Theoretical Computer Science 102 (1992) 283–305
Elsevier

Approximating the permanent of graphs with large factors

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Communicated by F. Yao
Received May 1989
Revised February 1991

Abstract

Dagum, P. and M. Luby, Approximating the permanent of graphs with large factors, Theoretical Computer Science 102 (1992) 283–305.

Let $G=(U,V,E)$ be a bipartite graph with $|U|=|V|=n$. The *factor size* of G, f, is the maximum number of edge disjoint perfect matchings in G. We characterize the complexity of counting the number of perfect matchings in classes of graphs parameterized by factor size. We describe the *simple algorithm*, which is an approximation algorithm for the permanent that is a natural simplification of the algorithm suggested by Broder (1986) and analyzed by Jerrum and Sinclair (1988a, b). Compared to the algorithm by Jerrum and Sinclair (1988a, b), the simple algorithm achieves a polynomial speed up in the running time to compute the permanent. A combinatorial lemma is used to prove that the simple algorithm runs in time $n^{O(n/f)}$. Thus: (1) for all constants $\alpha>0$, the simple algorithm runs in polynomial time for graphs with factor size at least αn; (2) for some constant c, the simple algorithm is the fastest known approximation for graphs with factor size at least $c \log n$. (Compare with the approximation algorithms described in Karmarkar et al. (1988).)

We prove the following complementary hardness results. For functions f such that $3 \leqslant f(n) \leqslant n-3$, the exact counting problem for $f(n)$-regular bipartite graphs is $\sharp$**P**-complete. For any $\varepsilon>0$, for any function f such that $3 \leqslant f(n) \leqslant n^{1-\varepsilon}$, approximate counting for $f(n)$-regular bipartite graphs is as hard as approximate counting for all bipartite graphs.

An announcement of these results appears in Dagum et al. (1988).

* Supported by NSERC of Canada and the International Computer Science Institute, Berkeley, California. This work was done while the author was at the University of Toronto and also while visiting ICSI.

** Research partially supported by NSERC of Canada operating grant A8092. A portion of this research was done while the author was at the University of Toronto.

0304-3975/92/$05.00

1. Introduction

The permanent function has a long history, having been first introduced by Cauchy in 1812 in his celebrated memoir on determinants and almost simultaneously by Binet. More recently, several problems in the physical sciences as well as enumeration problems in combinatorics and linear algebra have been reduced to the computation of a permanent. Unfortunately, the fastest known algorithm for computing the permanent of an $n \times n$ matrix runs in $O(n2^n)$ time, [20]. Moreover, strong evidence for the apparent intractability of the problem was provided in Valiant [22] who showed that evaluating the permanent is $\sharp\mathbf{P}$-complete even when restricted to 0/1 matrices. This has led to the search fo (ε, δ)-approximation algorithms, i.e. an algorithm, with input parameters ε and δ, that with probability greater than $1-\delta$ outputs an approximation of the permanent with relative error less than ε. We say the (ε, δ)-approximation algorithm is *fully polynomial* if its running time is polynomial in n, $1/\varepsilon$ and $\log 1/\delta$ [15].

In graph theoretic terms, evaluating the permanent of a 0/1 $n \times n$ matrix A is the same as counting the number of perfect matchings in the corresponding bipartite graph $G=(U, V, E)$, where U and V are each sets of n nodes and E is the set of edges such that $(u_i, v_j) \in E$ if and only if $A_{i,j}=1$. Broder [4] proposed the following (ε, δ)-approximation algorithm for estimating the number of perfect matchings in dense graphs G (G is *dense* if every vertex has degree at least $n/2$): Let $\mathcal{M}_k$ be the set of matchings of size k in G and let $\mathcal{N}_k = \mathcal{M}_k \cup \mathcal{M}_{k-1}$. Estimating $|\mathcal{M}_n|$ is efficiently reduced to estimating for $k=2, \ldots, n$ the ratio $|\mathcal{M}_k|/|\mathcal{M}_{k-1}|$, which is further efficiently reduced to estimating $|\mathcal{M}_n/|\mathcal{M}_{n-1}|$. Define the graph H_n on vertex set $\mathcal{N}_n$ (perfect and almost perfect matchings) by connecting any two such matchings if a local modification (deleting and adding at most 1 edge) is sufficient to transform one into the other. $\mathcal{N}_n$ is uniformly sampled by simulating a random walk on H_n and thus $|\mathcal{M}_n|/|\mathcal{M}_{n-1}|$ can be estimated. However, the efficiency of the whole algorithm, which depends on the rate of convergence of the random walk on H_n, was left open (see [5, 18]). For this latter problem Jerrum and Sinclair [11] obtained a bound by proving that for any dense graph G the graph H_n has magnification factor $\Omega(1/n^4)$ (in fact [12] shows magnification $\Omega((|\mathcal{M}_n|/|\mathcal{N}_n|)^2)$ for any graph G), which implies a fully polynomial (ε, δ)-approximation algorithm for the permanent of dense bipartite graphs. However, the problem of estimating the permanent for all bipartite graphs is still open.

