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Free and Residually Artinian Regular Rings

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INTRODUCTION

In a (von Neumann) regular ring, the existence of quasi-inverses (i.e., solutions x to equations $axa = a$) controls most aspects of the structure of the ring, in particular its abundance of idempotents. A specific choice of a quasi-inverse for each ring element defines a unary operation q satisfying the identity $aq(a)a = a$. Equipping regular rings with such unary operations allows the class of regular rings to be viewed as a variety in the sense of universal algebra. While it is a trivial observation that free regular rings—i.e., free objects in the category of regular rings with specified quasi-inversions—exist, their structure has remained unknown. (In contrast, free commutative regular rings have been studied; see, e.g., [2].) Our goal here is to investigate the structure of such free regular rings in general, the main result being that they are residually artinian, i.e., subdirect products of artinian rings. In particular, any free regular ring R is directly finite (that is, the relation $xy = 1$ in R implies $yx = 1$) as are all the matrix rings over R. Since every regular ring is a homomorphic image of a free regular ring, we see in particular that factor rings of directly finite regular rings can exhibit arbitrarily pathological behavior. Further study of the structure of free regular rings leads to a determination of their centers, and shows that their socles are zero. Similar results are obtained for free regular algebras over a field K; in fact, any such algebra is a prime residually finite-

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dimensional K-algebra with zero socle and center K. Finally, parallel results are derived for free unit-regular rings and free unit-regular algebras. (A unit-regular ring is a regular ring in which every element has an invertible quasi-inverse.)

Unless explicitly stated otherwise, rings, algebras, subrings, and subalgebras in this paper are assumed to be unital. The term "regular ring" will be used exclusively in von Neumann's sense (introduced in the 1930s, two decades before the competing usage referring to "homologically regular" rings was established). We refer the reader to [7] for the general theory of regular rings.

1. Various Free Regular Rings

Several different objects deserve the title free regular ring, depending on the category in which they live. We begin by introducing these rings and pointing out some connections among them. In order to deal with various cases simultaneously, we fix a commutative ring Λ *throughout the paper* and work with regular Λ-algebras.

Let $\mathcal{R}eg_\Lambda$ be the category whose objects are pairs (R, q), where R is a Λ-algebra and $q: R \to R$ is a function (called a *quasi-inversion*) such that $rq(r)r = r$ for all $r \in R$. A morphism between two objects (R, q) and (R', q') in $\mathcal{R}eg_\Lambda$ is any Λ-algebra homomorphism $\phi: R \to R'$ such that $q'\phi = \phi q$. When we consider arbitrary regular rings we work in $\mathcal{R}eg_{\mathbf{Z}}$ and abbreviate this category as $\mathcal{R}eg$. Note that we put no restriction on the quasi-inversion q for an object (R, q) in $\mathcal{R}eg_\Lambda$; in particular, we do not assume that $q(0) = 0$. This is done partly to avoid unnecessary special cases, and partly so that the category of unit-regular Λ-algebras will be a subcategory of $\mathcal{R}eg_\Lambda$. Readers who prefer a different point of view will find it straightforward to transfer our results to the full subcategory $\mathcal{R}eg_\Lambda^0$ of $\mathcal{R}eg_\Lambda$ whose objects are those pairs (R, q) for which $q(0) = 0$. Indeed, if (R, q) is a free object on a set X in $\mathcal{R}eg_\Lambda$, there is a corresponding object (R, q^0) in $\mathcal{R}eg_\Lambda^0$ obtained by setting $q^0(0) = 0$ and $q^0(r) = q(r)$ for all nonzero $r \in R$; it is easily checked that (R, q^0) is free on $X \cup \{q(0)\}$ in $\mathcal{R}eg_\Lambda^0$. Similarly, a free object on a nonempty set in $\mathcal{R}eg_\Lambda^0$ corresponds to a free object in $\mathcal{R}eg_\Lambda$ on a set with one less element.

There is a useful alternate way to view $\mathcal{R}eg$, namely as a category of regular algebras over a commutative *regular* ring. More precisely, let $\mathcal{Z}$ be the subring of the direct product $\prod \mathbf{Z}/p\mathbf{Z}$, where p ranges through the prime positive integers, consisting of all sequences that are eventually of the form $(a + p\mathbf{Z})(b + p\mathbf{Z})^{-1}$, where a, b are constants from $\mathbf{Z}$. Then $\mathcal{Z}$ is a regular ring with the property that any regular ring is a $\mathcal{Z}$-algebra, cf. [6; 5, Sect. 125], and this in a unique way since $\mathcal{Z}$ is an epimorph of $\mathbf{Z}$, cf. [16, Corollaries to Theorem 1]. For the same reason, every ring homomorphism between regular rings is automatically a $\mathscr{Z}$-algebra homomorphism, and consequently the forgetful functor from $\mathscr{R}eg_{\mathscr{Z}}$ to $\mathscr{R}eg$ is an isomorphism of categories.

The category $\mathscr{R}eg_\Lambda$ is also a variety in the sense of universal algebra. Namely, $\mathscr{R}eg_\Lambda$ is a variety in the category of algebras with the following operations: two binary operations corresponding to addition and multiplication, two unary operations corresponding to additive inverse and quasi-inversion, two nullary operations corresponding to zero and one, and an additional set of unary operations corresponding to multiplication by λ for each $\lambda \in \Lambda$. Now for any set X, the existence of a free object on X in $\mathscr{R}eg_\Lambda$ follows from the existence of free objects in varieties [9, Theorem 2.10].

Suppose that an object (R, q) in $\mathscr{R}eg_\Lambda$, together with a map $j: X \to R$, is free on X. Since there exist regular Λ-algebras into which X can be embedded (e.g., rational function fields over residue fields of Λ), we see that j is injective. Thus we shall identify X with $j(X)$ and assume j to be the inclusion map.

Finally, we call a regular Λ-algebra R *free* if R is isomorphic to the underlying Λ-algebra of some free object in $\mathscr{R}eg_\Lambda$; that is, if there exists a quasi-inversion q on R such that (R, q) is a free object on some set X in $\mathscr{R}eg_\Lambda$. By a *free regular ring* we mean a free regular $\mathbf{Z}$-algebra. The underlying Λ-algebra of a free object on a set X in $\mathscr{R}eg_\Lambda$ will be denoted by $F(\mathscr{R}eg_\Lambda, X)$.

We begin with some very easy observations.

LEMMA 1.1. *Let (R, q) be a free object on a set X in $\mathscr{R}eg_\Lambda$.*

(a) *Any Λ-subalgebra of R which contains X and is closed under q must equal R.*

(b) *If Λ and X are countable, then R is countable.*

Proof. (a) If R' is a subalgebra of R such that $X \subseteq R'$ and $q(R') \subseteq R'$, then $(R', q') \in \mathscr{R}eg_\Lambda$, where $q' = q|_{R'}$. The identity map on X extends to a morphism $\phi: (R, q) \to (R', q')$, and the composition of ϕ with the inclusion morphism $\psi: (R', q') \to (R, q)$ extends the identity map on X, whence $\psi\phi = 1_R$. Thus $R' = R$.

(b) Define subalgebras $R_0 \subseteq R_1 \subseteq \cdots$ in R, where R_0 is generated by X and each R_{i+1} is generated by $R_i \cup q(R_i)$. The union of the R_i forms a countable subalgebra R' of R which contains X and is closed under q. Therefore $R' = R$, and hence R is countable. ∎

The second conclusion of Lemma 1.1 cannot be generalized to say that if X is countable then R is a countably generated Λ-algebra, as we can see from the next proposition.

PROPOSITION 1.2. *If R is a free regular algebra over a field K, then $\dim_K(R) \geqslant \operatorname{card}(K)$. In fact, R contains a subalgebra which is a free K-algebra on a set of the same cardinality as $K \setminus \{0\}$.*

Proof. There is a quasi-inversion q on R such that (R, q) is a free object in $\mathcal{R}eg_K$. Choose a set Y with a bijection $k \mapsto y_k$ from K onto Y. Let S be a free K-algebra on Y, set $T = \operatorname{End}_K(S)$, and embed S in T by left multiplication. Set $R' = K \times T$, and choose a quasi-inversion q' on R' such that $q'(0) = (1, 0)$ and $q'(k, 0) = (k^{-1}, y_k)$ for all nonzero $k \in K$. By freeness, there exists a morphism $\phi: (R, q) \to (R', q')$ in $\mathcal{R}eg_K$. Then $\phi q(0) = q'\phi(0) = (1, 0)$ and so $\phi q(kq(0)) = q'(k, 0) = (k^{-1}, y_k)$ for all nonzero $k \in K$. Thus the composition of ϕ with the projection $R' \to T$ yields a K-algebra homomorphism $\psi: R \to T$ such that $\psi q(kq(0)) = y_k$ for all nonzero $k \in K$. Therefore the K-subalgebra of R generated by $\{q(kq(0)) \mid k \in K \setminus \{0\}\}$ is isomorphic (via ψ) to the free algebra S. ∎

When we work with free regular algebras, the following reduction allows us to concentrate on free regular algebras on finite sets.

PROPOSITION 1.3. *Let (R, q) be a free object on an infinite set X in $\mathcal{R}eg_\Lambda$, and let $\mathbf{A}$ denote the collection of nonempty finite subsets of X. For $A \in \mathbf{A}$, let (R_A, q_A) be a free object on A in $\mathcal{R}eg_\Lambda$, and let $\alpha_A: (R_A, q_A) \to (R, q)$ be the unique morphism extending the inclusion map $A \to X$. For $A \subseteq B$ in $\mathbf{A}$, let $\beta_{BA}: (R_A, q_A) \to (R_B, q_B)$ be the unique morphism extending the inclusion map $A \to B$.*

(a) *The object (R, q) together with maps α_A is a direct limit in $\mathcal{R}eg_\Lambda$ for the direct system $\{(R_A, q_A), \beta_{BA}\}$.*

(b) *For each $A \in \mathbf{A}$, there is a morphism $\gamma_A: (R, q) \to (R_A, q_A)$ such that $\gamma_A \alpha_A$ is the identity morphism on (R_A, q_A). In particular, α_A is an embedding of R_A into R.*

(c) *The algebra R equals the union of the subalgebras $\alpha_A(R_A)$.*

(d) *The morphism $\gamma: (R, q) \to \prod_{A \in \mathbf{A}} (R_A, q_A)$ induced by the morphisms γ_A provides an embedding $R \to \prod_{A \in \mathbf{A}} R_A$.*

Proof. (a, b) These are immediate consequences of freeness.

