entrywise multiplication of matrices. Now $A_i \circ A_j = \delta_{ij} A_i$, and so the Bose–Mesner algebra is closed under this multiplication as well.

It can be shown (see for instance [1]) that the real vector space $\mathbb{R}\Omega$ has a unique orthogonal decomposition into $d+1$ subspaces V_j, all of them eigenspaces (or subspaces of eigenspaces) for the relations R_i of the association scheme. These subspaces are the *strata* of the association scheme. The $(d+1) \times (d+1)$-matrix P, where P_{ji} is the eigenvalue of the relation R_i for the eigenspace V_j, is the *matrix of eigenvalues* of the association scheme. The matrices E_j defining orthogonal projection onto the subspace V_j also span the Bose–Mesner algebra and are known as the *minimal idempotents*.

Now let Γ be a connected graph with diameter d on a set of vertices Ω. For every i in $\{0, \dots, d\}$, we let Γ_i denote the graph on the same set Ω, with two vertices adjacent if and only if they are at distance i in Γ, and we write R_i for the corresponding symmetric relation on V. The graph Γ is said to be *distance-regular* if the set of relations $\{R_0, R_1, \dots, R_d\}$ induces an association scheme on Ω. It can be shown (see Chapter 4 in [3]) that this is equivalent with the existence of parameters b_i and c_i, such that for every $(v, v_i) \in R_i$, there are c_i neighbors v_{i-1} of v_i with $(v, v_{i-1}) \in R_{i-1}$, for every $i \in \{1, \dots, d\}$, and b_i neighbors v_{i+1} with $(v, v_{i+1}) \in R_{i+1}$, for every $i \in \{0, \dots, d-1\}$. These parameters b_i and c_i are known as the *intersection numbers* of the distance-regular graph Γ.

Theorem 7. *(See [3], Theorem 9.4.3.) Let Γ be the dual polar graph of a polar space of rank N with parameters (q, q^e). This graph is distance-regular with diameter N, and two vertices are at distance i if and only if they meet in a subspace of codimension i. The intersection numbers are given by*:

$$b_i = q^{i+e} \begin{bmatrix} N-i \\ 1 \end{bmatrix}_q, \quad \forall i \in \{0, \dots, N-1\}, \qquad c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_q, \quad \forall i \in \{1, \dots, N\}.$$

The valency of Γ_i is given by $\begin{bmatrix} N \\ i \end{bmatrix}_q q^{i(i-1)/2} q^{ie}$.

The vector space $\mathbb{R}\Omega$ orthogonally decomposes as $V_0 \perp V_1 \perp \cdots \perp V_N$, where V_j is an eigenspace of the dual polar graph for the distinct eigenvalues $q^e \begin{bmatrix} N-j \\ 1 \end{bmatrix}_q - \begin{bmatrix} j \\ 1 \end{bmatrix}_q$.

Note that V_0 is, in this ordering, just the subspace spanned by the all-one vector χ_Ω. This means that for every subset S, the characteristic vector χ_S will have the component $\frac{(\chi_S)^t \chi_\Omega}{(\chi_\Omega)^t \chi_\Omega} \chi_\Omega = \frac{|S|}{|\Omega|} \chi_\Omega$ in this V_0.

A subset S of vertices in a regular graph is a *coclique* if two vertices in S are never adjacent. As EKR sets of generators are precisely the cocliques of the disjointness relation, we are especially interested in the graph Γ_N. The $N+1$ subspaces $V_j \subseteq \mathbb{R}\Omega$ from Theorem 7 are the strata of the induced d-class association scheme, and hence eigenspaces for all the relations $\Gamma_0, \Gamma_1, \ldots, \Gamma_N$. Stanton [14] calculated the eigenvalues of the disjointness graph Γ_N between generators in a polar space of rank N with parameters (q, q^e), and obtained the eigenvalue

$$(-1)^j q^{N(N+1)/2 + j(j-N-1) + (e-1)(N-j)}$$

for the subspace V_j.

We will use the following result on cocliques in regular graphs (see for instance Theorem 3.1 in [10]).

Theorem 8. *If S is a coclique of a regular graph Γ on a set Ω with valency k and smallest eigenvalue $\lambda < 0$, then*:

$$|S| \leqslant \frac{|\Omega|}{1 - k/\lambda}.$$

Moreover, if the bound is met, then χ_S can be written as a linear combination $\frac{|S|}{|\Omega|} \chi_\Omega + v$, with v an eigenvector for λ.

Stanton [14] used the inequality from Theorem 8 to obtain upper bounds on the size of EKR sets of generators in polar spaces.

Theorem 9. *Let S be an EKR set of generators in a polar space $\mathcal{P}$, and consider the decomposition $\mathbb{R}\Omega = V_0 \perp \cdots \perp V_N$ as in Theorem 7.*

- *If $\mathcal{P} = Q^+(2N-1, q)$, with N odd, then $|S|$ is at most half of the total number of generators, and if this bound is reached, then $\chi_S \in (V_0 \perp V_N)$.*
- *If $\mathcal{P} = Q^+(2N-1, q)$, with N even, then $|S|$ is at most the number of generators through a fixed point, and if this bound is reached, then $\chi_S \in (V_0 \perp V_1 \perp V_{N-1})$.*
- *If $\mathcal{P} = H(2N-1, q^2)$, with N odd, then $|S|$ is at most the number of generators in the polar space $\mathcal{P}$ divided by $q^N + 1$, and if this bound is reached, then $\chi_S \in (V_0 \perp V_N)$.*
- *If $\mathcal{P} = Q(2N, q)$ with N odd, or $\mathcal{P} = W(2N-1, q)$, with N odd, then $|S|$ is at most the number of generators through a fixed point, and if this bound is reached, then $\chi_S \in (V_0 \perp V_1 \perp V_N)$.*

For all other polar spaces, the size of S is at most the number of generators through a fixed point, and if this bound is reached, then $\chi_S \in (V_0 \perp V_1)$.

Proof. This follows from Theorem 8 and the formula for the eigenvalue

$$(-1)^j q^{N(N+1)/2 + j(j-N-1) + (e-1)(N-j)}$$

of the disjointness graph for V_j. For $j = 0$, one obtains the valency k. One must then consider the minimal eigenvalue for each possible value of e: 0, 1/2, 1, 3/2 and 2. This minimal eigenvalue is obtained only for $j = 1$, except in the following cases: if $e = 0$ (hence if $\mathcal{P} = Q^+(2N-1, q)$) and N is odd for $j = N$, if $e = 0$ (hence if $\mathcal{P} = Q^+(2N-1, q)$) and N is even for $j = 1$ and $j = N-1$, if

$e = 1/2$ (hence if $\mathcal{P} = H(2N-1, q^2)$) and N is odd for $j = N$, and if $e = 1$ (hence if $\mathcal{P} = W(2N-1, q)$ or $\mathcal{P} = Q(2N, q)$) and N is odd for $j = 1$ and $j = N$. $\square$

The phenomenon observed in Theorem 9 also has important consequences for the opposite problem: finding the maximum size of cliques of the disjointness relation. Sets of generators that are all mutually disjoint are known as *partial spreads* of the polar spaces. In [18], this was used to obtain a tight upper bound for partial spreads in $H(2N-1, q^2)$ with N odd.

In light of the description of the characteristic vector from Theorem 8, we give the following lemma.

Lemma 10. *Suppose S is a set of vertices in a regular graph Γ with valency k, with χ_S a sum of a multiple of the all-one vector and an eigenvector v of some eigenvalue λ.*

- *If $p \in S$, then p has $\frac{|S|}{|\Omega|}(k-\lambda)+\lambda$ neighbors in S.*
- *If $p \notin S$, then p has $\frac{|S|}{|\Omega|}(k-\lambda)$ neighbors in S.*

Proof. Let A be the adjacency matrix of the graph Γ. The results follow immediately from:

$$A\chi_S = A\left(\frac{|S|}{|\Omega|}\chi_\Omega + v\right) = k\frac{|S|}{|\Omega|}\chi_\Omega + \lambda v = \left(\frac{|S|}{|\Omega|}(k-\lambda)+\lambda\right)\chi_S + \frac{|S|}{|\Omega|}(k-\lambda)(\chi_\Omega - \chi_S),$$

recalling that $\chi_S = \frac{|S|}{|\Omega|}\chi_\Omega + v$. $\square$

4. Classification of the Erdős–Ko–Rado sets of maximum size in most polar spaces

As we will often compare the cardinalities of sets of geometric objects, we need some analytic tools.

Lemma 11. *If $n \geqslant 1$, $q \geqslant 2$, $e \geqslant \frac{1}{2}$ and $q^e \geqslant 2$, then $(1+\frac{1}{q^e})\cdots(1+\frac{1}{q^{n+e}}) < 2+\frac{1}{q^e}$, and in particular $(1+\frac{1}{3})\cdots(1+\frac{1}{3^{n+1}}) < 2$.*

Proof. As $1+\frac{1}{q^{e+i}}$ is at most $\exp(\frac{1}{q^{e+i}})$, the product $(1+\frac{1}{q^e})\cdots(1+\frac{1}{q^{n+e}})$ is at most

$$\left(1+\frac{1}{q^e}\right)\exp\left(\frac{1}{q^e}\frac{1}{q-1}\right) \leqslant \left(1+\frac{1}{q^e}\right)\exp\left(\frac{1}{q^e}\right).$$

As $(1+x)\exp(x) < 2+x$, $\forall x$ with $0 \leqslant x \leqslant 1/2$, this completes the proof of the general statement.

After taking $q = 3$ and $e = 1$, one easily proves the last part as well by verifying that $\frac{4}{3}\exp(\frac{1}{3}) < 2$. $\square$

Lemma 12.