First we introduce and analyze a natural simplification of the original algorithm suggested in [4] and analyzed in [11, 12] that we call the *simple algorithm*. For all $k=2, \ldots, n$ let H_k on node set $\mathcal{N}_k$ be defined analogously to how H_n is defined on $\mathcal{N}_n$. We show that for all $k=2, \ldots, n$, the magnification of H_k is $\Omega((|\mathcal{M}_k|/|\mathcal{N}_k|)^2)$. Therefore, in the case where $|\mathcal{M}_k|$ and $|\mathcal{N}_k|$ are polynomially related, a random walk on H_k converges to the uniform distribution on $\mathcal{N}_k$ in polynomial time and gives us a straightforward and direct method for estimating the ratio $|\mathcal{M}_k|/|\mathcal{M}_{k-1}|$. The simple algorithm estimates these ratios for all k and multiplies together the results. By directly estimating the ratios $|\mathcal{M}_k|/|\mathcal{M}_{k-1}|$ the simple algorithm achieves an $\mathrm{O}(n\cdot\min(|\mathcal{M}_{n-1}|/|\mathcal{M}_n|, |E|))$ improvement on the running time to estimate $|\mathcal{M}_n|$ over the algorithm in [11, 12]. Thus, for example, for dense graphs the simple algorithm achieves an $\mathrm{O}(n^3)$ improvement on the running time.

We apply the simple algorithm to the problem of counting perfect matchings in classes of bipartite graphs parameterized by factor size. The *factor size* of G, f, is the maximum number of edge disjoint perfect matchings in G. Using network flow techniques, f can be computed from G in polynomial time. We show that $|\mathcal{M}_{n-1}|/|\mathcal{M}_n| \leqslant n^{3n/f}$. This theorem provides an a priori upper bound on the running time of the simple algorithm in terms of factor size that is computable in polynomial time; the running time of the simple algorithm on input G is polynomial in $n^{n/f}$, $1/\varepsilon$ and $\log 1/\delta$. In particular, the simple algorithm is a fully polynomial (ε, δ)-approximation algorithm for every *large factor class* of bipartite graphs, i.e. if there is a constant $\alpha > 0$ such that all graphs in the class have factor size at least αn. If a graph has an αn factor then every vertex has degree at least αn, but not necessarily vice versa. In contrast to the behavior of the simple algorithm on a large factor class, Broder [4] shows that, for all $\alpha < 1/2$, $|\mathcal{M}_{n-1}|/|\mathcal{M}_n|$ is exponential in n for the class of graphs with minimum vertex degree αn, and thus the simple algorithm does not run in polynomial time on this class of graphs.

The paper of Karmarkar et al. [14] describes an (ε, δ)-approximation algorithm for all bipartite graphs that has running time $poly(n)\cdot 2^{n/2}\cdot 1/\varepsilon\cdot\log 1/\delta$ where $poly(n)$ is a function that grows polynomially with n. For some constant $c > 0$, the upper bound on the running time of the simple algorithm is better than that for the algorithm in [14] for all bipartite graphs with factor size at least $c\log n$.