(c) The union of the subalgebras $\alpha_A(R_A)$ forms a subalgebra R' of R which contains X. Since $q\alpha_A = \alpha_A q_A$ for all A, we see that R' is closed under q, and so $R' = R$ by Lemma 1.1.

(d) Since each $\gamma_A \alpha_A$ is the identity on R_A, we see that $\alpha_A(R_A) \cap \ker(\gamma) = 0$. Now $\ker(\gamma) = 0$ because of (c). ∎

Another useful reduction is to embed a free regular algebra into a direct product of free regular algebras over fields. In proving this we make use of the following form of the Brown–McCoy Lemma [1, Lemma 1]: If I is a regular ideal in a ring R, and $a, b, \in R$ with $a - aba \in I$, then there exists $b' \in R$ such that $ab'a = a$ and $b' - b \in I$. (In fact, if $z \in I$ is a quasi-inverse for $a - aba$, we may choose $b' = z - zab - baz + bazab + b$.)

LEMMA 1.4. *Let ϕ: $R' \to R$ be a surjective Λ-algebra homomorphism with R' regular and R free regular. Then there exists a Λ-algebra homomorphism ψ: $R \to R'$ such that $\phi\psi = 1_R$.*

Proof. We may assume that ϕ is the quotient map $R' \to R'/I$ for some ideal I of R'. Let q be a quasi-inversion on R such that (R, q) is free on a set X in $\mathcal{R}eg_\Lambda$. If $s \in R'$, choose $t \in R'$ such that $\phi(t) = q\phi(s)$. Then $sts - s \in I$, and by the Brown–McCoy Lemma there exists an element $q'(s)$ in R' such that $sq'(s)s = s$ and $q'(s) - t \in I$; that is, $\phi q'(s) = q\phi(s)$. This defines a map q': $R' \to R'$ such that (R', q') is an object in $\mathcal{R}eg_\Lambda$ and ϕ is a morphism from (R', q') to (R, q). Fix a map α: $X \to R'$ such that $\phi\alpha = 1_X$. By the freeness of (R, q), there is a unique morphism ψ from (R, q) to (R', q') that agrees with α on X. The morphism $\phi\psi$ is the identity on X and so must be the identity on (R, q). Therefore $\phi\psi = 1_R$. ∎

PROPOSITION 1.5. *Any free regular Λ-algebra can be embedded (as a Λ-algebra) in a direct product of free regular algebras over quotient fields of prime factors of Λ.*

Proof. Let R be a free regular Λ-algebra, and let $\{R_\alpha\}$ be the set of indecomposable factor rings of R. Then R is isomorphic to a subdirect product of the R_α. Since each R_α is regular, its center must be a field, whence the kernel of the natural map $\Lambda \to R_\alpha$ is a prime ideal P_α of Λ and R_α is an algebra over the quotient field K_α of Λ/P_α. Now $R_\alpha \cong S_\alpha/I_\alpha$, where S_α is a free regular K_α-algebra and I_α is an ideal in S_α.

Since R embeds in the direct product of the Λ-algebras S_α/I_α, there is a subalgebra R' of $\prod_\alpha S_\alpha$ containing $\prod_\alpha I_\alpha$ such that $R'/(\prod_\alpha I_\alpha) \cong R$. Noting that $\prod_\alpha I_\alpha$ is a regular ideal and R is a regular ring, we see that R' is regular. Therefore by Lemma 1.4, R can be embedded in R' and so in $\prod_\alpha S_\alpha$. ∎

We conclude this section by introducing another class of free regular rings, which take into account that a quasi-inverse s for an element r in a regular ring can always be chosen to satisfy $srs = s$ as well as $rsr = r$. We consider the full subcategory $\mathcal{R}eg'_\Lambda$ of $\mathcal{R}eg_\Lambda$ whose objects are those objects (R, q) from $\mathcal{R}eg_\Lambda$ satisfying the additional condition $q(r)rq(r) = q(r)$ for all $r \in R$. As before, these categories are varieties, and hence free objects exist in them. Continuing our previous notation, we write $F(\mathscr{R}eg'_\Lambda, X)$ for the underlying Λ-algebra of a free object on a set X in $\mathscr{R}eg'_\Lambda$.

Free objects in $\mathscr{R}eg'_\Lambda$ are closely related to those in $\mathscr{R}eg_\Lambda$, as the following proposition shows.

PROPOSITION 1.6. *Let (R, q) be a free object on a set X in $\mathscr{R}eg_\Lambda$, and let $q': R \to R$ be the map given by the rule $q'(r) = q(r)rq(r)$. Then (R, q') is an object in $\mathscr{R}eg'_\Lambda$. If $\phi: (R, q) \to (R, q')$ is the unique morphism in $\mathscr{R}eg_\Lambda$ extending the identity map on X, then $(\phi(R), q')$ is a free object on X in $\mathscr{R}eg'_\Lambda$. Moreover, ϕ restricts to the identity map on $\phi(R)$, and so $\phi(R)$ is a retraction of R.*

Proof. It is a trivial computation to see that (R, q') and $(\phi(R), q')$ are objects in $\mathscr{R}eg'_\Lambda$.

For any $(S, g) \in \mathscr{R}eg'_\Lambda$ and any map $\alpha: X \to S$, there is a unique morphism $\beta: (R, q) \to (S, g)$ in $\mathscr{R}eg_\Lambda$ such that β agrees with α on X. If γ is the restriction of β to $\phi(R)$, then

$$\gamma q'(r) = \beta(q(r)rq(r)) = g\beta(r)\beta(r)\,g\beta(r) = g\beta(r) = g\gamma(r)$$

for all $r \in \phi(R)$. Thus γ is a morphism from $(\phi(R), q')$ to (S, g) in $\mathscr{R}eg'_\Lambda$, extending α. Moreover, if $\gamma': (\phi(R), q') \to (S, g)$ is another morphism in $\mathscr{R}eg'_\Lambda$ extending α, then $\gamma'\phi$ and $\gamma\phi$ coincide on X, so $\gamma'\phi = \gamma\phi$ and hence $\gamma' = \gamma$. Therefore $(\phi(R), q')$ is free on X in $\mathscr{R}eg'_\Lambda$.

Finally, if δ is the restriction of ϕ to $\phi(R)$, we note as above that δ is an endomorphism of $(\phi(R), q')$ in $\mathscr{R}eg'_\Lambda$. Since δ extends the identity map on X, we conclude from the freeness of $(\phi(R), q')$ that δ equals the identity map on $\phi(R)$. ∎

Using Proposition 1.6, the main results of the following section imply correponding results for the free objects in $\mathscr{R}eg'_\Lambda$. We leave to the reader the formulation and proofs of analogs for the results of Section 4.

2. Free Regular Rings Are Residually Artinian

Adapting standard terminology from group theory, we say that a ring R is *residually artinian* if R has a collection $\{I_\alpha\}$ of ideals such that $\bigcap I_\alpha = 0$ and each factor ring R/I_α is artinian. In other words, R is residually artinian if and only if R is isomorphic to a subdirect product of artinian rings. Of course if R is regular, this occurs if and only if R isomorphic to a subdirect product of *simple* artinian rings. We prove that free regular algebras are residually artinian, and that they can be embedded in direct products of finite matrix rings over fields.

It follows that any regular ring is a homomorphic image of a residually artinian regular ring. In particular, although residually artinian regular rings are directly finite, they are far from being unit-regular. The first example of this phenomenon was constructed by Chuang and Lee [3], who built a specific residually artinian regular ring with a particular countable, regular, non-unit-regular homomorphic image. This construction was introduced (independently) for other purposes by Tyukavkin [13]. We use the construction to build residually artinian regular algebras over countable fields with countable free regular algebras as homomorphic images, and then we reach our desired conclusion using the results of the previous section.

To work with algebras over a field K, we need to stay within the algebra of those $\omega \times \omega$ matrices over K which are both row-finite and column-finite. We write $B(K)$ for this algebra.

PROPOSITION 2.1. *Any countable dimensional algebra T over a field K can be embedded (as a K-algebra) in $B(K)$.*

Proof. Since this is clear if T is finite dimensional, assume that $\dim_K(T)$ is countably infinite.

Let T_0 be a K-vector space copy of T, and let T be viewed as a subalgebra of $\mathrm{End}_K(T_0)$ via the left regular representation. Choose a basis $\{t_1, t_2, ...\}$ for T_0, and for all $i, j \in \mathbf{N}$ let f_{ij} be the linear transformation in $\mathrm{End}_K(T_0)$ such that $f_{ij}(t_j) = t_i$ and $f_{ij}(t_k) = 0$ for $k \neq j$. If T' is the subalgebra of $\mathrm{End}_K(T_0)$ generated by T and the f_{ij}, then T' is a prime countable dimensional K-algebra and $\dim_K(f_{11} T' f_{11}) = 1$. Thus there is no loss of generality in assuming that T is prime and that T contains an idempotent e with $\dim_K(eTe) = 1$.