1. *For any $n \geqslant 0$ and $q \geqslant 3$, we have:*

$$\prod_{i=1}^{n}\left(q^i+1\right) < 2q^{n(n+1)/2}.$$

2. *If $N \geqslant 3$, $q \geqslant 2$, $e \geqslant \frac{1}{2}$ and $q^e \geqslant 2$, then:*

$$\left(q^e+1\right)\cdots\left(q^{N+e-2}+1\right) - \begin{bmatrix} N-1 \\ 1 \end{bmatrix}_q q^{(N-2)(N-3)/2+(N-2)e} < 2q^{(N-1)(N-2)/2+(N-1)e}.$$

Proof. 1. The result is obvious if $n = 0$. Suppose $n \geqslant 1$. When dividing both sides by $q^{n(n+1)/2}$, we obtain the equivalent inequality:

$$\left(1+\frac{1}{q}\right)\cdots\left(1+\frac{1}{q^n}\right) < 2.$$

As $q \geqslant 3$, this is at most $(1+\frac{1}{3})\cdots(1+\frac{1}{3^n})$, which is in turn less than 2 because of Lemma 11.

2. As $\left[{N-1 \atop 1}\right]_q > q^{N-2}$, the left-hand side is at most

$$q^{(N-1)(N-2)/2+(N-1)e}\left(\left(1+\frac{1}{q^e}\right)\cdots\left(1+\frac{1}{q^{N+e-2}}\right)-\frac{1}{q^e}\right).$$

We can now use Lemma 11 to complete the proof. □

We also give the following general result on polar spaces (see for instance Lemma 9.4.2 in [3]).

Theorem 13. *Let $\mathcal{P}$ be a polar space of rank N with parameters (q, q^e). The number of generators meeting a fixed totally isotropic subspace π_m with dimension m in a subspace of codimension i in π_m is given by*:

$$q^{i(N-m+e+\frac{i-1}{2})}\begin{bmatrix} m \\ i \end{bmatrix}_q \prod_{j=0}^{N-m-1}\left(q^{N-m-j-1+e}+1\right).$$

For any generator, there are $q^{\frac{i(i-1)}{2}+ie}$ generators intersecting it in a fixed subspace of codimension i.

We know from Theorem 9 that in most polar spaces, the maximum size of an EKR set of generators is the number of generators through one point, and that the characteristic vector of such a set also satisfies strong conditions. We will now use this to obtain strong properties of such sets in these polar spaces.

Lemma 14. *Let $\mathcal{P}$ be a polar space of rank $N \geqslant 3$, either $H(2N, q^2)$, $H(2N-1, q^2)$ with N even, $Q(2N, q)$ with N even, $W(2N-1, q)$ with N even or $Q^-(2N+1, q)$. If S is an EKR set of generators of $\mathcal{P}$ with $|S|$ equal to the number of generators through a fixed point, then for every element $\pi \in S$, the number of elements of S meeting π in a subspace of codimension i is given by $a_i = \left[{N-1 \atop i}\right]_q q^{i(i-1)/2} q^{ie}$.*

Proof. Theorem 9 yields that in these polar spaces, S must have a characteristic vector in $V_0 \perp V_1$. We know that V_0 and V_1 are not only eigenspaces for Γ_N, but for all Γ_i. For each Γ_i, let k_i denote the valency and λ_i the eigenvalue for V_1. Lemma 10 yields that the number of elements of S, meeting a generator π in a subspace with codimension i, is $\frac{|S|}{|\Omega|}(k_i - \lambda_i) + \lambda_i$ if $\pi \in S$. Instead of explicitly calculating these eigenvalues, we will derive these numbers by considering the point-pencil construction, consisting of all generators through some fixed point p. If a generator π is through p, then the number of generators through p, meeting π in a subspace of codimension i, is the same as the valency of the i-distance relation Γ_i in the residual polar space of the same type and of rank $N-1$. Theorem 7 now gives us the desired values. □

As an example, we consider the elliptic quadric $Q^-(7, q)$ with $N = 3$ and $e = 2$. Here, the upper bound for EKR sets of generators is $(q^2+1)(q^3+1)$. If it is reached, the constants (a_0, a_1, a_2, a_3) will be given by $(1, (q+1)q^2, q^5, 0)$.

We will now characterize the EKR sets of generators of maximum size in almost all polar spaces. These polar spaces are also treated in Theorem 1 in [15], but we will use an alternative, more local approach.

Theorem 15. *Let $\mathcal{P}$ be a polar space as in Lemma 14. If S is an EKR set of generators of $\mathcal{P}$ with $|S|$ equal to the number of generators through a fixed point, then S must be the set of generators through a fixed point.*

Proof. Let us assume that the polar space has parameters (q, q^e).

Lemma 14 yields that for any $\pi \in S$, the number of elements of S meeting π in a subspace of codimension i is given by $q^{i(i-1)/2} q^{ie} \left[\begin{smallmatrix} N-1 \\ i \end{smallmatrix}\right]_q$. In particular, the number of neighbors of π in S in the dual polar graph is $q^e \left[\begin{smallmatrix} N-1 \\ 1 \end{smallmatrix}\right]_q$. Hence, Lemma 6 yields that for any generator π, the nucleus is a point. The number of elements of S meeting π in that point is $a_{N-1} = q^{(N-1)(N-2)/2+(N-1)e}$.

Let π_1 be an element of S with nucleus p. Suppose that there is an element $\pi' \in S$ not through p. Every $\pi'' \in S$ that intersects π_1 in just p has the point p as nucleus as well, and hence π' and π'' meet in at least a line. The generators through p that meet π' in at least a line, correspond in the residual geometry of p of rank $N-1$ with those generators meeting a fixed generator in a subspace of codimension different from $N-1$ and $N-2$. Their number is $(q^e+1)\cdots(q^{N-2+e}+1) - q^{(N-1)(N-2)/2+(N-1)e} - q^{(N-2)(N-3)/2+(N-2)e} \left[\begin{smallmatrix} N-1 \\ 1 \end{smallmatrix}\right]_q$ by Theorem 7, and hence we should have

$$\begin{aligned} &\left(q^e+1\right)\cdots\left(q^{N-2+e}+1\right) - q^{(N-1)(N-2)/2+(N-1)e} - q^{(N-2)(N-3)/2+(N-2)e} \begin{bmatrix} N-1 \\ 1 \end{bmatrix}_q \\ &\quad \geqslant a_{N-1} = q^{(N-1)(N-2)/2+(N-1)e}, \end{aligned}$$

but this is false by Lemma 12(2). This implies that all elements of S are through p. □

The next sections will be devoted to the remaining polar spaces.

5. Hyperbolic quadrics

In the case of the hyperbolic quadric $Q^+(2N-1, q)$, there are two systems of generators of the same size. We will refer to them as the *Latin and Greek* generators, and use the symbols Ω_1 and Ω_2 for these sets. They have the property that two generators are in the same system if and only if the algebraic dimension of their intersection has the same parity as N. Moreover, a totally isotropic subspace of dimension $N-1$ is contained in exactly two generators: one in Ω_1 and one in Ω_2.

The collineation group of $Q^+(2N-1, q)$ acts transitively on the generators, but the dual polar graph is bipartite with diameter N with the sets of Latins and Greeks as the two bipartite classes (see for instance Theorems 11.59 and 11.60 in [16]). Hence every collineation either stabilizes both systems, or switches them.

For this particular dual polar graph, the eigenvalue for the subspace V_j from Theorem 7, with $0 \leqslant j \leqslant N$, is given by $\left[\begin{smallmatrix} N-j \\ 1 \end{smallmatrix}\right]_q - \left[\begin{smallmatrix} j \\ 1 \end{smallmatrix}\right]_q$. Note that the eigenvalues for V_j and V_{N-j} are opposite. The following relation holds between eigenspaces:

$$V_{N-j} = \{v_1 - v_2 \mid v_1 \in \mathbb{R}\Omega_1,\ v_2 \in \mathbb{R}\Omega_2,\ v_1 + v_2 \in V_j\}.$$

In particular, V_0 and V_N are one-dimensional eigenspaces of this dual polar graph, with $V_0 = \langle \chi_{\Omega_1} + \chi_{\Omega_2} \rangle$ and $V_N = \langle \chi_{\Omega_1} - \chi_{\Omega_2} \rangle$.

Let us first consider the case where N is odd. Here, two generators of the same system cannot intersect trivially, so the set of all Latins and the set of all Greeks are both EKR sets, and their sizes meet the eigenvalue bound from Theorem 9. The following algebraic argument quickly establishes that this is the only possibility.

Theorem 16. *Let S be an EKR set of generators in $Q^+(2N-1, q)$ with N odd of size $|\Omega|/2$, then S is one of the two systems of the hyperbolic quadric.*

Proof. Theorem 9 yields that $\chi_S \in V_0 \perp V_N$. The eigenspace V_0 is spanned by $\chi_{\Omega_1} + \chi_{\Omega_2}$, while V_N is spanned by $\chi_{\Omega_1} - \chi_{\Omega_2}$. Hence χ_S can only be χ_{Ω_1} or χ_{Ω_2}. □

Next, we consider the case where N is even. Here, two generators of two different systems cannot intersect trivially, so if S_1 is an EKR set contained in Ω_1 and S_2 is an EKR set contained in Ω_2, $S_1 \cup S_2$ is still an EKR set for the polar space. From now on, we will also denote the rank N by

$2n+2$, which will simplify some of our calculations. The upper bound from Theorem 9 for an EKR set of generators S in $Q^+(4n+3,q)$ is $2(q+1)\cdots(q^{2n}+1)$. This bound can be reached by taking all generators through a single point, but one could for instance also take all Latins through one point, and all Greeks through another point to obtain an EKR set of generators. If S_1 is a set of Latins, no two of which disjoint, and σ is any automorphism of $Q^+(4n+3,q)$ switching the Latins and Greeks, then $S_1 \cup S_1^\sigma$ is an EKR set and hence $2|S_1| \leqslant 2(q+1)\cdots(q^{2n}+1)$. Hence the upper bound for an EKR set of one given system is $(q+1)\cdots(q^{2n}+1)$. The two systems of generators are projectively equivalent, so it is enough to classify the EKR sets of size $(q+1)\cdots(q^{2n}+1)$ of one system in $Q^+(4n+3,q)$. Therefore, we will consider the *half dual polar graph* Γ', the vertices of which are the generators of one system, with two of them adjacent when meeting in a subspace of codimension two. We refer to 9.4.C in [3] for a discussion of this graph.