On the negative side, we show the following. Let $\alpha < 1$ be any constant and let f be a function such that $3 \leqslant f(n) \leqslant n-3$. For a variety of functions f, counting the exact number of perfect matchings for $f(n)$-regular bipartite graphs is $\sharp\mathbf{P}$-complete. (A f-regular graph has factor size exactly f.) We say that a class of bipartite graphs is *approximation complete for the permanent* if an (ε, δ)-approximation algorithm for the class implies an (ε, δ)-approximation algorithm for all bipartite graphs. Let $\alpha < 1$ be any constant and let f be any function such that $3 \leqslant f(n) \leqslant n^{1-\alpha}$. The reductions are parsimonious in the sense that they also prove that the class of $f(n)$-regular bipartite graphs is approximation complete for the permanent.

2. Preliminaries

This section presents a brief overview of previous work and sets up a context for our results. Let $G=(V, E)$, $|V| = 2n$, be an undirected graph, let $\mathcal{M}_k$ be the set of matchings of size k in G and let $\mathcal{N}_k = \mathcal{M}_k \cup \mathcal{M}_{k-1}$, $k\in[n]$. Propositions 2.1 and 2.2 state that randomized approximate counting of $\mathcal{M}_n$ is efficiently reducible to almost uniform sampling of $\mathcal{N}_k$, $2 \leqslant k \leqslant n$.

Proposition 2.1 (Broder [4], Jerrum et al. [13]). *Suppose that for all $k=2,\ldots,n$ there is a fully polynomial (ε,δ)-approximation algorithm for the ratios $|\mathcal{M}_k|/|\mathcal{M}_{k-1}|$. Then, there is a fully polynomial (ε,δ)-approximation algorithm for $|\mathcal{M}_n|$.*

Proof. Let Y_k be the approximation of $|\mathcal{M}_k|/|\mathcal{M}_{k-1}|$. The idea is to let the approximation of $|\mathcal{M}_n|$ be $Y=\prod_{k=1}^{n} Y_k$. Let each Y_k approximate $|\mathcal{M}_k|/|\mathcal{M}_{k-1}|$ with relative error $\varepsilon/2n$, where $\varepsilon<1$. Then the relative error in the approximation of $|\mathcal{M}_n|$ by Y is $(1+\varepsilon/2n)^n<1+\varepsilon$. By assumption, for $k=2,\ldots,n$ there exist a polynomial $p_k(n,1/\varepsilon,\log 1/\delta)$ such that the (ε,δ)-approximation algorithm for $|\mathcal{M}_k|/|\mathcal{M}_{k-1}|$ runs in time $p_k(n,1/\varepsilon,\log 1/\delta)$. It follows that the (ε,δ)-approximation algorithm for $|\mathcal{M}_n|$ has running time $\sum_{k=2}^{n} p_k(n,2n/\varepsilon,\log 1/\delta)$. $\square$

A *fully polynomial ζ-sampling scheme* for $\mathcal{N}_k$ is an algorithm that runs in time polynomial in n and $\log 1/\zeta$ and outputs $M\in\mathcal{N}_k$ according to a probability distribution π on $\mathcal{N}_k$ that satisfies:

$$\max_{S\subset\mathcal{N}_k}\left|\pi(S)-\frac{|S|}{|\mathcal{N}_k|}\right|\leqslant\zeta.$$

Let $M_0,M_1,\ldots$ be the output of a ζ-sampling scheme for $\mathcal{N}_k$. Let $Y_0,Y_1,\ldots$ be independent and identically distributed 0–1 valued random variables where $Y_i=1$ if and only if $M_i\in\mathcal{M}^*$, and where $\mathcal{M}^*=\mathcal{M}_{k-1}$, if $|M_{k-1}|\leqslant|\mathcal{M}_k|$ and $\mathcal{M}^*=\mathcal{M}_k$ otherwise. Define $Y=1/N(\sum_{i=1}^{N} Y_i)$ and $\mu=\min(|\mathcal{M}_{k-1}|/|\mathcal{N}_k|,|\mathcal{M}_k|/|\mathcal{N}_k|)$.

Proposition 2.2 is a strong version of the well known Zero–One Estimator Theorem (e.g. see [23]).