Consider now the K-vector spaces $V = eT$ and $W = Te$. The function $\langle\ ,\ \rangle\colon V \times W \to F$ given by $\langle v, w \rangle e = vw$ defines a nondegenerate inner product (bilinear pairing) between V and W. Let $\mathscr{L}_W(V)$ be the ring of all K-linear transformations on V with an adjoint (with respect to $\langle\ ,\ \rangle$) in $\mathrm{End}_K(W)$. Then it follows from an argument of Kaplansky [8, Sect. IV.9, second proof that (1) implies (3)] that T can be embedded in $\mathscr{L}_W(V)$. Since V and W are countable dimensional over K we can find K-bases $\{v_1, v_2, ...\}$ and $\{w_1, w_2, ...\}$ for V and W, respectively, such that $\langle v_i, w_j \rangle = \delta_{ij}$ for all i, j [8, Sect. IV.15, Proposition 1]. By using matrix representations for the elements of $\mathscr{L}_W(V)$ with respect to the basis $\{v_i\}$ we obtain a K-algebra isomorphism $\mathscr{L}_W(V) \cong B(K)$ [8, Sect. IV.10, Example 2]. The result follows. ∎

Now we shall use a construction extracted from arguments due to Tyukavkin [13]. Let K be a field and consider the subset S of $B(K) \times \prod_{n \geq 1} M_n(K)$ such that $(a, (x_n)) \in S$ if and only if for each integer

$k \geqslant 1$ there exists an integer $m \geqslant k$ with $(x_n)_{ij} = a_{ij}$ whenever $n \geqslant m$ and $i, j \leqslant n$ with either $i \leqslant k$ or $j \geqslant k$. Define A to be the subset of $\prod_{n \geqslant 1} M_n(K)$ which consists of those sequences (x_n) for which there exists $a \in B(K)$ such that $(a, (x_n)) \in S$. From the definition of S it follows that such an a is uniquely determined by (x_n). It is not difficult to check that A is a K-subalgebra of $\prod_{n \geqslant 1} M_n(K)$, and that the map $\varphi: A \to B(K)$ defined by $(x_n) \mapsto a$ is a surjective K-algebra homomorphism. If we look at the kernel I of φ then we see by simple inspection (by using that $M_n(K)$ is unit-regular for all $n \geqslant 1$) that $I + K$ is a unit-regular ring and $I(\prod_{n \geqslant 1} M_n(K))I \subseteq I$. If we combine these observations and Proposition 2.1, then we have:

PROPOSITION 2.2. *Let T be a countable dimensional algebra over a field K. Then there exists a K-subalgebra A of $\prod_{n \geqslant 1} M_n(K)$ with an ideal I containing $\bigoplus_{n \geqslant 1} M_n(K)$ such that $I + K$ is unit-regular, $I(\prod_{n \geqslant 1} M_n(K))I \subseteq I$, and $A/I \cong T$.* ∎

COROLLARY 2.3. *If K is a countable field, any free regular K-algebra on a countable set can be embedded in $\prod_{n \geqslant 1} M_n(K)$. Moreover, every free regular K-algebra can be embedded in a direct product of finite matrix algebras over K.*

Proof. The first conclusion follows from Lemma 1.1, Proposition 2.2, and Lemma 1.4, and the second follows using Proposition 1.3. ∎

In particular, Corollary 2.3 shows that any free regular K-algebra R is *residually finite dimensional* over K, that is, R is isomorphic to a subdirect product of finite dimensional K-algebras. We can extend this result to free regular algebras over arbitrary fields as follows.

THEOREM 2.4. *Any free regular algebra R over a field K is residually finite dimensional over K.*

Proof. We may assume that K is uncountable. Choose a quasi-inversion q such that (R, q) is free on some set X in $\mathcal{R}eg_K$. Let $\{K_\alpha\}$ be the collection of countable subfields of K, and for each α let (R_α, q_α) be a free object on X in $\mathcal{R}eg_{K_\alpha}$.

For any α, the identity map on X extends uniquely to a morphism ϕ_α from (R_α, q_α) to (R, q) in $\mathcal{R}eg_{K_\alpha}$. By Corollary 2.3, R_α can be embedded in a direct product of finite matrix algebras over K_α, and this in turn can be embedded in a direct product S_α of finite matrix algebras over K. Hence, there exist a quasi-inversion g_α on S_α and an injective morphism $\psi_\alpha: (R_\alpha, q_\alpha) \to (S_\alpha, g_\alpha)$ in $\mathcal{R}eg_{K_\alpha}$. Since S_α is a K-algebra, there is a morphism $\rho_\alpha: (R, q) \to (S_\alpha, g_\alpha)$ in $\mathcal{R}eg_K$ which agrees with ψ_α on X. Then

$\rho_\alpha\phi_\alpha$ is a morphism from (R_α, q_α) to (S_α, g_α) in $\mathscr{R}eg_{K_\alpha}$ which agrees with ψ_α on X, and so $\rho_\alpha\phi_\alpha = \psi_\alpha$. Consequently, $\ker(\rho_\alpha) \cap \phi_\alpha(R_\alpha) = 0$.

The maps ρ_α induce a K-algebra homomorphism $\rho\colon R \to \prod_\alpha S_\alpha$ such that

$$\ker(\rho) \cap \phi_\alpha(R_\alpha) = 0$$

for all α. We will be done if $\ker(\rho) = 0$. Hence, it is enough to show that R is the union of the subrings $\phi_\alpha(R_\alpha)$.

Whenever $K_\alpha \subseteq K_\beta$, the identity map on X extends to a morphism $\sigma_{\beta\alpha}$ from (R_α, q_α) to (R_β, q_β) in $\mathscr{R}eg_{K_\alpha}$. Then $\phi_\beta\sigma_{\beta\alpha}$ is a morphism from (R_α, q_α) to (R, q) in $\mathscr{R}eg_{K_\alpha}$ which is the identity on X, and so $\phi_\beta\sigma_{\beta\alpha} = \phi_\alpha$. Thus $\phi_\alpha(R_\alpha) \subseteq \phi_\beta(R_\beta)$. Therefore the subrings $\phi_\alpha(R_\alpha)$ form a directed family under inclusion, and hence their union is a subring R' of R. Obviously R' contains X and is closed under q, and since K is the union of the K_α, we see that R' is a K-subalgebra of R. Therefore $R' = R$ (by Lemma 1.1), as desired. ∎

Now our main result follows by combining Theorem 2.4 with Proposition 1.5. The final conclusion relies on the well known fact that any regular subring of an artinian ring is artinian. (An artinian ring contains no infinite sequences of nonzero pairwise orthogonal idempotents.)

THEOREM 2.5. *Any free regular Λ-algebra R can be embedded (as a Λ-algebra) in a direct product of finite matrix algebras over quotient fields of prime factors of Λ. Thus R is residually artinian.* ∎

In particular, Theorem 2.5 shows that every free regular Λ-algebra is directly finite.

COROLLARY 2.6. *Any regular Λ-algebra is a homomorphic image of a directly finite regular Λ-algebra which is a subdirect product of simple artinian Λ-algebras.* ∎

COROLLARY 2.7. *Let R be a regular Λ-algebra. Then there exist a collection $\{R_\alpha\}$ of finite matrix algebras over quotient fields of prime factors of Λ and a regular Λ-subalgebra A of $\Pi_\alpha R_\alpha$ with an ideal I containing $\bigoplus_\alpha R_\alpha$ such that $A/I \cong R$.*

Proof. First write $R \cong S/J$, where S is a free regular Λ-algebra. By Theorem 2.5 we may assume that S is a Λ-subalgebra of a direct product $\prod_\beta S_\beta$, where each S_β is a finite matrix algebra over the quotient field of a prime factor of Λ. Let $S_{\beta,n} = S_\beta$ for all β and all $n \in \mathbf{N}$, embed $\prod_\beta S_\beta$ in $\prod_{\beta,n} S_{\beta,n}$ via the map $(s_\beta) \mapsto (s_{\beta,n} = s_\beta)$, and identify S with its image in $\prod_{\beta,n} S_{\beta,n}$ under this embedding. If $J' = \bigoplus_{\beta,n} S_{\beta,n}$, then J' is a regular ideal of $\prod_{\beta,n} S_{\beta,n}$ and $S \cap J' = 0$. Finally, set $A = S + J'$. Since J' is a regular ideal of A and $A/J' \cong S$ is regular, A is a regular Λ-algebra. Moreover, $A/(J+J') \cong S/J \cong R$. ∎

Corollaries 2.6 and 2.7 show that directly finite regular rings can be almost arbitrarily bad, and they immediately yield further counterexamples of the Chuang–Lee type. To settle one further conjecture, choose a simple regular ring which is not directly finite, and apply Corollary 2.6. This yields a directly finite regular ring R with a maximal ideal M such that R/M is not directly finite, and provides a negative answer to the first part of [7, Problem 2]. (We have been informed by D. V. Tyukavkin that he also has found a counterexample for this problem.)

Recall that a ring R satisfies the *n-stable range condition* (for a given positive integer n) if whenever $a_1, ..., a_{n+1} \in R$ with $a_1 R + \cdots + a_{n+1} R = R$, there exist elements $b_1, ..., b_n \in R$ such that

$$(a_1 + a_{n+1} b_1) R + \cdots + (a_n + a_{n+1} b_n) R = R.$$

If n is the least positive integer such that R satisfies the n-stable range condition, then R is said to have *stable range n*. A ring with stable range 1 is directly finite [15, Theorem 2.6]. Hence R has stable range 1 provided that for any $a, b \in R$ satisfying $aR + bR = R$, the coset $a + bR$ contains a unit. It is well-known that a regular ring satisfies stable range 1 if and only if it is unit-regular [7, Proposition 4.12]. Some examples of regular rings having stable range 2 were given in [10, 11]. In the latter it was asked [11, Question 4] whether there are examples of directly finite regular rings which do not satisfy any stable range condition. Corollaries 2.6 and 2.7 tell us how to construct many such rings by using the fact that there are regular rings which do not satisfy any stable range condition (see below). The question of whether or not there exist regular rings with stable range n for $3 \leqslant n < \infty$ seems to be open.

The easiest way to obtain a (regular) ring not satisfying any stable range condition is to choose a nonzero (regular) ring R such that $R \cong R^2$ as right R-modules. Given a positive integer n, observe that $R \cong R^{n+1}$; hence, there exist pairwise orthogonal idempotents $e_1, ..., e_{n+1} \in R$ such that $e_1 + \cdots + e_{n+1} = 1$ and each $e_i R \cong R$. For each i, there exist elements $a_i \in e_i R$ and $c_i \in Re_i$ such that $a_i c_i = e_i$ and $c_i a_i = 1$. Then $a_1 R + \cdots + a_{n+1} R = R$. If R satisfied the n-stable range condition, there would exist elements $b_1, ..., b_n$ and $x_1, ..., x_n$ in R such that

$$(a_1 + a_{n+1} b_1) x_1 + \cdots + (a_n + a_{n+1} b_n) x_n = 1.$$

Multiplying this equation on the left by c_i, we obtain $x_i = c_i$ for $i = 1, ..., n$. But then $x_i e_{n+1} = 0$ for $i = 1, ..., n$, which is impossible because $e_{n+1} \neq 0$. Therefore R does not satisfy any stable range condition.