We will use similar techniques as those applied by Tanaka for the dual polar graph in [15]. Therefore, we will also need some more algebraic background. The association scheme induced by the dual polar graph has a natural ordering for its relations, determined by the corresponding distance from a given generator. One formulates this by saying it is P-polynomial. We will now introduce the dual notion of Q-polynomiality, which gives a special meaning to a certain ordering of the strata instead. Let $(\Omega,\{R_0,\dots,R_d\})$ be an association scheme with strata $V_0,\dots,V_d$. Let E_j denote orthogonal projection in $\mathbb{R}\Omega$ onto V_j, and again let $\circ$ denote entrywise multiplication of elements of the Bose–Mesner algebra. We say that the ordering $V_0,\dots,V_d$ is *Q-polynomial* if $E_1 \circ E_j$ is a linear combination of E_{j-1}, E_j and E_{j+1} with non-zero coefficients for E_{j-1} and E_{j+1} for every j with $0 \leqslant j \leqslant d$, where we let E_{-1} and E_{d+1} simply be zero.

For the following properties of the half dual polar graph, we refer to Theorem 9.4.8 and Corollary 8.4.2 in [3].

Theorem 17. *Let Γ' be the half dual polar graph, with set of vertices Ω being the set of the generators of one system in the hyperbolic quadric $Q^+(2N-1,q)$. This graph is distance-regular with diameter $d = \lfloor \frac{N}{2} \rfloor$, and two vertices are at distance i if and only if they meet in a subspace of codimension $2i$.*

The valency of Γ_i' is given by $\begin{bmatrix} N \\ 2i \end{bmatrix}_q q^{i(2i-1)}$, and the intersection numbers by: $b_i = q^{4i+1}\begin{bmatrix} N-2i \\ 2 \end{bmatrix}_q$ $(0 \leqslant i \leqslant d-1)$ and $c_i = \begin{bmatrix} 2i \\ 2 \end{bmatrix}_q$ $(1 \leqslant i \leqslant d)$.

The vector space $\mathbb{R}\Omega$ orthogonally decomposes as $W_0 \perp W_1 \perp \cdots \perp W_d$, where W_j is an eigenspace of the half dual polar graph for the eigenvalue $q^{2j+1}\begin{bmatrix} N-2j \\ 2 \end{bmatrix}_q - \frac{q^{2j}-1}{q^2-1}$, and all $d+1$ eigenvalues are distinct. This ordering of the spaces W_j is Q-polynomial.

If Γ' is the half dual polar graph in $Q^+(2N-1,q)$, then the eigenvalue of Γ_i' for the subspace W_j is the same as the eigenvalue of the $2i$-distance graph Γ_{2i} of the original dual polar graph Γ for both the subspace V_j and V_{N-j}. Hence, the ratio $1-k/\lambda$ from Theorem 8 remains the same, and we find that an EKR set of generators of the same system has size at most $(q+1)\cdots(q^{2n}+1)$, and this bound can only be reached if the characteristic vector is in $W_0 \perp W_1$.

It is our aim to show that an EKR set of generators of one system in the hyperbolic quadric $Q^+(2N-1,q)$, for even $N \geqslant 6$, consists of all generators of that system through one point, and that for $N=4$, there is only one extra construction.

A consequence of Theorem 2 from [4] and Proposition 2 from [15] yields the following.

Theorem 18. *Let Γ be a distance-regular graph with diameter d on a set Ω. Suppose $\mathbb{R}\Omega$ has an orthogonal decomposition $V_0 \perp \cdots \perp V_d$ into strata which are Q-polynomially ordered. For each subset S of Ω with $\chi_S \in V_0 \perp V_1$, the maximum distance between elements of S is at least $d-1$, and if it is $d-1$, then the i-distance relations, with $i \in \{0,\dots,d-1\}$, induce an association scheme with one class less, when restricted to the subset S. The parameters of this scheme are independent of the subset S.*

This allows us to obtain a result regarding convexity of EKR sets of generators in the half dual polar graph, just as in [15].

Corollary 19. *Let S be a set of $(q+1)\cdots(q^{2n}+1)$ generators of one system in $Q^+(4n+3,q)$ pairwise not disjoint.*

(1) *The i-distance relations of the half dual polar graph with $0 \leqslant i \leqslant n$ induce an association scheme on S with the same parameters as the scheme induced by the half dual polar graph of $Q^+(4n+1,q)$.*
(2) *Suppose that π_a and π_b are two elements of S at distance i in the associated half dual polar graph. If π is a neighbor of π_a in the half dual polar graph and at distance $i-1$ from π_b, then π must be in S as well.*

Proof. We know that the assumptions imply that $\chi_S \in W_0 \perp W_1$, with the same notation as in Theorem 17. Theorem 18 yields that the i-distance relations induce an association scheme on S.

Now let S' be the set of all generators of the same system through a fixed point. This set satisfies the same assumptions. We know that the association schemes induced on S and on S' have the same parameters, and the latter is isomorphic to that on generators of one system in the hyperbolic quadric $Q^+(4n+1,q)$.

Hence if two generators π_a and π_b in S are at distance i, the number of generators in S at distance $i-1$ from π_a and at distance one from π_b is given by: $c_i = \left[{2i \atop 2}\right]_q$ (see Theorem 17).

Moreover, $\left[{2i \atop 2}\right]_q$ is also the number of generators in the full half dual polar graph, at distance $i-1$ from π_a, and distance one from π_b. Hence every such generator in the half dual polar graph must belong to S. □

The proof of the following lemma is similar to the proof of Theorem 1 in [15].

Lemma 20. *Let S be a set of $(q+1)\cdots(q^{2n}+1)$ generators of one system of $Q^+(4n+3,q)$ pairwise intersecting. If π_1 and π_2 are elements of S that meet in just a line ℓ, all the elements of S cannot be disjoint from ℓ.*

Proof. Let $\perp$ denote the polarity associated to the polar space. Suppose π_1 and π_2 are elements of S meeting in just the line ℓ. Suppose $\pi \in S$ intersects π_1 in a subspace μ of codimension $2i$ in π_1, skew to ℓ. Let m be any line in π, skew to $\ell^\perp \cap \pi$. Consider the generator $\pi' = \langle m, m^\perp \cap \pi_1\rangle$. This generator meets π_1 in a subspace of codimension two, skew to ℓ, and is at distance $i-1$ with respect to the half dual polar graph from π. Hence π' is in S as well, because of Corollary 19, so π' and π_2 must also meet non-trivially and the triple (π_1, π_2, π') would contradict Lemma 4. □

We now come to the main result concerning hyperbolic quadrics.

Theorem 21. *Let S be a set of Latins in $Q^+(4n+3,q)$ pairwise intersecting and $n \geqslant 2$ of size $(q+1)\cdots(q^{2n}+1)$, then S is the set of Latins through a fixed point.*

Proof. Let π be in S. We know from Corollary 19 that the number of elements of S meeting π in exactly a line is the same as the number of generators in $Q^+(4n+1,q)$ that meet a fixed generator in exactly a point, so it is $\left[{2n+1 \atop 1}\right]_q q^{n(2n-1)}$ (Theorem 17). On the other hand, Theorem 13 yields that there are exactly $q^{n(2n-1)}$ generators of $Q^+(4n+3,q)$, intersecting π in just a fixed line. Hence the set of lines A that are intersections of π with some element of S has size at least $\left[{2n+1 \atop 1}\right]_q$, and we know from Lemma 20 that no two of them can be disjoint. As $n \geqslant 2$, we can now apply Theorem 2 to see that A is precisely the set of $\left[{2n+1 \atop 1}\right]_q$ lines through some fixed point p in π.

Now suppose π' is an element of S not through p. This means that $\mu = \pi \cap \pi'$ is a subspace of codimension at least two in π and not through p. Let ℓ be a line in A skew to μ. Now ℓ is the intersection of two elements of S, while π' is disjoint from ℓ, contradicting Lemma 20. □

The hyperbolic quadric $Q^+(7,q)$ must be treated separately. Let $\mathcal{P}_0$ be the set of $(q+1)(q^2+1)\times(q^3+1)$ points in $Q^+(7,q)$, $\mathcal{P}_1$ the set of $(q+1)(q^2+1)(q^3+1)$ Latins, $\mathcal{P}_2$ the set of $(q+1)(q^2+1)\times(q^3+1)$ Greeks, and $\mathcal{L}$ the set of lines of $Q^+(7,q)$. We can define an incidence relation between two elements belonging to any couple of sets: a Greek and a Latin are incident if they intersect in a plane, and in all the other cases it is just symmetrized inclusion. There is always a *triality* (see for instance Section 2.4 in [17]): an incidence preserving map τ of order three that maps $\mathcal{P}_0$ to $\mathcal{P}_1$, $\mathcal{P}_1$ to $\mathcal{P}_2$, $\mathcal{P}_2$ to $\mathcal{P}_0$, and $\mathcal{L}$ to $\mathcal{L}$.

Theorem 22. *If S is a set of Latin generators of $Q^+(7,q)$ pairwise intersecting and $|S| = (q+1)(q^2+1)$, then S consists of all the Latins through one point, or of all the Latins meeting a fixed Greek in a plane.*

Proof. Let S be a set of pairwise intersecting Latins, and let τ be any triality. Then $S^{\tau^{-1}}$ is a set of mutually collinear points. It is well known that in every polar space the largest set of pairwise collinear points is the set of points in a generator (see for instance Lemma 9.2 in [8]). Hence, there is a generator π containing all the $(q+1)(q^2+1)$ points of $S^{\tau^{-1}}$.