Proposition 2.2 (Broder [4], Jerrum et al. [13]). *Suppose that for all $k=2,\ldots,n$ there is a fully polynomial ζ-sampling scheme for $\mathcal{N}_k$. When $N=c\cdot 1/\varepsilon^2\cdot 1/\mu\cdot\log 1/\delta$, for some $c>0$, and $\zeta\leqslant\varepsilon\mu/3$ then Y is an (ε,δ)-approximation of μ.*

Let Δ denote the symmetric difference of two sets.

Definition 2.3. The $(k,k-1)$-*exchange graph* H_k has vertex set $\mathcal{N}_k$ and there is an edge between $M_A,M_B\in\mathcal{N}_k$ if and only if: either $M_A\in\mathcal{M}_k$, $M_B\in\mathcal{M}_{k-1}$ and $|M_A\Delta M_B|=1$, or $M_A,M_B\in\mathcal{M}_{k-1}$ and $|M_A\Delta M_B|=2$ (H_k is undirected).

Definition 2.4. The Markov chain $\mathcal{MC}_k$ is a random walk on H_k where transitions between distinct states are taken with probability $1/(2|E|)$ and the self-loop probability of a state M is $1-\deg(M)/(2|E|)$, where $\deg(M)$ is the degree of M in H_k.

It is easily verified that $\mathcal{MC}_k$ is ergodic, and because it is symmetric it converges to the uniform distribution over $\mathcal{N}_k$. For this reason, we propose the simulation of $\mathcal{MC}_k$ as an ε-sampling scheme for $\mathcal{N}_k$.

We bound the rate of convergence of $\mathcal{MC}_k$ to the uniform distribution in terms of its magnification properties. For this we rely on the following proposition (similar results have been obtained in [2, 3, 6]):

Proposition 2.5 (Sinclair and Jerrum [21]). *For a time reversible, ergodic Markov chain $(X_t)_{t=0}^{\infty}$ on a finite state space $\mathcal{N}$ with transition matrix $P=(p_{ij})_{i,j\in\mathcal{N}}$ and stationary distribution π:*

$$\max_{X_0\in\mathcal{N}}\max_{S\subseteq\mathcal{N}}\left|Pr(X_t\in S)-\sum_{i\in S}\pi(i)\right|\leqslant\frac{(1-\Phi(H)^2/8)^t}{\min_{i\in\mathcal{N}}\pi_i}$$

where X_0 is the initial state of the Markov chain and where $\Phi(H)$ is the conductance of the underlying graph H and is defined as follows:

$$\Phi(H)=\min_{S\subseteq\mathcal{N}}\frac{\sum_{i\in S,\,j\in\bar{S}}\pi_i p_{ij}}{\sum_{i\in S}\pi_i\sum_{j\in\bar{S}}\pi_j}.$$

Proposition 2.6 (see [17, Exercise 8.5.10]) is given below for the sake of completeness.

Proposition 2.6 (Lovász, Plummer [17]). *For all $k=1,\ldots,n$, $|\mathcal{N}_k|/|\mathcal{M}_k|\leqslant|\mathcal{N}_n|/|\mathcal{M}_n|$.*

Proof. It is equivalent to show that $|\mathcal{M}_n|\cdot|\mathcal{M}_{k-1}|\leqslant|\mathcal{M}_{n-1}|\cdot|\mathcal{M}_k|$. Let Δ denote the symmetric difference of two sets. For each pair $M_A\in\mathcal{M}_{k-1}$, $M_B\in\mathcal{M}_n$ it is the case that $M_A\Delta M_B$ can be viewed as $n-k+1$ vertex disjoint augmenting paths for M_A, which defines a mapping into $n-k+1$ distinct pairs $M_{A'}\in\mathcal{M}_k$, $M_{B'}\in\mathcal{M}_{n-1}$. Moreover, for each pair $M_{A'}\in\mathcal{M}_k$, $M_{B'}\in\mathcal{M}_{n-1}$, there is at most one pair $M_A\in\mathcal{M}_{k-1}$, $M_B\in\mathcal{M}_n$ which maps into the pair $M_{A'}, M_{B'}$. Consequently, the stronger result $|\mathcal{M}_n|\cdot|\mathcal{M}_{k-1}|\leqslant|\mathcal{M}_{n-1}|\cdot|\mathcal{M}_k|/(n-k+1)$ holds. □