Now suppose that R is a nonzero regular Λ-algebra such that $R \cong R^2$

(for instance, any infinite dimensional full linear ring over a quotient field of a prime factor of Λ). Then there exist $a, b, c, d \in R$ such that

$$(a \quad b)\begin{pmatrix} c \\ d \end{pmatrix} = (1) \quad \text{and} \quad \begin{pmatrix} c \\ d \end{pmatrix}(a \quad b) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now consider any quasi-inversion $q: R \to R$ such that $q(0) = a$ and $q(a) = c + d$, while $q(c+d) = b$ and $q(b) = d$. Any free regular Λ-algebra $S = F(\mathscr{R}eg_\Lambda, X)$ has a homomorphism ϕ to R whose image, being closed under q, contains a, b, c, d. Then $\phi(S) \cong \phi(S)^2$ and so $\phi(S)$ satisfies no stable range condition, whence S does not either.

3. Free Unit-Regular Rings Are Residually Artinian

We introduce free unit-regular rings and show, in parallel with the previous section, that they too are residually artinian.

We use $U(R)$ to denote the group of units of a ring R. Now form the full subcategory $\mathscr{U}\mathscr{R}eg_\Lambda$ of $\mathscr{R}eg_\Lambda$ whose objects are those objects (R, u) from $\mathscr{R}eg_\Lambda$ satisfying $u(R) \subseteq U(R)$. Once again, we obtain a variety in a category of algebras (here the algebras need an additional unary operation corresponding to multiplicative inverses of quasi-inverses). Thus free objects exist in $\mathscr{U}\mathscr{R}eg_\Lambda$. We write the underlying Λ-algebras of these free objects in the form $F(\mathscr{U}\mathscr{R}eg_\Lambda, X)$, and we refer to them as *free unit-regular Λ-algebras*.

Free unit-regular Λ-algebras can be obtained as quotients of free regular Λ-algebras in the following manner. Let (R, q) be a free object on a set X in $\mathscr{R}eg_\Lambda$, let

$$\{\phi_\alpha : (R, q) \to (R_\alpha, u_\alpha) \mid \alpha \in \Omega\}$$

be the class of all morphisms in $\mathscr{R}eg_\Lambda$ such that $(R_\alpha, u_\alpha) \in \mathscr{U}\mathscr{R}eg_\Lambda$, and write I for $\bigcap_{\alpha \in \Omega} \ker(\phi_\alpha)$. If $r, r' \in R$ and $r - r' \in I$, then $q(r) - q(r') \in I$ because $\phi_\alpha q = u_\alpha \phi$ for all α. Thus q induces a map $u: R/I \to R/I$ such that $u(r + I) = q(r) + I$ for all $r \in R$. Choose a subset $A \subseteq \Omega$ such that $I = \bigcap_{\alpha \in A} \ker(\phi_\alpha)$. Then the maps ϕ_α for $\alpha \in A$ induce a Λ-algebra embedding $\phi: R/I \to \prod_{\alpha \in A} R_\alpha$. If $r \in R$, then $\phi u(r + I) = (u_\alpha \phi(r))$ is a unit in $\prod_\alpha R_\alpha$, whence $u(r + I)$ is a non-zero-divisor in R/I and thus a unit, because R/I is regular. Therefore $(R/I, u) \in \mathscr{U}\mathscr{R}eg_\Lambda$. It is clear that $(R/I, u)$ is free on X in $\mathscr{U}\mathscr{R}eg_\Lambda$, with respect to the obvious map $X \to R \to R/I$.

LEMMA 3.1. *Let (R, u) be a free object on a set X in $\mathscr{U}\mathscr{R}eg_\Lambda$.*

(a) *Any A-subalgebra of R which contains X and is closed under u must equal R.*

(b) *If A and X are countable, then R is countable.*

Proof. (a) If R' is a subalgebra of R such that $X \subseteq R'$ and $u(R') \subseteq R'$, then $(R', u') \in \mathscr{R}eg_A$, where $u' = u|_{R'}$. For $r \in R'$, note that $u'(r)$ is a non-zero-divisor (because it is a unit in R), whence $u'(r)$ is a unit in R' (because R' is regular). Thus $(R', u') \in \mathscr{U}\mathscr{R}eg_A$, and the proof is completed as in Lemma 1.1.

(b) As Lemma 1.1. ∎

The unit-regular analog of Proposition 1.3 can be obtained with the same proof:

PROPOSITION 3.2. *Let (R, u) be a free object on an infinite set X in $\mathscr{U}\mathscr{R}eg_A$, and let $\mathbf{A}$ denote the collection of nonempty finite subsets of X. For $A \in \mathbf{A}$, let (R_A, u_A) be a free object on A in $\mathscr{U}\mathscr{R}eg_A$, and let $\alpha_A: (R_A, u_A) \to (R, u)$ be the unique morphism extending the inclusion map $A \to X$. For $A \subseteq B$ in $\mathbf{A}$, let $\beta_{BA}: (R_A, u_A) \to (R_B, u_B)$ be the unique morphism extending the inclusion map $A \to B$.*

(a) *The object (R, u) together with the maps α_A is a direct limit in $\mathscr{U}\mathscr{R}eg_A$ for the direct system $\{(R_A, u_A), \beta_{BA}\}$.*

(b) *For each $A \in \mathbf{A}$, there is a morphism $\gamma_A: (R, u) \to (R_A, u_A)$ such that $\gamma_A \alpha_A$ is the identity morphism on (R_A, u_A). In particular, α_A is an embedding of R_A into R.*

(c) *The algebra R equals the union of the subalgebras $\alpha_A(R_A)$.*

(d) *The morphism $\gamma: (R, u) \to \prod_{A \in \mathbf{A}} (R_A, u_A)$ induced by the morphisms γ_A provides an embedding $R \to \prod_{A \in \mathbf{A}} R_A$.* ∎

To prove a unit-regular analog of Lemma 1.4, we need the following unit-regular version of the Brown–McCoy Lemma.

LEMMA 3.3. *Let R be a unit-regular ring and I an ideal in R. If $x, y \in R$ are such that $xyx - x \in I$ and $y + I$ is a unit of R/I, then there exists a unit u of R such that $xux = x$ and $u - y \in I$.*

Proof. By the Brown–McCoy Lemma, there exists $y' \in R$ such that $xy'x = x$ and $y' - y \in I$. Hence, after replacing y by y', we may assume that $xyx = x$. Since $y + I$ is a unit in R/I, there exist $a \in R$ and $b \in I$ such that $ya + b = 1$. Hence, $y(a + xb) + (1 - yx)b = 1$. Since R satisfies stable range 1, there exists $c \in R$ such that the element $u = y + (1 - yx)bc$ is a unit. Then $u - y \in I$ because $b \in I$, and $xux = xyx = x$. ∎

LEMMA 3.4. *Let $\phi: R' \to R$ be a surjective Λ-algebra homomorphism with R' unit-regular and R free unit-regular. Then there exists a Λ-algebra homomorphism $\psi: R \to R'$ such that $\phi\psi = 1_R$.*

Proof. As Lemma 1.4, using Lemma 3.3 in place of the Brown–McCoy Lemma. ∎

PROPOSITION 3.5. *Any free unit-regular Λ-algebra can be embedded (as a Λ-algebra) in a direct product of free unit-regular algebras over quotient fields of prime factors of Λ.*

Proof. Let R be a free unit-regular Λ-algebra. As in the proof of Proposition 1.5, we can find free unit-regular algebras S_α over quotient fields of prime factors of Λ, ideals I_α in S_α, and a regular subalgebra R' of $\prod_\alpha S_\alpha$ containing $\prod_\alpha I_\alpha$ such that $R'/(\prod_\alpha I_\alpha) \cong R$. Since R and $\prod_\alpha S_\alpha$ are unit-regular, R' is unit-regular by [7, Lemma 4.16]. Therefore by Lemma 3.4, R can be embedded in R' and so in $\prod_\alpha S_\alpha$. ∎

In order to prove the unit-regular version of Theorem 2.4 we require some results which will be established in a more general setting.

Recall that two idempotents e and f in a ring R are said to be *equivalent*, written $e \sim_R f$, if there exist $x \in eRf$ and $y \in fRe$ such that $e = xy$ and $f = yx$. If there exists a unit u of R such that $u^{-1}eu = f$, then e and f are said to be *conjugate*; it is well known that this happens if and only if $e \sim_R f$ and $1-e \sim_R 1-f$. (We recall one implication. If $e \sim_R f$ with elements x, y as above, and if $1-e \sim_R 1-f$ with corresponding elements x', y', then $u = x + x'$ is a unit, with inverse $y + y'$, such that $u^{-1}eu = f$.)

LEMMA 3.6. *Let S be a ring such that any two equivalent idempotents of S are conjugate. If A is a subring of S and I is a regular ideal of A such that $ISI \subseteq I$, then the natural map $U(A) \to U(A/I)$ is surjective.*

Proof. Suppose $a, b \in A$ are such that $ab - 1$, $ba - 1 \in I$. We must prove that $a + I$ contains a unit of A. Since I is regular there exists an idempotent $e \in I$ such that $(1-ab)A = eA$. Thus $1 - e = (1-e)ab$. Then $b(1-e)a = 1-f$, where $f \in I$ is an idempotent. Set $a_1 = (1-e)a(1-f)$ and $b_1 = (1-f)b(1-e)$. It is easily seen that $a_1b_1 = 1-e$ and $b_1a_1 = 1-f$, and so $1-e \sim_A 1-f$. By hypothesis, $1-e$ and $1-f$ are conjugate in S and hence $e \sim_S f$. Since $e, f \in I$, it follows from our assumptions that $eSf + fSe \subseteq I$, and therefore $e \sim_A f$. Hence, there exists a unit $u \in A$ such that $u^{-1}eu = f$. Define $a' = a_1 + eu$ and $b' = b_1 + fu^{-1}$. Clearly $a' \in a + I$ and $a'b' = b'a' = 1$. ∎

We also require the following result which is a special case of a theorem Vaserstein [14, Theorem 4].