If π is a Latin, then S itself consists of all Latins incident with the Greek π^τ, or hence of all Latins meeting π^τ in a plane. If π is a Greek, then S itself consists of all Latins through the point π^τ. $\square$

6. The case $Q(4n+2,q)$ for all q, and $W(4n+1,q)$ for even q

We will now treat the problem in the parabolic quadrics of odd rank. The bound from Theorem 9 is still reached by the point-pencil construction, but the properties of the characteristic vector are a bit weaker. We will make use of its embedding in the hyperbolic quadric. We have seen in Section 5 that $Q^+(7,q)$ is a special case, and therefore $Q(6,q)$ will also be exceptional. If q is even, parabolic and symplectic spaces with the same parameters are isomorphic (see for instance Chapter 11 in [16]), and hence we will also be able to obtain the classification in those spaces.

Theorem 23. *Let S be an EKR set of generators in $Q(4n+2,q)$, with $n \geqslant 1$ and $|S| = (q+1)\cdots(q^{2n}+1)$, then one of the following cases must occur:*

- *S is the set of all generators through a fixed point,*
- *S is the set of all generators of one system of an embedded $Q^+(4n+1,q)$,*
- *$n = 1$ and S consists of one fixed generator and all generators meeting it in a line.*

Proof. Consider the embedding of $Q(4n+2,q)$ in $Q^+(4n+3,q)$ as a non-singular hyperplane section. Every generator of $Q(4n+2,q)$ is contained in a unique generator of a fixed system of $Q^+(4n+3,q)$, so let $\bar{S}$ be the set of Latin generators in $Q^+(4n+3,q)$ through an element of S. The elements of $\bar{S}$ cannot be disjoint either and $|\bar{S}| = |S| = (q+1)\cdots(q^{2n}+1)$. Theorems 21 and 22 then yield that $\bar{S}$ is either the set of all Latins through a point p in $Q^+(4n+3,q)$, or $\bar{S}$ is the set of all Latins meeting a fixed Greek γ in a plane with $n=1$. Suppose that we are in the first case. If p is in the hyperplane H, then S is simply the set of all generators through p in $Q(4n+2,q)$. If p is not in H, then $p^\perp \cap H$ meets the parabolic quadric in a non-singular hyperbolic quadric $Q^+(4n+1,q)$. Then S is one system of generators of that hyperbolic quadric. Finally, in the second case we see that S consists of the plane $\gamma \cap H$ and the $(q^2+q+1)q$ planes of $Q(6,q)$ meeting that plane in a line. $\square$

We now consider $Q(4n+2,q)$ and $W(4n+1,q)$ with q even. It is well known (see for example Corollary 2 of Lemma 22.3.1 in [11]) that the projection from a point of $\mathsf{PG}(4n+2,q)$, called the *nucleus of the parabolic quadric*, on any non-singular hyperplane H gives an isomorphism between the polar spaces $Q(4n+2,q)$ and $W(4n+1,q)$. In particular, if H is a non-singular hyperplane intersecting the parabolic quadric in a hyperbolic quadric $Q^+(4n+1,q)$, then the generators of $Q(4n+2,q)$ in H will correspond with those of an embedded $Q^+(4n+1,q)$ in $W(4n+1,q)$.

Theorem 24. *Let S be an EKR set of generators in $W(4n+1,q)$, with q even, $n \geqslant 1$ and $|S| = (q+1)\cdots(q^{2n}+1)$. One of the following must hold:*

- *S is the set of all generators through a fixed point,*
- *S is the set of all generators of one system of a hyperbolic quadric $Q^+(4n+1,q)$ embedded in $W(4n+1,q)$,*
- *$n=1$ and S consists of one fixed generator and all generators meeting it in a line.*

Proof. This follows immediately from Theorem 23. □

7. The case $W(4n+1,q)$ for odd q

In the polar space $W(4n+1,q)$, Theorem 9 does not yield that the characteristic vector χ_S of an EKR set of generators of maximum size is in the span of the subspaces V_0 and V_1. This significantly weakens our control over this set. We also don't have an isomorphism between $Q(4n+2,q)$ and $W(4n+1,q)$ if q is odd, but the parameters and the eigenvalues of the association schemes on generators are still the same (see for instance Section 9.4 in [3]).

With respect to the disjointness relation, we can still prove a strong property.

Lemma 25. *Let S be an EKR set of generators in $Q(4n+2,q)$ or $W(4n+1,q)$ of size $(q+1)\cdots(q^{2n}+1)$. Every generator $\pi\notin S$ is disjoint from exactly $q^{n(2n+1)}$ elements of S.*

Proof. It follows from Theorem 9 that if $|S|=(q+1)\cdots(q^{2n}+1)$, then $\chi_S\in V_0\perp V_1\perp V_N$, with V_j as defined in Theorem 7. Here, $V_1\perp V_N$ is an eigenspace for the eigenvalue $\lambda=-q^{n(2n+1)}$ of the disjointness relation. We can now use Lemma 10 with respect to the disjointness relation. Let k denote the valency of the disjointness relation. We obtain that each generator not in S is disjoint from exactly $\frac{|S|}{|\Omega|}(k-\lambda)=\frac{|\Omega|}{|\Omega|(1-k/\lambda)}(k-\lambda)=-\lambda=q^{n(2n+1)}$ elements of S. □

For any generator π and any subset S of generators in $W(4n+1,q)$, we let $(v_{\pi,S})_i$ denote the number of generators in S meeting π in a subspace of codimension i. Note that $(v_{\pi,S})_0$ is 1 if $\pi\in S$. We now use algebraic techniques to obtain information on these vectors $v_{\pi,S}$.

We first consider the two known constructions of EKR sets of maximum size S in $W(4n+1,q)$ (q even) or $Q(4n+2,q)$, together with some element $\pi\in S$:

- Point-pencil construction: $v_{\pi,S}=v_1$ with $(v_1)_i=\begin{bmatrix}2n\\ i\end{bmatrix}_q q^{i(i+1)/2}$ (this follows from Theorem 7).
  For instance, in $W(9,q)$ or $Q(10,q)$:
$$v_1=\left(1,\begin{bmatrix}4\\1\end{bmatrix}_q q,\begin{bmatrix}4\\2\end{bmatrix}_q q^3,\begin{bmatrix}4\\3\end{bmatrix}_q q^6,\begin{bmatrix}4\\4\end{bmatrix}_q q^{10},0\right).$$
- All Latins of an embedded $Q^+(4n+1,q)$: $v_{\pi,S}=v_2$ with $(v_2)_i=\begin{bmatrix}2n+1\\ i\end{bmatrix}_q q^{i(i-1)/2}$ if i is even, 0 if i is odd (this follows from Theorem 7).
  For instance, in $W(9,q)$ (q even) or $Q(10,q)$:
$$v_2=\left(1,0,\begin{bmatrix}5\\2\end{bmatrix}_q q,0,\begin{bmatrix}5\\4\end{bmatrix}_q q^6,0\right).$$

Theorem 26. *Let S be an EKR set of generators in $Q(4n+2,q)$ or $W(4n+1,q)$ of size $|S|=(q+1)\cdots(q^{2n}+1)$. Then for every $\pi\in S$, there is a parameter τ such that $v_{\pi,S}=\tau v_1+(1-\tau)v_2$.*

Proof. We know that if $|S|$ reaches the bound from Theorem 9, then $\chi_S\in V_0\perp V_1\perp V_N$, with $N=2n+1$ in this case. Let Ω be the full set of generators, and let P be the matrix of eigenvalues of the association scheme. Let E_j denote the orthogonal projection onto the space of eigenvectors V_j. Lemma 2.5.1(iii) from [3] implies that $(v_{\pi,S}P^{-1})_j=(E_j\chi_S)_\pi$, where the latter denotes the entry of the vector $E_j\chi_S$, corresponding to the generator π. We note that E_0 is just projection onto the all-one vector, so the first entry $(v_{\pi,S}P^{-1})_0 = (E_0\chi_S)_\pi = (\frac{|S|}{|\Omega|}\chi_\Omega)_\pi = |S|/|\Omega|$. On the other hand, $E_0 + \cdots + E_N$ is the identity matrix, which means that the sum of all entries of $v_{\pi,S}P^{-1}$ is given by $((E_0 + \cdots + E_N)\chi_S)_\pi = (\chi_S)_\pi$, which is 1 as $\pi \in S$. Finally, as we know that $E_j\chi_S = 0$ unless $j = 0, 1$, or $j = N$ $(= 2n + 1)$, we have that $(v_{\pi,S}P^{-1})_j = 0$ if j is not 0, 1 or $2n + 1$. Hence, we know that $v_{\pi,S}$ is given by $(|S|/|\Omega|, t, 0, \ldots, 0, 1 - t - |S|/|\Omega|)P$ for some real number t.

As the parameters and the eigenvalues for generators in $W(4n + 1, q)$ and $Q(4n + 2, q)$ are the same, the vectors v_1 and v_2 that were given are both of that form. This means that $(v_1 - v_2)P^{-1}$ is of the form $(0, a, 0, \ldots, 0, -a)$ for some $a \neq 0$, and that $(v_{\pi,S} - v_2)P^{-1}$ is of the form $(0, b, 0, \ldots, 0, -b)$. Hence $(v_{\pi,S} - v_2)P^{-1}$ is a scalar multiple of $(v_1 - v_2)P^{-1}$. This means $v_{\pi,S} - v_2$ can be written as $\tau(v_1 - v_2)$ for some $\tau \in \mathbb{R}$. $\square$

Theorem 27. *Let S be an EKR set of generators in $W(4n + 1, q)$ of size $(q + 1)\cdots(q^{2n} + 1)$, with $n \geqslant 1$. Let π be any element of S with nucleus π_s.*

The number of elements of S meeting π in a subspace of codimension i, is given by

$$\frac{q^{2n-s} - 1}{q^{2n} - 1}\begin{bmatrix}2n\\ i\end{bmatrix}_q q^{i(i+1)/2}$$

if i is odd, and by

$$\begin{bmatrix}2n\\ i\end{bmatrix}_q q^{i(i+1)/2} + \frac{q^{2n-s}(q^s - 1)}{q^{2n} - 1}\begin{bmatrix}2n\\ i-1\end{bmatrix}_q q^{i(i-1)/2}$$

if i is even.