An ideal I of a ring R satisfies *stable range* 1 provided that for any

$a \in 1+I$ and $b \in I$ satisfying $aR+bR=R$, the coset $a+bI$ contains a unit of R. It is easily seen that if R satisfies stable range 1, then every ideal of R satisfies stable range 1.

LEMMA 3.7 (Vaserstein). *Let I be an ideal of a ring R. Then R has stable range 1 if and only if R/I and I have stable range 1 and the natural map $U(R) \to U(R/I)$ is surjective.* ∎

PROPOSITION 3.8. *Let T be a countable dimensional algebra over a field K, satisfying stable range 1. Then there exists a K-subalgebra A of $\prod_{n \geq 1} M_n(K)$ with a regular ideal I containing $\bigoplus_{n \geq 1} M_n(K)$ such that A has stable range 1 and $A/I \cong T$.*

Proof. By Proposition 2.2, there exists a K-subalgebra A of $\prod_{n \geq 1} M_n(K)$ with an ideal I containing $\bigoplus_{n \geq 1} M_n(K)$ such that $I+K$ is unit-regular, $I(\prod_{n \geq 1} M_n(K))I \subseteq I$, and $A/I \cong T$. It is clear that I is regular and has stable range 1, because it is an ideal in the unit-regular ring $I+K$. By Lemma 3.6, the natural map $U(A) \to U(A/I)$ is surjective, and so we conclude from Lemma 3.7 that A has stable range 1. ∎

COROLLARY 3.9. *Let T be a countable dimensional unit-regular algebra over a field K. Then there exists a unit-regular K-subalgebra A of $\prod_{n \geq 1} M_n(K)$ with an ideal I containing $\bigoplus_{n \geq 1} M_n(K)$ such that $A/I \cong T$.* ∎

COROLLARY 3.10. *If K is a countable field, any free unit-regular K-algebra on a countable set can be embedded in $\prod_{n \geq 1} M_n(K)$. Moreover, every free unit-regular K-algebra can be embedded in a direct product of finite matrix algebras over K.*

Proof. Lemma 3.1, Corollary 3.9, Lemma 3.4, and Proposition 3.2. ∎

Just as in Theorem 2.4 (and using the same proof), we can extend Corollary 3.10 to free unit-regular algebras over arbitrary fields:

THEOREM 3.11. *Any free unit-regular algebra R over a field K is residually finite dimensional over K.* ∎

THEOREM 3.12. *Any free unit-regular Λ-algebra R can be embedded (as a Λ-algebra) in a direct product of finite matrix algebras over quotient fields of prime factors of Λ. Thus R is residually artinian.*

Proof. Proposition 3.5 and Theorem 3.11. ∎

Finally, we give the unit-regular analog of Corollary 2.7.

COROLLARY 3.13. *Let R be a unit regular Λ-algebra. Then there exist a collection $\{R_\alpha\}$ of finite matrix algebras over quotient fields of prime factors of Λ and a unit-regular Λ-subalgebra A of $\prod_\alpha R_\alpha$ with an ideal I containing $\bigoplus_\alpha R_\alpha$ such that $A/I \cong R$.*

Proof. Write $R \cong S/J$, where S is a free unit-regular Λ-algebra. As in the proof of Corollary 2.7, we can identify S with a Λ-subalgebra of a direct product $\prod_{\beta,n} S_{\beta,n}$, where each $S_{\beta,n}$ is a finite matrix algebra over the quotient field of a prime factor of Λ, such that if $J' = \bigoplus_{\beta,n} S_{\beta,n}$ then $S \cap J' = 0$. Set $A = S + J'$; then $A/J' \cong S$ and $A/(J+J') \cong R$. Since S and $\prod_{\beta,n} S_{\beta,n}$ are unit-regular, we conclude from [7, Lemma 4.16] that A is unit-regular. ∎

4. CENTERS AND SOCLES

In this section we investigate the structure of free regular and free unit-regular Λ-algebras R. We assume *for this section only* that Λ is a commutative *regular* ring. We show that the center of R equals Λ, and that the socle of R is zero. Moreover, in case Λ is a field we show that R is a prime ring.

To get a better grip on free regular Λ-algebras, we need to write down some particular construction of them. We do this using coproducts in the category of Λ-algebras, which exist by, e.g., [9, Theorem 2.11]. Here all we need are coproducts of two or three Λ-algebras, R_1, R_2, R_3, which we denote $R_1 \coprod_\Lambda R_2$ and $R_1 \coprod_\Lambda R_2 \coprod_\Lambda R_3$.

Now we describe a universal process for adjoining quasi-inverses for the elements of a given subset Y of a given Λ-algebra S. Choose a Λ-algebra T with a map $c\colon Y \to T$ making T free on Y. Form the coproduct $S \coprod_\Lambda T$, let σ and τ be the canonical maps from S and T to $S \coprod_\Lambda T$, and let I be the ideal of $S \coprod_\Lambda T$ generated by the set

$$\{\sigma(y)\, \tau c(y)\, \sigma(y) - \sigma(y) \mid y \in Y\}.$$

Let us define $A(S, Y) = (S \coprod_\Lambda T)/I$, and let $\alpha\colon S \to A(S, Y)$ and $\beta\colon Y \to A(S, Y)$ denote the compositions of σ and τc with the quotient map $S \coprod_\Lambda T \to A(S, Y)$. We refer to α and β as the *canonical maps* from S and Y to $A(S, Y)$. Clearly $A(S, Y)$ is a Λ-algebra satisfying $\alpha(y)\, \beta(y)\, \alpha(y) = \alpha(y)$ for all $y \in Y$, and $A(S, Y)$ is universal for this property in the following sense.

LEMMA 4.1. *Let S be a Λ-algebra and $Y \subseteq S$, and construct $A(S, Y)$ with canonical maps α, β as above.*

(a) *If* $\phi: S \to S'$ *is a* Λ*-algebra homomorphism and* $f: Y \to S'$ *is a map such that* $\phi(y) f(y) \phi(y) = \phi(y)$ *for all* $y \in Y$, *there exists a unique* Λ*-algebra homomorphism* $\psi: A(S, Y) \to S'$ *such that* $\psi\alpha = \phi$ *and* $\psi\beta = f$.

(b) *The* Λ*-algebra homomorphism* $\alpha: S \to A(S, Y)$ *is injective.*

Proof. (a) This is clear from the construction.

(b) It suffices to find a Λ-algebra embedding $\phi: S \to S'$ with S' regular, since then by (a) there exists a Λ-algebra homomorphism $\psi: A(S, Y) \to S'$ with $\psi\alpha = \phi$. Since S is isomorphic to a subdirect product of its indecomposable factor rings, it is enough to consider the case that S itself is indecomposable. Then S is an algebra over some indecomposable factor ring K of Λ, and K is a field because of our assumption that Λ is regular. Now S can be embedded (via left multiplication) in the regular Λ-algebra $\mathrm{End}_K(S)$. ∎

By iterating the $A(S, Y)$ construction, we can give a construction of the free regular Λ-algebra on a set X. First let S_0 be the free Λ-algebra on X. Set $Y_0 = S_0$ and form $S_1 = A(S_0, Y_0)$ with canonical maps $\alpha_0: S_0 \to S_1$ and $\beta_0: Y_0 \to S_1$. Once we have constructed $S_0, ..., S_i, \alpha_0, ..., \alpha_{i-1}, \beta_0, ..., \beta_{i-1}$ for some $i \geq 1$, set $Y_i = S_i \setminus \alpha_{i-1}(S_{i-1})$ and form $S_{i+1} = A(S_i, Y_i)$ with canonical maps $\alpha_i: S_i \to S_{i+1}$ and $\beta_i: Y_i \to S_{i+1}$. Let S be the direct limit of the sequence

$$S_0 \xrightarrow{\alpha_0} S_1 \xrightarrow{\alpha_1} \cdots,$$

with canonical maps $\gamma_i: S_i \to S$.

By Lemma 4.1 all the maps α_i are injective, and so the γ_i are injective too. Hence, each $\gamma_i(S_i)$ equals the disjoint union of $\gamma_0(Y_0), ..., \gamma_i(Y_i)$, and so S equals the disjoint union of the subsets $\gamma_i(Y_i)$. Thus there exists a unique map $\beta: S \to S$ such that $\beta\gamma_i|_{Y_i} = \gamma_{i+1}\beta_i$ for all i. It is clear from the construction that $(S, \beta) \in \mathscr{R}eg_\Lambda$, and that (S, β) is a free object on X with respect to the map $\gamma_0|_X: X \to S$.

Since S_0 is generated as a Λ-algebra by X, we see that $\gamma_0(S_0)$ equals the Λ-subalgebra of S generated by $\gamma_0(X)$. Moreover, each S_{i+1} is generated as a Λ-algebra by $\alpha_i(S_i) \cup \beta_i(Y_i)$, and so $\gamma_{i+1}(S_{i+1})$ equals the Λ-subalgebra of S generated by $\gamma_i(S_i) \cup \beta\gamma_i(Y_i)$, which equals $\gamma_i(S_i) \cup \beta\gamma_i(S_i)$. We can transfer the universal property of each S_i to $\gamma_i(S_i)$ because γ_i is injective, and by the uniqueness of free objects the corresponding universal properties carry over to all free regular Λ-algebras. Therefore we obtain the following:

PROPOSITION 4.2. *Let* (R, q) *be a free object on a set* X *in* $\mathscr{R}eg_\Lambda$, *and let* $R_0 \subseteq R_1 \subseteq \cdots$ *be the* Λ*-subalgebras of* R *defined as follows:* R_0 *is generated by* X, *and each* R_{i+1} *is generated by* $R_i \cup q(R_i)$. *Set* $R_{-1} = \varnothing$.

(a) *The algebra* R_0 *is a free* Λ*-algebra on* X.

(b) *Suppose that* $\phi: R_i \to T$ *is a* Λ*-algebra homomorphism (for some* $i \geq 0$*) and that* $f: R_i \setminus R_{i-1} \to T$ *is a map such that* $\phi(r) f(r) \phi(r) = \phi(r)$ *for all* $r \in R_i \setminus R_{i-1}$. *Then* ϕ *extends uniquely to a* Λ*-algebra homomorphism* $\psi: R_{i+1} \to T$ *such that* $\psi q = f$ *on* $R_i \setminus R_{i-1}$. *Moreover, if* T *is regular then* ψ *can be extended (nonuniquely) to a* Λ*-algebra homomorphism from* R *to* T.

Proof. Only the last statement of (b) still needs to be proved. Given ψ, use the regularity of T to choose a map $f_{i+1}: R_{i+1} \setminus R_i \to T$ such that $\psi(r) f_{i+1}(r) \psi(r) = \psi(r)$ for all $r \in R_{i+1} \setminus R_i$. By what we have just proved, ψ extends to a Λ-algebra homomorphism $\psi_{i+2}: R_{i+2} \to T$. Continuing inductively, we obtain compatible Λ-algebra homomorphisms $\psi_j: R_j \to T$ for all $j \geq i+2$, and these ψ_j induce the desired homomorphism $R \to T$. ∎

In studying a free regular Λ-algebra R on a set X, we would like to be able to reduce to the case that Λ is a field, for instance by passing to a free regular (Λ/M)-algebra on X, where M is a maximal ideal of Λ. However, since the free regular (Λ/M)-algebra on X is usually much smaller than R/MR (see Section 5), we need to work instead with R/MR.

PROPOSITION 4.3. *Let* R *be a free regular* Λ*-algebra and* M *a maximal ideal of* Λ. *Then* R/MR *is a prime ring whose center is* Λ/M *and whose socle is zero.*

Proof. Choose a quasi-inversion q such (R, q) is free on a set X in $\mathcal{R}eg_\Lambda$, and define the subalgebras R_i of R as in Proposition 4.2. Let S be the direct product of an infinite sequence of copies of R/MR, and let T denote the prime regular Λ-algebra $\mathrm{End}_{\Lambda/M}(S)$. Letting R act on S via left multiplication, we obtain a Λ-algebra homomorphism $\phi: R \to T$ such that $\ker(\phi) = MR$ and $\phi(e) T\phi(e)$ is not artinian for any idempotent $e \in R \setminus MR$.

To show that R/MR is prime, it is enough to show that $eRf \not\subseteq MR$ for any idempotents $e, f \in R \setminus MR$. We may assume that $ef \in MR$, and since R/MR is semiprime we may assume that $fe \in MR$ as well. Let i be the least index such that $e, f \in R_i$. Now e and f cannot both be in R_{i-1}; say $e \notin R_{i-1}$. Since T is prime, there exists $x \in T$ with $\phi(e) x\phi(f) \neq 0$. Set $u = 1 + \phi(e) x\phi(f)$, which is a unit in T such that $\phi(e) u\phi(e) = \phi(e)$ and $\phi(e) u\phi(f) = \phi(e) x\phi(f) \neq 0$. Since T is regular, we may choose a map $g: R_i \setminus R_{i-1} \to T$ such that $g(e) = u$ and $\phi(r) g(r) \phi(r) = \phi(r)$ for all $r \in R_i \setminus R_{i-1}$. By Proposition 4.2, the map $\phi|_{R_i}$ extends to a Λ-algebra homomorphism $\psi: R \to T$ such that $\psi q(e) = u$. Then $\psi(eq(e) f) = \phi(e) u\phi(f) \neq 0$. Since $MT = 0$ we see that $MR \subseteq \ker(\psi)$, and so $eq(e) f \notin MR$. Thus R/MR is prime.

Next let $\alpha: R \to \Lambda/M$ be some Λ-algebra homomorphism. (These exist because R is free.) Then α induces a Λ-algebra homomorphism

β: $R/MR \to \Lambda/M$. If $\ker(\beta)$ has nonzero intersection with the center C of R/M, there is a nonzero idempotent in $\ker(\beta) \cap C$. However, this is a central idempotent different from 0 and 1, which is impossible since R/MR is prime. Hence, β is injective on C, and therefore $C = \Lambda/M$.

Finally, to show that the socle of R/MR is zero, it is enough to show that the right ideal $(eR + MR)/MR$ is not simple for any idempotent $e \in R \setminus MR$. Equivalently, it suffices to show that the endomorphism ring of any such right ideal, namely $(eRe + MR)/MR$, is not a division ring. This is obvious if $1 - e \in MR$. (E.g., there exist Λ-algebra homomorphisms from R to some (Λ/M)-algebras sending $q(0)$ to nonzero nonunits, so the coset $q(0) + MR$ is neither zero nor a unit in R/MR.) Hence, we may assume that $1 - e \notin MR$. By construction, $\phi(e)\, T\phi(e)$ is not a division ring, and thus $\phi(e)\, T\phi(e)$ contains an idempotent f different from 0 and $\phi(e)$. Since T is prime there are elements $x, y \in T$ such that $\phi(e)\, y\phi(1-e)\, xf \neq 0$. Then the elements $u = 1 + \phi(1-e)\, xf$ and $v = 1 + \phi(e)\, y\phi(1-e)$ are units of T satisfying

$$\phi(1-e)\, u\phi(1-e) = \phi(1-e) \qquad \text{and} \qquad \phi(e)\, v\phi(e) = \phi(e).$$

If i is the least index such that $e \in R_i$, then using Proposition 4.2 the map $\phi|_{R_i}$ extends to a Λ-algebra homomorphism ψ: $R \to T$ such that $\psi q(e) = v$ and $\psi q(1-e) = u$. Then

$$\psi(eq(e)(1-e)\, q(1-e)e) = \phi(e)\, v\phi(1-e)\, u\phi(e) = \phi(e)\, y\phi(1-e)\, xf,$$

which is nonzero and a nonunit in $\phi(e)\, T\phi(e)$. Thus the coset

$$eq(e)(1-e)\, q(1-e)e + MR$$

is nonzero and a nonunit in $(eRe + MR)/MR$, and therefore $(eRe + MR)/MR$ is not a division ring, as desired. ▮

In particular, it follows from Proposition 4.3 that the minimal prime ideals of a free regular Λ-algebra R are precisely the ideals MR for maximal ideals M of Λ.

THEOREM 4.4. *Let R be a free regular Λ-algebra. Then the socle of R is zero and the center of R is Λ. Moreover, if Λ is a field then R is a prime ring.*

Proof. The final statement is immediate from Proposition 4.3.

Since Λ is regular, each prime ideal of Λ is maximal. In particular, the kernel of the natural map from Λ to any prime factor of R is a maximal ideal of Λ, and so any prime ideal of R contains MR for some $M \in \operatorname{Spec}(\Lambda)$. Hence, $\bigcap_M MR = 0$, because R is semiprime. Thus R is isomorphic to a subdirect product of the algebras R/MR. Since $\operatorname{Soc}(R/MR) = 0$ for all $M \in \operatorname{Spec}(\Lambda)$ by Proposition 4.3, it follows that $\operatorname{Soc}(R) = 0$.

Finally, since Λ is regular there exists a Λ-algebra homomorphism $\phi: R \to \Lambda$. For each $M \in \mathrm{Spec}(\Lambda)$, let $\pi_M: \Lambda \to \Lambda/M$ denote the quotient map, and note that $MR \subseteq \ker(\pi_M \phi)$. Since the algebra R/MR is prime by Proposition 4.3, the ideal $\ker(\pi_M \phi)/MR$ contains no nonzero central idempotents, and hence any central idempotent of R which lies in $\ker(\pi_M \phi)$ must already lie in MR. Hence, any central idempotent of R which lies in $\ker(\phi)$ must lie in MR for all $M \in \mathrm{Spec}(\Lambda)$ and so must be zero. Therefore $\ker(\phi) \cap C = 0$, where C is the center of R, and we conclude that $C = \Lambda$. ∎

To prove a corresponding theorem for unit-regular algebras, we need a unit-regular analog of the construction of $A(S, Y)$. We want to construct Λ-algebras that will embed in unit-regular Λ-algebras. Since arbitrary Λ-algebras need not embed in unit-regular ones, we factor out an extra ideal in our construction to ensure this embeddability.

Start again with an arbitrary Λ-algebra S and a subset $Y \subseteq S$. Choose Λ-algebras T_1 and T_2 with maps $c_i: Y \to T_i$ making T_i free on Y. Form the coproduct $C = S \coprod_\Lambda T_1 \coprod_\Lambda T_2$, with canonical maps σ, τ_1, τ_2 from S, T_1, T_2 to C. Let I be the ideal of C generated by all the elements

$$\sigma(y)\,\tau_1 c_1(y)\,\sigma(y) - \sigma(y), \qquad \tau_1 c_1(y)\,\tau_2 c_2(y) - 1, \qquad \tau_2 c_2(y)\,\tau_1 c_1(y) - 1,$$

where $y \in Y$, and let J be the intersection of the kernels of all those Λ-algebra homomorphisms from C to unit-regular Λ-algebras that vanish on I. Finally, define $B(S, Y) = C/J$, and let $\gamma: S \to B(S, Y)$ and $\delta: Y \to B(S, Y)$ denote the compositions of σ and $\tau_1 c_1$ with the quotient map $C \to B(S, Y)$. We of course refer to γ and δ as the *canonical maps*. Now $B(S, Y)$ is a Λ-algebra and γ is a Λ-algebra homomorphism, δ maps Y to $U(B(S, Y))$, and $\gamma(y)\,\delta(y)\,\gamma(y) = \gamma(y)$ for all $y \in Y$.

LEMMA 4.5. *Let S be a Λ-algebra and $Y \subseteq S$, and construct $B(S, Y)$ with canonical maps γ, δ as above.*

(a) *The Λ-algebra $B(S, Y)$ can be embedded in a unit-regular Λ-algebra.*

(b) *Suppose that $\phi: S \to S'$ is a Λ-algebra homomorphism with S' unit-regular, and that $f: Y \to U(S')$ is a map such that $\phi(y) f(y) \phi(y) = \phi(y)$ for all $y \in Y$. Then there exists a unique Λ-algebra homomorphism $\psi: B(S, Y) \to S'$ such that $\psi\gamma = \phi$ and $\psi\delta = f$.*

(c) *If S can be embedded in a unit-regular Λ-algebra, then γ is injective.*

Proof. (a, b) These are clear from the construction.