For every point of π_s, there are exactly $q^{n(2n+1)-s}$ elements of S meeting π in just that point.

Proof. We already know from Lemma 6 that if an element of S meets π in exactly one point p, then $p \in \pi_s$, and that $(v_{\pi,S})_1$, the number of elements of S meeting π in a dual line, is exactly $\begin{bmatrix}2n-s\\ 1\end{bmatrix}_q q$.

Theorem 26 also yields that $v_{\pi,S}$ can be written as $\tau v_1 + (1 - \tau)v_2$ for some parameter τ. In particular, $(v_{\pi,S})_1$ gives us the following equation: $\tau\begin{bmatrix}2n\\ 1\end{bmatrix}_q q + (1 - \tau)0 = \begin{bmatrix}2n-s\\ 1\end{bmatrix}_q q$, or hence: $\tau = \frac{q^{2n-s}-1}{q^{2n}-1}$. For an odd i in general, this means that $(v_{\pi,S})_i = \frac{q^{2n-s}-1}{q^{2n}-1}\begin{bmatrix}2n\\ i\end{bmatrix}_q q^{i(i+1)/2}$ since the corresponding entry of v_2 is zero. If i is even, then $(v_{\pi,S})_i = \tau\begin{bmatrix}2n\\ i\end{bmatrix}_q q^{i(i+1)/2} + (1 - \tau)\begin{bmatrix}2n+1\\ i\end{bmatrix}_q q^{i(i-1)/2}$. Using the identity $\begin{bmatrix}2n+1\\ i\end{bmatrix}_q = \begin{bmatrix}2n\\ i\end{bmatrix}_q q^i + \begin{bmatrix}2n\\ i-1\end{bmatrix}_q$, the latter can also be written as: $\begin{bmatrix}2n\\ i\end{bmatrix}_q q^{i(i+1)/2} + \frac{q^{2n-s}(q^s-1)}{q^{2n}-1}\begin{bmatrix}2n\\ i-1\end{bmatrix}_q q^{i(i-1)/2}$. In particular, we find that $(v_{\pi,S})_{2n}$, the number of elements of S meeting π in just a point, is exactly $\begin{bmatrix}s+1\\ 1\end{bmatrix}_q q^{n(2n+1)-s}$.

For any point p in π_s, let $f(p)$ denote the number of elements of S meeting π in just p. Consider any hyperplane π_{s-1} of π_s. We want to obtain $\sum_{p\in\pi_{s-1}} f(p)$. Consider any generator π' meeting π in a dual line but meeting π_s in just π_{s-1}. As π' is not in S, Lemma 25 implies that π' is disjoint from exactly $q^{n(2n+1)}$ elements of S, all necessarily meeting π in just a point in $\pi_s \backslash \pi_{s-1}$. Conversely, any generator of S that meets π in just a point of $\pi_s \backslash \pi_{s-1}$ must be disjoint from π' because of Lemma 4. Hence $\sum_{p\in\pi_{s-1}} f(p)$, the number of elements of S that meet π in just a point of π_{s-1}, is given by $\begin{bmatrix}s+1\\ 1\end{bmatrix}_q q^{n(2n+1)-s} - q^{n(2n+1)} = \begin{bmatrix}s\\ 1\end{bmatrix}_q q^{n(2n+1)-s}$. Now let $\mathcal{H}$ denote the set of all hyperplanes in π_s, and consider any point p_0 in π_s. We obtain:

$$\sum_{h\in\mathcal{H}, p_0\notin h}\left(\sum_{p\in h} f(p)\right) = q^{s-1}\left(\sum_{p\in\pi_s\backslash\{p_0\}} f(p)\right)$$

$$\Downarrow$$

$$q^s\left(\begin{bmatrix}s\\ 1\end{bmatrix}_q q^{n(2n+1)-s}\right) = q^{s-1}\left(\sum_{p\in\pi_s\backslash\{p_0\}} f(p)\right).$$

Hence

$$\begin{aligned} f(p_0) &= \sum_{p\in\pi_s} f(p) - \sum_{p\in\pi_s\setminus\{p_0\}} f(p) \\ &= \begin{bmatrix} s+1 \\ 1 \end{bmatrix}_q q^{n(2n+1)-s} - \begin{bmatrix} s \\ 1 \end{bmatrix}_q q^{n(2n+1)-s+1} \\ &= q^{n(2n+1)-s}. \qquad \square \end{aligned}$$

We will characterize the EKR sets of generators in $W(4n+1,q)$ by excluding values for the dimension of the nucleus of an element of S. For that purpose, we will also need parameters with respect to dual lines instead of generators. Therefore, we require more properties of the strata of the association scheme induced by the dual polar graph. Our techniques will be similar to those used for other association schemes in [5] and [7].

Consider a general polar space of rank N with parameters (q,q^e). We define the incidence matrix $C_{(N,N-1)}$ as follows. The columns are indexed by the generators and the rows by the dual lines. The entry $(C_{(N,N-1)})_{ij}$ is 1 if the corresponding dual line and generator are incident, and 0 otherwise.

We already know from Theorem 7 that the eigenvalues of the dual polar graph are given by $q^e\left[{N-j \atop 1}\right]_q - \left[{j \atop 1}\right]_q$, with $0 \leqslant j \leqslant N$. We denote each corresponding eigenspace by V_j. In particular, $-\left[{N \atop 1}\right]_q$ is an eigenvalue, obtained when $j=N$. The following lemma characterizes the eigenvectors of this last eigenvalue.

Lemma 28. *The eigenspace V_N of the eigenvalue $-\left[{N \atop 1}\right]_q$ of the dual polar graph is the kernel of the incidence matrix $C_{(N,N-1)}$.*

Proof. Let A denote the adjacency matrix of the dual polar graph. We consider the product $(C_{(N,N-1)})^t C_{(N,N-1)}$, which has the same kernel as $C_{(N,N-1)}$. As every generator contains $\left[{N \atop 1}\right]_q$ dual lines, while two distinct generators contain either a unique common dual line (if they are adjacent in the dual polar graph) or none (if they are not adjacent), we can write: $(C_{(N,N-1)})^t C_{(N,N-1)} = \left[{N \atop 1}\right]_q \mathrm{I} + A$. This means that

$$V_N = \ker\left(A - \left(-\begin{bmatrix} N \\ 1 \end{bmatrix}_q\right)\mathrm{I}\right) = \ker\big((C_{(N,N-1)})^t C_{(N,N-1)}\big) = \ker(C_{(N,N-1)}). \qquad \square$$

We now generalize our incidence matrix. For each $i \in \{0,\dots,N-1\}$, we let $C^i_{(N,N-1)}$ denote the $(0,1)$-matrix, the columns of which are indexed by the generators of the polar space, and the rows by the dual lines. An entry is 1 if the corresponding generator and dual line meet in a subspace of codimension i in the dual line, and zero otherwise. In particular: $C^0_{(N,N-1)} = C_{(N,N-1)}$. Note that the matrices $C^i_{(N,N-1)}$ add up to the all-one matrix.

Lemma 29. *Consider a polar space of rank N and an eigenspace V_j of the dual polar graph. There exist scalars λ^i_j such that $C^i_{(N,N-1)}v = \lambda^i_j C_{(N,N-1)}v$, $\forall v \in V_j$. In particular: $C^i_{(N,N-1)}v = 0$, $\forall v \in V_N$.*

Proof. We again let A denote the adjacency matrix of the dual polar graph. For any $i \in \{0,\dots,N-2\}$, consider the product $C^i_{(N,N-1)}A$. Consider a generator π and a dual line π_{N-1}. There can only be neighbors of π meeting π_{N-1} in a subspace of codimension i, if π and π_{N-1} meet in a subspace of codimension $i-1, i, i+1$. We denote the number of such neighbors in these cases by x_i, y_i and z_i, respectively (we let x_0 be zero), and these numbers only depend on i and not on π or π_{N-1}. For every $i \in \{0,\dots,(N-1)-1\}$, we have $z_i \neq 0$, and we can write $C^i_{(N,N-1)}A = x_i C^{i-1}_{(N,N-1)} + y_i C^i_{(N,N-1)} + z_i C^{i+1}_{(N,N-1)}$, with $C^{-1}_{(N,N-1)}$ defined as zero.

If $v \in V_j$, then v is an eigenvector of A for some eigenvalue λ of the dual polar graph, and hence:

$$C^{i+1}_{(N,N-1)}v = \left(C^i_{(N,N-1)}(\lambda v) - x_i C^{i-1}_{(N,N-1)}v - y_i C^i_{(N,N-1)}v\right)/z_i.$$

Induction on i now allows us to prove that there are scalars λ^i_j such that $C^i_{(N,N-1)}v = \lambda^i_j C_{(N,N-1)}v$, $\forall v \in V_j$ for every $i \in \{0,\dots,N-1\}$. If $j = N$, then it follows from Lemma 28 that $C^i_{(N,N-1)}v = 0$. □

The scalars λ^i_j from Lemma 29 will play a role that is somewhat similar to that of eigenvalues.

Now let S be a set of generators in a polar space, and let π_{N-1} be any dual line in the polar space. We define the vector $v_{\pi_{N-1},S}$ as follows: $(v_{\pi_{N-1},S})_i = |\{\pi \in S \mid \dim(\pi \cap \pi_{N-1}) = (N-1) - i\}|$. Note that the entries of $v_{\pi_{N-1},S}$ add up to $|S|$.