(c) If there is a Λ-algebra embedding $\phi: S \to S'$, where S' is unit-regular, we can choose a map $f: Y \to U(S')$ such that $\phi(y) f(y) \phi(y) = \phi(y)$

for all $y \in Y$. Then there exists a Λ-algebra homomorphism $\psi: B(S, Y) \to S'$ such that $\psi\gamma = \phi$, and therefore γ is injective. ∎

Using this new construction, we can form the free unit-regular Λ-algebra on a set X as a direct limit of Λ-algebras $S_0 \to S_1 \to \cdots$, where each S_{i+1} is of the form $B(S_i, Y_i)$. The maps $S_i \to S_{i+1}$ for $i \geqslant 1$ will all be injective by Lemma 4.5. In order to make sure that the map $S_0 \to S_1$ is injective, we must start with a Λ-algebra S_0 that can be embedded in a unit-regular Λ-algebra. Hence, we start with $S_0 = S_0'/I$, where S_0' is the free Λ-algebra on X and I is the intersection of the kernels of all Λ-algebra homomorphisms from S_0' to unit-regular Λ-algebras. In parallel with Proposition 4.2, the universal properties of this construction carry over to all free unit-regular Λ-algebras, as follows.

PROPOSITION 4.6. *Let (R, u) be a free object on a set X in $\mathcal{UR}eg_\Lambda$, and let $R_0 \subseteq R_1 \subseteq \cdots$ be the Λ-subalgebras of R defined as follows: R_0 is generated by X, and each R_{i+1} is generated by $R_i \cup u(R_i)$. Set $R_{-1} = \varnothing$.*

(a) *Any function from X to a unit-regular Λ-algebra T extends uniquely to a Λ-algebra homomorphism from R_0 to T.*

(b) *Suppose that $\phi: R_i \to T$ is a Λ-algebra homomorphism (for some $i \geqslant 0$) with T unit-regular, and that $f: R_i \setminus R_{i-1} \to U(T)$ is a map such that $\phi(r) f(r) \phi(r) = \phi(r)$ for all $r \in R_i \setminus R_{i-1}$. Then ϕ extends uniquely to a Λ-algebra homomorphism $\psi: R_{i+1} \to T$ such that $\psi u = f$ on $R_i \setminus R_{i-1}$. Moreover, ψ can be extended (nonuniquely) to a Λ-algebra homomorphism from R to T.* ∎

If Λ is a field, then any free Λ-algebra can be embedded in a unit-regular Λ-algebra, for example its universal skew field of fractions (cf. [4, Corollaries 2.4.3, 7.5.11]). Thus in this case part (a) of Proposition 4.6 shows that R_0 is a free Λ-algebra on X.

In order to deal with a free unit-regular Λ-algebra R using the method of proof of Proposition 4.3, we need to be able to embed the direct product of a sequence of copies of R/MR into a prime unit-regular Λ-algebra for any maximal ideal M of Λ. The next lemmas show how to do this.

LEMMA 4.7. *Let R be a free unit-regular Λ-algebra and M a maximal ideal of Λ. Then R/MR is residually finite dimensional over Λ/M.*

Proof. Note (as in the proof of Theorem 4.4) that the intersection of the ideals NR, for $N \in \operatorname{Spec}(\Lambda)$, is zero. Hence, the quotient maps $\rho_N: R \to R/NR$ induce a Λ-algebra embedding $\rho: R \to \prod_N R/NR$. In particular, if π_M is the projection of $\prod_N R/NR$ onto R/MR, then $\pi_M \rho = \rho_M$.

For each $N \in \operatorname{Spec}(\Lambda)$, choose a free unit-regular (Λ/N)-algebra S_N and a surjective Λ-algebra homomorphism $\phi_N: S_N \to R/NR$. Then let ϕ be the Λ-algebra homomorphism from $\prod_N S_N$ to $\prod_N R/NR$ induced by the ϕ_N. In particular, if τ_M is the projection of $\prod_N S_N$ onto S_M, then $\phi_M \tau_M = \pi_M \phi$.

Next let $S = \phi^{-1}\rho(R)$. Then S is a Λ-subalgebra of $\prod_N S_N$, containing the ideal $\ker(\phi)$, such that $S/\ker(\phi) \cong R$. Since $\prod_N S_N$ and R are unit-regular, S is unit-regular by [7, Lemma 4.16]. Now since ρ is injective, there exists a surjective Λ-algebra homomorphism $\sigma: S \to R$ such that $\rho\sigma = \phi|_S$. Then we use Lemma 3.4 to obtain a Λ-algebra homomorphism $\psi: R \to S$ such that $\sigma\psi = 1_R$. We compute (just following a commutative diagram) that

$$\rho_M = \pi_M \rho = \pi_M \rho\sigma\psi = \pi_M(\phi|_S)\psi = \phi_M(\tau_M|_S)\psi.$$

Thus $\alpha = (\tau_M|_S)\psi: R \to S_M$ is a Λ-algebra homomorphism with $\phi_M \alpha = \rho_M$, and so $\ker(\alpha) \subseteq \ker(\rho_M) = MR$. On the other hand, $MR \subseteq \ker(\alpha)$ because $MS_M = 0$, and so α induces a (Λ/M)-algebra embedding of R/MR into S_M. By Theorem 3.11, S_M is residually finite dimensional over Λ/M, and therefore so is R/MR. ∎

LEMMA 4.8. *If S is a residually finite dimensional algebra over a field K, then S can be embedded (as a K-algebra) in a prime unit-regular K-algebra.*

Proof. We first embed S in a direct product $S_0 = \prod_{\alpha \in A} M_{m(\alpha)}(K)$ for some set A and some map $m: A \to \mathbf{N}$. We then construct a Λ-algebra embedding $\phi_0: S_0 \to S_1$ such that S_1 is again a direct product of finite matrix algebras over K and $\phi_0(x)S_1\phi_0(y) \neq 0$ for all nonzero $x, y \in S_0$. Iterating this construction and forming the direct limit of the resulting sequence

$$S_0 \xrightarrow{\phi_0} S_1 \xrightarrow{\phi_1} \cdots,$$

we obtain a prime unit-regular K-algebra in which S embeds.

For $\alpha \in A$, let $\pi_\alpha: S_0 \to M_{m(\alpha)}(K)$ be the projection on the αth component. For $\alpha, \beta \in A$, set $n(\alpha, \beta) = m(\alpha) + m(\beta)$ and define $\phi_{\alpha,\beta}: S_0 \to M_{n(\alpha,\beta)}(K)$ to be the composition of the map (π_α, π_β) from S_0 to $M_{m(\alpha)}(K) \times M_{m(\beta)}(K)$ with the block diagonal embedding of $M_{m(\alpha)}(K) \times M_{m(\beta)}(K)$ into $M_{n(\alpha,\beta)}(K)$. Then the map

$$\phi_0: S_0 \to S_1 = \prod_{(\alpha,\beta) \in A \times A} M_{n(\alpha,\beta)}(K)$$

induced by the $\phi_{\alpha,\beta}$ has the desired properties. ∎

Now we use Proposition 4.6 in the same way as Proposition 4.2 to prove the following analog of Proposition 4.3, except that in place of $\text{End}_{A/M}(S)$ in the proof we use Lemmas 4.7 and 4.8 to embed S in a prime unit-regular (A/M)-algebra T.

PROPOSITION 4.9. *Let R be a free unit-regular A-algebra and M a maximal ideal of A. Then R/MR is a prime ring whose center is A/M and whose socle is zero.* ▮

With Proposition 4.9, we get the unit-regular of Theorem 4.4:

THEOREM 4.10. *Let R be a free unit-regular A-algebra. Then the socle of R is zero and the center of R is A. Moreover, if A is a field then R is a prime ring.* ▮

5. Quotient Algebras

We conclude the paper with some observations about quotient algebras of a free regular A-algebra (R, q) modulo various ideals. In particular, we discuss some relations between R and free regular algebras over residue fields of A, and between R and free commutative regular algebras.

Let us say that an ideal I of R is *q-compatible* provided that whenever $a, b \in R$ with $a-b \in I$, then $q(a)-q(b) \in I$. Observe that the kernel of any morphism $\phi: (R, q) \to (R', q')$ in $\mathcal{R}eg_A$ is q-compatible: if $a, b \in R$ and $a-b \in \ker(\phi)$, then $\phi q(a) = q'\phi(a) = q'\phi(b) = \phi q(b)$ and so $q(a)-q(b) \in \ker(\phi)$. Conversely, any q-compatible ideal I of R is the kernel of a morphism: the rule $q'(r+I) = q(r)+I$ gives a well-defined quasi-inversion q' on R/I, and the quotient map gives a morphism $(R, q) \to (R/I, q')$ in $\mathcal{R}eg_A$ with kernel I.

It is easy to obtain from R a free regular algebra over a residue field A/M, where M is any maximal ideal of A. Let I be the smallest q-compatible ideal of R containing MR, and let q' be the unique quasi-inversion on R/I such that the quotient map $\pi: R \to R/I$ gives a morphism from (R, q) to $(R/I, q')$ in $\mathcal{R}eg_A$. If (R, q) is free on a set X in $\mathcal{R}eg_A$, then $(R/I, q')$ is free on X in $\mathcal{R}eg_{A/M}$, with respect to the function $\pi|_X: X \to R/I$. However, as we hinted in the previous section, the ideal I is usually larger than MR, the reason being that MR is usually not q-compatible. For instance, suppose that A has a maximal ideal $N \neq M$. Choose $a \in M$ such that $1-a \in N$. Set $R'' = (A/M) \times (A/N)$, and note that $a \cdot 1 = (0, 1)$ in R''. Next, choose a quasi-inversion q'' on R'' such that $q''(0) = (0, 1)$ and $q''(0, 1) = 1$. There exists a morphism $\phi: (R, q) \to (R'', q'')$ in $\mathcal{R}eg_A$, and

$$\phi q(a) - \phi q(0) = q''(0, 1) - q''(0) = (1, 0) \notin MR''.$$

Thus $q(a)-q(0)\notin MR$, even though $a-0\in MR$. Actually, $q(b)-q(0)\notin MR$ for any nonzero $b\in MR$, as the following proposition shows.

PROPOSITION 5.1. *Let (R, q) be either a free regular or a free unit-regular Λ-algebra, and let M be a maximal ideal of Λ. Then the subalgebra of R/MR generated by the cosets $q(a)+MR$, for $a\in MR$, is a free (Λ/M)-algebra on the set MR.*

Proof. Choose a set Y with a bijection $r\mapsto y_r$ from R onto Y, let S be the free (Λ/M)-algebra on Y, and let T be the universal skew field of fractions of S (e.g., [4, Corollaries 2.4.3, 7.5.11]). Then S is embedded in T, and since T is generated by S as a skew field, T is a (Λ/M)-algebra. Now set $R'=R\times T$, and define a quasi-inversion q' on R' as follows: $q'(r, 0)=(q(r), y_r)$ for all $r\in R$, while $q'(r, t)=(q(r), t^{-1})$ for all $r\in R$ and all nonzero $t\in T$. Note that if $(R, q)\in \mathscr{U}\mathscr{R}eg_\Lambda$, then so is (R', q'). Also, the projection map $\pi: R'\to R$ gives a morphism from (R', q') to (R, q) in $\mathscr{R}eg_\Lambda$.

If (R, q) is free on a set X, there is a unique morphism $\phi: (R, q)\to(R', q')$ in $\mathscr{R}eg_\Lambda$ such that $\phi(x)=(x, 0)$ for all $x\in X$. Then $\pi\phi$ is a morphism from (R, q) to itself which is the identity on X, and so $\pi\phi=1_R$.

Given any $a\in MR$, we have $\phi(a)\in MR'=MR\times 0$, and hence $\phi(a)=(a, 0)$. Thus $\phi q(a)=q'\phi(a)=(q(a), y_a)$. Now the composition of ϕ with the projection $R'\to T$ induces a (Λ/M)-algebra homomorphism $\psi: R/MR\to T$, and $\psi(q(a)+MR)=y_a$ for all $a\in MR$. Therefore the subalgebra of R/MR generated by $\{q(a)+MR \mid a\in MR\}$ is isomorphic (via ψ) to the free algebra S. ∎

In some instances, Proposition 5.1 can be used to show that R/MR is not even isomorphic to a free regular (Λ/M)-algebra. For example, suppose that (R, q) is free on a countable set X and that Λ/M is countable. Then, by Lemma 1.1, the free regular (Λ/M)-algebra $F(\mathscr{R}eg_{\Lambda/M}, X)$ is countable. However, if Λ has another maximal ideal N such that Λ/N is uncountable, the uncountable algebra R/NR is a homomorphic image of MR, and so MR is uncountable. In this case, R/MR is uncountable by Proposition 5.1, and therefore $R/MR\not\cong F(\mathscr{R}eg_{\Lambda/M}, X)$.

Free commutative regular rings were constructed by Burgess [2] by using the universal regular ring functor studied by Olivier [12] and Wiegand [16], which assigns to each commutative ring a commutative regular ring. More generally, there exist *free commutative regular Λ-algebras*; i.e., free objects in the full subcategory $\mathscr{C}\mathscr{R}eg_\Lambda$ of $\mathscr{R}eg_\Lambda$ whose objects are all pairs (R, q) where the ring R is commutative.

To obtain a free commutative regular Λ-algebra from a free regular Λ-algebra (R, q), we would expect to pass to the abelianization of R, which we denote by R_{ab}. Now proceed as in the passage to a free (Λ/M)-algebra.

Namely, let I be the smallest q-compatible ideal of R containing the commutator ideal, and let q' be the unique quasi-inversion on R/I such that the quotient map $\pi: R \to R/I$ gives a morphism from (R, q) to $(R/I, q')$ in $\mathcal{R}eg_\Lambda$. If (R, q) is free on a set X in $\mathcal{R}eg_\Lambda$, then $(R/I, q')$ is free on X in $\mathcal{C}\mathcal{R}eg_\Lambda$, with respect to the function $\pi|_X: X \to R/I$. The algebra R/I is always a proper homomorphic image of R_{ab}, since the commutator ideal is never q-compatible, as we see using the next proposition.

PROPOSITION 5.2. *Let (R, q) be either a free regular or a free unit-regular Λ-algebra, let H be the commutator ideal of R, and let M be a maximal ideal of Λ. Then the cosets $q(a) + H + MR$, for $a \in H$, are algebraically independent over Λ/M.*

Proof. Choose a field extension $T \supseteq \Lambda/M$ of transcendence degree at least card(R), and choose algebraically independent elements $y_r \in T$ for $r \in R$. Then define R', q', π, ϕ as in the proof of Proposition 5.1.

Given any $a \in H$, note that $\phi(a)$ must lie in the commutator ideal of R', and hence $\phi(a) = (a, 0)$; thus $\phi q(a) = (q(a), y_a)$. Since the composition of ϕ with the projection $R' \to T$ induces a (Λ/M)-algebra homomorphism $R/(H + MR) \to T$, we conclude that the cosets $q(a) + H + MR$ are indeed algebraically independent over Λ/M. ∎

In view of Proposition 5.2, the commutator ideal H cannot be q-compatible: obviously $H \neq 0$, and $q(a) - q(0) \notin H$ for all nonzero $a \in H$.

COROLLARY 5.3. *Let (R, q) be either a free regular or a free unit-regular Λ-algebra, and let H be the commutator ideal of R. If Λ is semiprimitive, then the cosets $q(a) + H$, for $a \in H$, are linearly independent over Λ.*

Proof. Suppose that $\lambda_1 q(a_1) + \cdots + \lambda_n q(a_n) \in H$ for some distinct $a_i \in H$ and some $\lambda_i \in \Lambda$. For each maximal ideal M of Λ, Proposition 5.2 shows that the cosets $q(a_i) + H + MR$ are linearly independent over Λ/M, and hence $\lambda_i \in M$ for all i. Since Λ is semiprimitive, it follows that $\lambda_i = 0$ for all i. ∎

Another way to obtain a free commutative regular Λ-algebra from a free regular Λ-algebra (R, q) is as a subalgebra of R_{ab}, provided we choose a quasi-inversion on R_{ab} by applying q to selected coset representatives, as follows.

PROPOSITION 5.4. *Let (R, q) be a free regular Λ-algebra on a set X, and let $\pi: R \to R_{ab}$ be the quotient map. For each $s \in R_{ab}$, choose a coset representative $r_s \in \pi^{-1}(s)$, and set $\bar{q}(s) = \pi q(r_s)$. If $p: (R, q) \to (R_{ab}, \bar{q})$ is the unique morphism in $\mathcal{R}eg_\Lambda$ that agrees with π on X, then $(p(R), \bar{q})$ is free on*

X in $\mathscr{CReg}_\Lambda$, with respect to the map $\pi|_X\colon X \to p(R)$. However, $(R_{\mathrm{ab}}, \bar{q})$ is not *free on X in $\mathscr{CReg}_\Lambda$.*

Proof. Given $(R'', q'') \in \mathscr{CReg}_\Lambda$ and a map $f\colon X \to R''$, there is a unique morphism $\psi\colon (R, q) \to (R'', q'')$ in $\mathscr{Reg}_\Lambda$ such that $\psi|_X = f$. Since R'' is commutative, ψ induces a Λ-algebra homomorphism $\bar{\psi}\colon R_{\mathrm{ab}} \to R''$ such that $\bar{\psi}\pi = \psi$. In particular, $\bar{\psi}(\pi|_X) = \psi|_X = f$. For any $s \in R_{\mathrm{ab}}$, we have

$$\bar{\psi}\bar{q}(s) = \bar{\psi}\pi q(r_s) = \psi q(r_s) = q''\psi(r_s) = q''\bar{\psi}\pi(r_s) = q''\bar{\psi}(s);$$

thus $\bar{\psi}$ is a morphism from $(R_{\mathrm{ab}}, \bar{q})$ to (R'', q'') in $\mathscr{CReg}_\Lambda$. Consequently, $\bar{\psi}p$ is a morphism from (R, q) to (R'', q'') in $\mathscr{Reg}_\Lambda$ such that $\bar{\psi}p|_X = \bar{\psi}\pi|_X = f$, whence $\bar{\psi}p = \psi$.

Now $\bar{\psi}|_{p(R)}$ is a morphism from $(p(R), \bar{q})$ to (R'', q'') in $\mathscr{CReg}_\Lambda$ such that $(\bar{\psi}|_{p(R)})(\pi|_X) = f$. If $\phi\colon (p(R), \bar{q}) \to (R'', q'')$ is any morphism in $\mathscr{CReg}_\Lambda$ such that $\phi\pi|_X = f$, then $\phi p|_X = f$. Since ϕp is a morphism in $\mathscr{Reg}_\Lambda$, it follows from the uniqueness of ψ that $\phi p = \psi = \bar{\psi}p$. Hence, $\phi = \bar{\psi}|_{p(R)}$, completing the proof that $(p(R), \bar{q})$ is free in $\mathscr{CReg}_\Lambda$.

Finally, suppose that $(R_{\mathrm{ab}}, \bar{q})$ is free on X in $\mathscr{CReg}_\Lambda$, with respect to the map $\pi|_X$. Now p induces a Λ-algebra homomorphism $\bar{p}\colon R_{\mathrm{ab}} \to R_{\mathrm{ab}}$ such that $\bar{p}\pi = p$, and as above we check that $\bar{p}$ is a morphism from $(R_{\mathrm{ab}}, \bar{q})$ to itself in $\mathscr{CReg}_\Lambda$. Moreover, $\bar{p}$ is the identity on $\pi(X)$, and so the freeness assumption on $(R_{\mathrm{ab}}, \bar{q})$ implies that $\bar{p}$ is the identity map on R_{ab}. Hence, $p = \pi$, and so $\ker(p)$ equals the commutator ideal of R. However, $\ker(p)$ is q-compatible because p is a morphism in $\mathscr{Reg}_\Lambda$, while Proposition 5.2 shows that the commutator ideal of R is not q-compatible. Therefore $(R_{\mathrm{ab}}, \bar{q})$ cannot be free on X in $\mathscr{CReg}_\Lambda$. ∎

We close with a question: is the free commutative regular Λ-algebra $F(\mathscr{CReg}_\Lambda, X)$ ever isomorphic to the abelianization of the free regular Λ-algebra $F(\mathscr{Reg}_\Lambda, X)$?

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