Theorem 30. *Let S be a set of generators in a polar space of rank N, and let λ^i_j be scalars as provided by Lemma* 29. *Suppose $\chi_S \in \langle V_j \mid j \in J\rangle$ with $J \subseteq \{0,\dots,N\}$. Then for every dual line π_{N-1}, the vector $v_{\pi_{N-1},S}$ can be written as a linear combination of the vectors of scalars $(\lambda^0_j,\dots,\lambda^{N-1}_j)$, with $j \in J\backslash\{N\}$.*

Proof. We first note that $(v_{\pi_{N-1},S})_i$ can be expressed algebraically as $(\chi_{\{\pi_{N-1}\}})^t C^i_{(N,N-1)}\chi_S$. Now suppose χ_S has the following orthogonal decomposition into eigenvectors of the dual polar graph: $\chi_S = \sum_{j\in J} v_j$ with $v_j \in V_j$. Lemma 29 yields that we can write for every $i \in \{0,\dots,N-1\}$:

$$C^i_{(N,N-1)}\chi_S = \sum_{j\in J} C^i_{(N,N-1)}v_j = \sum_{j\in J\backslash\{N\}} \lambda^i_j C_{(N,N-1)}v_j = \sum_{j\in J\backslash\{N\}} \lambda^i_j w_j,$$

with $w_j = C^0_{(N,N-1)}v_j = C_{(N,N-1)}v_j$. This allows us to write:

$$(v_{\pi_{N-1},S})_i = (\chi_{\{\pi_{N-1}\}})^t\left(\sum_{j\in J\backslash\{N\}} \lambda^i_j w_j\right) = \sum_{j\in J\backslash\{N\}} \lambda^i_j\big((\chi_{\{\pi_{N-1}\}})^t w_j\big). \qquad \square$$

Theorem 31. *Let S be an EKR set of generators of size $(q+1)\cdots(q^{2n}+1)$ in $W(4n+1,q)$. For each secant dual line π_{N-1}, the entry $(v_{\pi_{N-1},S})_i$ is given by $\left[{2n-1 \atop i}\right]_q (q+1)q^{i(i+3)/2}$.*

Proof. We know from Theorem 9 that $\chi_S \in V_0 \perp V_1 \perp V_N$, with $N = 2n+1$. Applying Lemma 30, this means that for any dual line π_{N-1} the vector $v_{\pi_{N-1},S}$ can be written as a linear combination of two certain vectors of scalars $(\lambda^0_0,\dots,\lambda^{2n}_0)$ and $(\lambda^0_1,\dots,\lambda^{2n}_1)$. Instead of explicitly calculating these scalars, we consider two particular vectors spanned by these two vectors. Let S_0 be the set of all generators through a fixed point p_0. This is certainly an EKR set of generators of the maximum size. Let v_1 denote the vector v_{π_{N-1},S_0} for some dual line through p_0, and let v_2 denote that vector for some dual line not through p_0 and not spanning a generator with p_0 either. In the first case, the dual line is secant, and in the second case it is external, so $(v_1)_0 = q+1$ and $(v_2)_0 = 0$. Hence the vectors v_1 and v_2 are certainly different. So for any EKR set of generators S of the maximum size, and for any dual line π_{N-1} we can write: $v_{\pi_{N-1},S} = \tau v_1 + \tau' v_2$ for some parameters τ and τ'. We know that the entries of $v_{\pi_{N-1},S}$, v_1 and v_2 must all add up to $|S|$, and hence $\tau' = 1-\tau$. If the dual line π_{N-1} is assumed to be secant, then $(v_{\pi_{N-1},S})_0 = q+1$, and hence $\tau = 1$ and so $v_{\pi_{N-1},S} = v_1$.

Now we explicitly calculate $(v_1)_i$. Considering the residual geometry of p_0, which is isomorphic to $W(4n-1,q)$, we can obtain this using Theorem 13. □

We will also need the following lemma on projective geometries.

Lemma 32. *(See [3], Lemma 9.3.2(ii).) If W is a subspace of dimension m in $V(n,q)$, then the number of subspaces of dimension $n-m$ meeting W trivially is $q^{m(n-m)}$.*

The following result is a first step towards our characterization of EKR sets of maximum size of generators in $W(4n+1,q)$, q odd. We prove that if S is an EKR set of maximum size and $\pi \in S$, then not all the neighbors of π are in S, except in the smallest case $W(5,q)$. In other words, if π_s is the nucleus of π, then $s \neq -1$, unless $n=1$.

Lemma 33. *Let S be an EKR set of generators of size $(q+1)\cdots(q^{2n}+1)$ in $W(4n+1,q)$, with $n \geqslant 2$. There is no element $\pi \in S$ with a trivial nucleus.*

Proof. Let $\pi \in S$ and let π_s be the nucleus of π. Suppose that $s=-1$. Theorem 27 yields that no element of S meets π in just a point, exactly $q^{n(2n-1)}\left[\begin{smallmatrix}2n+1\\1\end{smallmatrix}\right]_q$ meet π in a line, and exactly $q^3\left[\begin{smallmatrix}2n\\2\end{smallmatrix}\right]_q - q^{2n+1}$ meet π in a subspace of codimension two. We know from Theorem 13 that there are $q^{n(2n-1)}$ generators meeting π in a fixed line, and hence the set of lines A in π appearing as such an intersection has size at least $\left[\begin{smallmatrix}2n+1\\1\end{smallmatrix}\right]_q$. Now consider any subspace ρ with codimension 2 in π. There are exactly q^3 generators meeting π in just ρ. This implies that B, the set of all subspaces with codimension two in π arising from the intersection with an element of S, has cardinality at least $\left[\begin{smallmatrix}2n\\2\end{smallmatrix}\right]_q - q^{2n-2}$. Lemma 4 also yields that every element of A meets every element of B.

As there are only $\left[\begin{smallmatrix}2n\\1\end{smallmatrix}\right]_q$ lines through a point in $\mathrm{PG}(2n,q)$, no point can be on all lines in A. If a point p is not on a line $\ell \in A$, then it follows from Lemma 32 that there are precisely $\left[\begin{smallmatrix}2n\\2\end{smallmatrix}\right]_q - q^{4n-4}$ subspaces with codimension two in π through p that meet ℓ, which is less than $|B|$ as $n \geqslant 2$. Hence we can conclude that no point on π is on all elements of B. Since all elements of A must meet every element of B, there can be at most $\left[\begin{smallmatrix}2n-1\\1\end{smallmatrix}\right]_q$ elements of A through each point of π.

Now let μ be any hyperplane of π. Let X denote the subset of elements of B contained in μ. We know from Theorem 31 that exactly $\left[\begin{smallmatrix}2n-1\\1\end{smallmatrix}\right]_q(q+1)q^2$ elements of S meet μ in a hyperplane of μ. These elements of S either meet π in some element of X, or meet π in some hyperplane, different from μ. Hence, we have:

$$\begin{bmatrix}2n-1\\1\end{bmatrix}_q (q+1)q^2 \leqslant |X|q^3 + \left(\begin{bmatrix}2n+1\\1\end{bmatrix}_q - 1\right)q,$$

yielding: $|X| \geqslant \left[\begin{smallmatrix}2n-2\\1\end{smallmatrix}\right]_q$.

Next, consider two distinct lines ℓ_1 and ℓ_2 in A and a hyperplane μ of π, meeting these lines in the points p_1 and p_2 respectively. We know from the above that μ contains at least $\left[\begin{smallmatrix}2n-2\\1\end{smallmatrix}\right]_q$ elements of B. These elements must hence contain the points p_1 and p_2, and hence they are precisely the $\left[\begin{smallmatrix}2n-2\\1\end{smallmatrix}\right]_q$ hyperplanes of μ through the line $\langle p_1, p_2\rangle$. Let ρ be a fixed hyperplane of μ not through the line $\langle p_1, p_2\rangle$. As $\rho \notin B$, there is certainly a generator π' with $\rho = \pi \cap \pi'$ and $\pi' \notin S$. Lemma 25 implies that there are exactly $q^{n(2n+1)}$ elements of S that are disjoint from π'. These elements must meet π in a line, disjoint from ρ. Hence we obtain at least $q^{n(2n+1)}/q^{n(2n-1)} = q^{2n}$ lines of A in π that meet μ in just a point. As these lines must meet all elements of B, and hence certainly all hyperplanes of μ through $\langle p_1, p_2\rangle$, they must meet μ in a point of that line, not on ρ. But through each of those q points on $\langle p_1, p_2\rangle$, there are at most $\left[\begin{smallmatrix}2n-1\\1\end{smallmatrix}\right]_q$ elements of A. This yields $q^{2n} \leqslant q\left[\begin{smallmatrix}2n-1\\1\end{smallmatrix}\right]_q$, which is clearly a contradiction. □

Even though $W(4n+1,q)$ and $Q(4n+2,q)$ are isomorphic if and only if q is even, the parameters of the corresponding association scheme are the same, regardless of the parity of q. However, we want to prove that the construction using an embedded $Q^+(4n+1,q)$ in $Q(4n+2,q)$, which appeared in Theorem 23 has no analog for $W(4n+1,q)$ if q is odd. We will need the following fundamental result on the associated classical generalized quadrangle $W(3,q)$ (see 1.3.6, 3.2.1 and 3.3.1 in [13]).

Theorem 34. *If three lines are pairwise skew in $W(3,q)$, then the number of lines of $W(3,q)$ meeting all three is 0 or 2 if q is odd, and 1 or $q+1$ if q is even.*

This can be used to prove the following lemma.

Lemma 35. *Suppose S is an EKR set of generators in $W(4n+1,q)$ of size $|S|=(q+1)\cdots(q^{2n}+1)$, with $n \geqslant 1$ and q odd. Then there are at least two elements of S intersecting in a space of codimension one.*

Proof. Suppose that the elements of S never intersect in a subspace of codimension one. Lemma 6 yields that in this case each element of S is its own nucleus. Hence Theorem 27 implies that for each $\pi \in S$, the vector $v_{\pi,S}$ is v_2. In particular, the elements of S cannot meet in a subspace with odd codimension. Let π be any element in S. We know that exactly $(v_2)_2 = \left[{2n+1 \atop 2}\right]_q q$ elements of S meet π in exactly a subspace of codimension two. As there are only $\left[{2n+1 \atop 2}\right]_q$ subspaces with codimension two in π, there must certainly be a subspace μ of codimension two in π, such that at least $q \geqslant 3$ elements of S meet π in just μ. Let π_1 and π_2 be two such elements. Note that they cannot meet in more than just μ, because their intersection cannot be a dual line as they are both in S. Hence the three generators π, π_1 and π_2 correspond with three mutually skew lines ℓ, ℓ_1 and ℓ_2, respectively, in the residual geometry $W(3,q)$ of μ.

Now let S_0 denote the subset of generators in S, meeting π in just a point, not in μ. Such a generator must meet both π_1 and π_2 in a subspace of even codimension and skew to μ, thus in just a point not in μ. For every $\pi_0 \in S_0$, the generator $\langle \mu, \mu^\perp \cap \pi_0\rangle$ through μ corresponds with a line meeting ℓ, ℓ_1 and ℓ_2 in $W(3,q)$. As q is odd, there are at most two such lines, by Theorem 34. Hence, there are at most two possibilities for the generator $\langle \mu, \mu^\perp \cap \pi_0\rangle$. As π_0 is skew to μ, it must meet $\langle \mu, \mu^\perp \cap \pi_0\rangle$ in a line. There are precisely q^{4n-2} lines in a $\mathsf{PG}(2n,q)$, skew to a given subspace with codimension two (Lemma 32). Finally, we consider the generators of S_0 that can go through that line. Since the elements of S pairwise meet in a subspace with even codimension, all these generators must meet in at least a plane, and hence in the residue of that line, which is isomorphic to $W(4n-3,q)$, we obtain a set of generators, all meeting in at least a point. This implies that we can apply the upper bound from Theorem 9 for EKR sets of generators in $W(4n-3,q)$, and see that there are at most $\prod_{i=1}^{2n-2}(q^i+1)$ elements of S_0 through each such line. Hence, we see that $|S_0| \leqslant 2q^{4n-2}\prod_{i=1}^{2n-2}(q^i+1)$.

Let us now explicitly calculate $|S_0|$. Theorem 27 yields that through each point of π, not in μ, there are precisely $q^{n(2n+1)-2n}$ elements of S that meet π in just that point. Hence, $|S_0| = (\left[{2n+1 \atop 1}\right]_q - \left[{2n-1 \atop 1}\right]_q)q^{2n^2-n} = (q^{2n}+q^{2n-1})q^{2n^2-n}$, and thus we obtain the inequality: $(q^{2n}+q^{2n-1})q^{2n^2-n} \leqslant 2q^{4n-2}\prod_{i=1}^{2n-2}(q^i+1)$, which is equivalent to $\frac{q+1}{2}q^{2n^2-3n+1} \leqslant \prod_{i=1}^{2n-2}(q^i+1)$. As $2 \leqslant \frac{q+1}{2}$, this contradicts Lemma 12(1). $\square$

We now prove a result on the nuclei of two neighbors of an EKR set of generators in the dual polar graph.

Lemma 36. *Let S be an EKR set of generators in $W(4n+1,q)$ of size $(q+1)\cdots(q^{2n}+1)$, with $n \geqslant 1$. If π_1 and π_2 are neighbors and both are elements of S with a non-trivial nucleus, then they have the same nucleus.*

Proof. Let π_1 and π_2 have nuclei π_s and π_t with projective dimensions $s \geqslant 0$ and $t \geqslant 0$, respectively. It follows from the definition of nuclei that π_s and π_t are both in $\pi_1 \cap \pi_2$. If π_s is not contained in π_t, then $|\pi_s \backslash \pi_t| \geqslant q^s$. We know from Theorem 27 that for every $p \in \pi_s \backslash \pi_t$ there are $q^{n(2n+1)-s}$ elements of S meeting π_1 in just p, and by Lemma 6 these elements cannot meet π_2 in just p. As $\pi_1 \cap \pi_2$ is a hyperplane in π_2, we see that these elements meet π_2 in exactly a line. We also know from Theorem 27 that there are exactly $\frac{q^{2n-t}-1}{q-1}q^{n(2n-1)}$ elements of S meeting π_2 in a line, and hence:

$$q^s q^{n(2n+1)-s} \leqslant \frac{q^{2n-t}-1}{q-1}q^{n(2n-1)},$$

which yields: $q^{2n} \leqslant \frac{q^{2n-t}-1}{q-1}$, and that is a contradiction as $t \geqslant 0$. Hence $\pi_s \subseteq \pi_t$, and in a completely similar way we can show that $\pi_t \subseteq \pi_s$. $\square$

Theorem 37. *Let S be an EKR set of generators in $W(4n+1,q)$, q odd, of size $(q+1)\cdots(q^{2n}+1)$. Suppose that there is a $\pi \in S$ with a point $p := \pi_0$ as nucleus. Then S is the set of generators through p.*

Proof. By Theorem 27, there are $q^{n(2n+1)}$ elements of S meeting π exactly in p. Suppose that there exists a generator π' through p not in S, then by Lemma 25, there are $q^{n(2n+1)}$ elements of S disjoint from π' that hence cannot be through p. So there are $q^{n(2n+1)}$ elements in S not through p and $q^{n(2n+1)}$ through p, hence $|S| \geqslant 2q^{n(2n+1)} > (q+1)\cdots(q^{2n}+1)$ for $q \geqslant 3$ by Lemma 12, which is a contradiction. □

Lemma 38. *Let S be an EKR set of generators in $W(4n+1,q)$, q odd, of size $(q+1)\cdots(q^{2n}+1)$, with $n \geqslant 1$. If $\pi \in S$ has nucleus π_s with projective dimension s, then $s \in \{-1,0,1,2n\}$, and if $s=1$, then for every dual line μ with $\pi_s \subseteq \mu \subset \pi$, an element of S meets π in just a point if and only if it meets μ in just a point.*

Proof. Suppose $s < 2n$. Then $\pi_s \neq \pi$, and consider any dual line μ with $\pi_s \subseteq \mu \subset \pi$. Lemma 6 implies that μ is secant. We know from Theorem 27 that $\left[{s+1 \atop 1}\right]_q q^{n(2n+1)-s}$ elements of S meet π in exactly a point of π_s, and hence must meet μ in exactly a point as well. Theorem 31 also yields that exactly $(q+1)q^{(2n-1)(n+1)}$ elements of S meet μ in just a point. Hence we obtain:

$$\frac{q^{s+1}-1}{q-1}q^{n(2n+1)-s} \leqslant (q+1)q^{(2n-1)(n+1)},$$

which is equivalent with $s \leqslant 1$. If $s=1$, then the two sizes are equal, and hence generators in S meeting μ in just a point must be precisely those meeting π in just a point. □

We can now finally complete the classification of EKR sets of generators in $W(4n+1,q)$ with q odd.

Theorem 39. *Let S be an EKR set of generators in $W(4n+1,q)$ of size $(q+1)\cdots(q^{2n}+1)$, with q odd and $n \geqslant 2$. Then S is the set of all generators through some point.*

Proof. Let $\pi \in S$ and let π_s be the nucleus of π. By Lemmas 33 and 38, $s \in \{0,1,2n\}$. If $s=0$, then by Theorem 37, S consists of all the generators through a point. Hence, from now on we can assume that $s \in \{1,2n\}$ for every $\pi \in S$. First suppose some $\pi \in S$ has nucleus π_s with $s=1$. Now consider any dual line μ with $\pi_s \subseteq \mu \subset \pi$. Theorem 27 yields that there is certainly an element $\pi' \in S$ meeting π in just a point of π_s. Consider the generator $\pi'' = \langle \mu, \mu^\perp \cap \pi' \rangle$, which meets π' in a line. As π'' is through the secant dual line μ, it is also in S. Since by Lemma 33 π'' has a non-trivial nucleus, Lemma 36 yields that π'' also has π_s as nucleus. But this contradicts Lemma 38, as we now have the generator $\pi' \in S$ meeting μ in just a point, while it meets $\pi'' \in S$ in a line.

Hence the dimension of the nucleus is $2n$ for every element of S, which contradicts Lemma 35 as q is odd. □

Just as for $Q(6,q)$, there is an extra construction for EKR sets of generators of the maximum size for $W(5,q)$, and hence this case must be treated separately.

Theorem 40. *Suppose S is an EKR set of $(q+1)(q^2+1)$ planes in $W(5,q)$, q odd. Then the elements of S are either all generators through a fixed point, or S consists of the plane π and all the planes meeting it in a line.*

Proof. By Lemma 35, there are at least two generators π and π_1 intersecting in a subspace of codimension one, hence the nucleus π_s of π is at most a line. Lemma 6 yields that if an element of S meets π in a point p, then $p \in \pi_s$, and that the elements of S meeting π in a line, are precisely those meeting π in a line through π_s. Obviously, $s \in \{-1,0,1\}$. If $s=-1$, then all $q(q^2+q+1) = |S|-1$ planes meeting π in a line, are in S, and hence S consists of these planes and π itself. If $s=0$, then π_s is a point contained in all elements of S, and hence we are done again. Finally, suppose $s=1$. Let a and b be distinct points on the line π_s. Theorem 27 yields that through both points, there are precisely q^2 elements of S, meeting π in just that point. Suppose $\pi_a, \pi_b \in S$ with $\pi_a \cap \pi = \{a\}$ and $\pi_b \cap \pi = \{b\}$. As π_a and π_b cannot be disjoint, by Lemma 4 they must meet in precisely one point c, necessarily outside of π. The points a, b and c span a plane π' in S by Lemma 6, as $\pi_s = \langle a, b\rangle \subseteq \pi'$. Since $\pi' \cap \pi = \langle a, b\rangle$, $\pi' \cap \pi_a = \langle a, c\rangle$ and $\pi' \cap \pi_b = \langle b, c\rangle$, π' has trivial nucleus and hence S consists of π' and all the planes meeting π' in a line. □

8. The case $H(4n+1, q^2)$

In $H(4n+1, q^2)$, the set of generators Ω is of size $(q+1)(q^3+1)\cdots(q^{4n+1}+1)$. The number of generators through one point is $|\Omega|/(q^{4n+1}+1)$, but the eigenvalue bound from Theorem 9 is $|\Omega|/(q^{2n+1}+1)$ in this case, which is much larger.

In $H(5, q^2)$, there are $(q+1)(q^3+1)(q^5+1)$ generators, and $(q+1)(q^3+1)$ generators through one point. The upper bound arising from eigenvalue techniques in this case is $(q+1)(q^5+1)$. The following example shows that the point-pencil construction is in this case indeed not of maximum size. Let π be a plane in $H(5, q^2)$. Let S consist of π, together with all planes meeting π in a line. Now $|S| = q(q^4+q^2+1)+1$, and in particular: $(q+1)(q^3+1) < |S| < (q+1)(q^5+1)$.

It is possible that there is no simple answer for $H(4n+1, q^2)$ in general. However, we can already exclude the possibility of reaching the upper bound from Theorem 9.

Delsarte developed a very general theory of regular semilattices in [6], which gives a special meaning to the eigenspaces of the dual polar graph. We now give a short proof for a very specific case.

Theorem 41. *Let Ω be the set of generators of a polar space of rank N with parameters (q, q^e), and consider the orthogonal decomposition of $\mathbb{R}\Omega$ into eigenspaces of the dual polar graph*:

$$\mathbb{R}\Omega = V_0 \perp V_1 \perp \cdots \perp V_N,$$

with the same ordering of the eigenspaces V_j as in Theorem 7. If the characteristic vector of a set of generators S satisfies $\chi_S \in V_0 \perp V_N$, then every dual line is in exactly $\frac{|S|}{|\Omega|}(q^e+1)$ elements of S.

Proof. We know from Lemma 28 that V_N is the kernel of the incidence matrix $C_{(N,N-1)}$. If Ω_{N-1} denotes the full set of dual lines, and χ_S decomposes as $\frac{|S|}{|\Omega|}\chi_\Omega + v_N$ with $v_N \in V_N$, then:

$$C_{(N,N-1)}\chi_S = C_{(N,N-1)}\left(\frac{|S|}{|\Omega|}\chi_\Omega + v_N\right) = \frac{|S|}{|\Omega|}\left(q^e+1\right)\chi_{\Omega_{N-1}}.$$

This means that every dual line is in exactly $\frac{|S|}{|\Omega|}(q^e+1)$ elements of S. □

Theorem 42. *Let Ω be the set of generators in $H(2N-1, q^2)$ with $N \geqslant 3$ odd. Let S be a set of generators, all meeting in at least a point. Then $|S| < |\Omega|/(q^N+1)$.*

Proof. We already know from Theorem 9 that $|S| \leqslant |\Omega|/(q^N+1)$, with equality if and only if $\chi_S \in V_0 \perp V_N$. Suppose equality holds. In that case, every dual line would be incident with exactly $\frac{q+1}{q^N+1}$ elements of S, because of Theorem 41. As $N \geqslant 3$, this yields a contradiction as this number is not an integer. □

Nevertheless, we can determine the maximum size of an EKR set of planes in $H(5, q^2)$. We first state a general theorem on generalized quadrangles (see for instance 1.2.4 in [13]).

Theorem 43. *Let a, b and c be three mutually non-collinear points in a generalized quadrangle with parameters (s, s^2). The number of points collinear with a, b and c is exactly $s+1$.*

Dualizing, this yields the following result for the generalized quadrangle $H(3, q^2)$ with parameters (q^2, q).

Corollary 44. *If ℓ_1, ℓ_2 and ℓ_3 are three mutually skew lines in $H(3,q^2)$, then there are precisely $q+1$ lines of $H(3,q^2)$ meeting all of them.*

Theorem 45. *Let S be an EKR set of planes in $H(5,q^2)$. Then $|S| \leqslant q^5+q^3+q+1$, and this bound can only be reached if S consists of a plane π and all planes meeting π in a line.*

Proof. Assume that S is a maximal EKR set of generators.

Suppose that $\pi \in S$ meets some element of S in a line. Lemma 6 yields that the nucleus π_s of π has projective dimension $s < 2$.

If $s = -1$, then Lemma 6 yields that S contains all $q(q^4+q^2+1)+1$ planes that are equal to or meeting π in a line, while there are no planes in S meeting π in just a point.

If $s = 0$, then all elements of S must be through the point π_s, and hence $|S| \leqslant (q+1)(q^3+1)$, which is less than q^5+q^3+q+1.

Now suppose $s = 1$. If no element of S meets π in a point, then all other elements of S meet π in the line π_1 and hence $|S| \leqslant q+1$. Similarly, if all elements of S either contain π_1 or meet π in the same point p, then again $|S| \leqslant (q+1)(q^3+1)$. Finally, suppose that π' and π'' are elements of S, meeting π in different points p' and p'' of the nucleus of π, respectively. Lemma 4 yields that π' and π'' meet in just a point, say p. Consider the plane $\langle p', p'', p\rangle$, which is in S since it is through the nucleus of π. Its nucleus is the empty set since π, π' and π'' intersect it in three non-concurrent lines. Lemma 6 again yields that S consists of $\langle p', p'', p\rangle$ and all the planes intersecting it in a line.

In the remainder of this proof, we can suppose that all elements of S meet in just a point. We will also assume that $|S|$ is at least the desired bound q^5+q^3+q+1, and prove that this leads to a contradiction. Suppose $\pi \in S$ and let p be a point on π. In the residual geometry of p, isomorphic to $H(3,q^2)$, the elements of S through p correspond with different mutually skew lines $\ell_1,\dots,\ell_t$. Lemma 3 yields that there are $(q^2+1)(q^3+1)$ points in $H(3,q^2)$ and hence $t \leqslant q^3+1$. Hence there are at least q^5+q elements of S not through p. The elements of S not through p are projected onto lines in the residual geometry $H(3,q^2)$. In each plane of $H(5,q^2)$ through p, there are q^4 lines skew to p. Since two elements of S cannot meet in a line, at most q^4 elements of S can be projected on the same line of $H(3,q^2)$, so we have at least $q+1$ lines, namely m_j, $j = 1,\dots,q+1$. Since the elements of S pairwise intersect only in one point, an element of S through p cannot be projected onto any of the m_j, so $\ell_i \neq m_j$, $\forall i, j$. As the elements of S cannot be pairwise disjoint, the lines l_i and m_j must intersect in the residual geometry $H(3,q^2)$. If $m_j \cap m_k$ is a point, then a line ℓ_i in $H(3,q^2)$ intersecting both of them must pass through their intersection point. As the lines ℓ_i are pairwise skew, t is thus 1 in this case and so through p there can be at most one element of S. If all these lines m_j are pairwise skew, then there are at most $q+1$ lines meeting all of them because of Corollary 44, so there are in this case at most $q+1$ elements of S through p. Hence through every point of π there are at most $q+1$ elements of S, but $|S| \geqslant q^5+q^3+q+1$ implies that through every point of π there are exactly $q+1$ elements of S and $|S| = q^5+q^3+q+1$. So now we can consider a point $p \in \pi \in S$ and two other elements of $\pi_1, \pi_2 \in S$ through p, such that π, π_1 and π_2 correspond with three skew lines ℓ, ℓ_1 and ℓ_2, respectively, in the residual geometry of p. In this geometry, only $q+1$ points of ℓ, corresponding to the plane π, are on a line meeting ℓ, ℓ_1 and ℓ_2, and hence only the points on the corresponding $q+1$ lines through p in π can be on a plane meeting the planes π, π_1 and π_2. This contradicts the fact that there are $q+1$ elements of S through each point in the plane π. □

9. Summary

We will now summarize our results for polar spaces of rank at least three. In Table 1, we will write p.-p. to denote the point-pencil construction of an EKR set of generators consisting of all generators through a fixed point. We also let Ω denote the full set of generators and for the hyperbolic space of even rank we will focus only on one system of generators, namely the *Latins*. Finally, *base* will refer to the construction in a polar space of rank three, consisting of one base plane and all those meeting it in a line.

Table 1

Polar space	Maximum size	Classification
$Q^-(2n+1,q)$	$(q^2+1)\cdots(q^n+1)$	p.-p., Theorem 15
$Q(4n,q)$	$(q+1)\cdots(q^{2n-1}+1)$	p.-p., Theorem 15
$Q(4n+2,q)$, $n\geqslant 2$	$(q+1)\cdots(q^{2n}+1)$	p.-p., Latins $Q^+(4n+1,q)$, Theorem 23
$Q(6,q)$	$(q+1)(q^2+1)$	p.-p., Latins $Q^+(5,q)$, base, Theorem 23
$Q^+(4n+1,q)$	$(q+1)\cdots(q^{2n}+1)$	one system, Theorem 16
Latins $Q^+(4n+3,q)$, $n\geqslant 2$	$(q+1)\cdots(q^{2n}+1)$	p.-p., Theorem 21
Latins $Q^+(7,q)$	$(q+1)(q^2+1)$	p.-p., meeting Greek in plane, Theorem 22
$W(4n+1,q)$, $n\geqslant 2$, q odd	$(q+1)\cdots(q^{2n}+1)$	p.-p., Theorem 39
$W(4n+1,q)$, $n\geqslant 2$, q even	$(q+1)\cdots(q^{2n}+1)$	p.-p., Latins $Q^+(4n+1,q)$, Theorem 24
$W(5,q)$, q odd	$(q+1)(q^2+1)$	p.-p., base, Theorem 40
$W(5,q)$, q even	$(q+1)(q^2+1)$	p.-p., base, Latins $Q^+(5,q)$, Theorem 24
$W(4n+3,q)$	$(q+1)\cdots(q^{2n+1}+1)$	p.-p., Theorem 15
$H(2n,q^2)$	$(q^3+1)(q^5+1)\cdots(q^{2n-1}+1)$	p.-p., Theorem 15
$H(4n+3,q^2)$	$(q+1)(q^3+1)\cdots(q^{4n+1}+1)$	p.-p., Theorem 15
$H(4n+1,q^2)$, $n\geqslant 2$	$<\lvert\Omega\rvert/(q^{2n+1}+1)$	?, Theorem 42
$H(5,q^2)$	$q(q^4+q^2+1)+1$	base, Theorem 45

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