3. The magnification of the $(k,k-1)$-exchange graph

In Theorem 3.4, for $\mathcal{MC}_k$ we show that $\Phi(H_k)=\Omega(1/|E|\cdot|\mathcal{M}_k|/|\mathcal{N}_k|\cdot|\mathcal{M}_{k-1}|/|\mathcal{N}_{k-1}|)$. In Section 4 we use the lower bound on the conductance of H_k to analyze the time complexity of the simple (ε,δ)-approximation algorithm for the permanent.

Definition 3.1. For a graph $G=(V,E)$ the *magnification factor* of G is: $\min_{\emptyset\subset S\subset V}|C(S)||V|/|S||\bar{S}|$, where $C(S)=\{(u,v)\in E\colon v\in S,\ u\in\bar{S}\}$.

This is the edge-cutset analogue to the usual notion of magnification and it is the natural quantity to consider in the context of this paper. In this section we obtain a lower bound for the magnification of the $(k,k-1)$-exchange graph H_k.

Definition 3.2. For $M_A, M_B \in \mathcal{N}_k$, $M_A \Delta M_B$ is a set of vertex disjoint alternating even length cycles, even length segments and odd length segments. We say an odd length alternating segment is *M_A-augmenting* (*M_A-deaugmenting*) if the first edge in the segment is not in M_A (in M_A). To *fix* an alternating even length cycle or segment, or alternating odd length segment, with respect to M_A, we take the symmetric difference of the cycle or segment with M_A.

Lemma 3.3 proves a lower bound for the magnification factor of H_k. The proof uses the idea in [11] of constructing paths in the underlying graph of the Markov chain between all pairs of states and encoding the paths that go through an edge by an element of the state space. This gives an upper bound on the number of paths that go through an edge in the underlying graph and therefore it also gives a lower bound on the size of the cut set $C(S)$, for S any subset of the state space. From this and Definition 3.1, a lower bound on the magnification follows.

Lemma 3.3. *The magnification factor of the $(k, k-1)$-exchange graph H_k is bounded by*

$$\min_{S \subseteq \mathcal{N}_k} \frac{|C(S)||\mathcal{N}_k|}{|S||\bar{S}|} = \Omega\left(\frac{|\mathcal{M}_k|}{|\mathcal{N}_k|}\frac{|\mathcal{M}_{k-1}|}{|\mathcal{N}_{k-1}|}\right).$$

Proof. Let $S \subseteq \mathcal{N}_k$. We construct paths in H_k between all pairs $M_A \in \mathcal{M}_{k-1}$ and $M_B \in \mathcal{M}_k$, and prove that for each edge in H_k the expected number of these paths that go through the edge is $O(|\mathcal{M}_{k-2} \cup \mathcal{M}_{k-1}|) = O(|\mathcal{N}_{k-1}|)$. Therefore there exists a way of choosing paths between $S \cap \mathcal{M}_{k-1}$ and $\bar{S} \cap \mathcal{M}_k$ such that $O(|\mathcal{M}_{k-2} \cup \mathcal{M}_{k-1}||C(S)|)$ of these go through $C(S)$. On the other hand, without loss of generality, we may assume that $|S \cap \mathcal{M}_{k-1}|/|S| \geqslant |\mathcal{M}_{k-1}|/|\mathcal{N}_k|$ and hence $|\bar{S} \cap \mathcal{M}_k|/|\bar{S}| \geqslant |\mathcal{M}_k|/|\mathcal{N}_k|$. Therefore, at least $|S||\bar{S}||\mathcal{M}_k||\mathcal{M}_{k-1}|/|\mathcal{N}_k|^2$ paths between $S \cap \mathcal{M}_{k-1}$ and $\bar{S} \cap \mathcal{M}_k$ go through $C(S)$. Thus, $|C(S)| = \Omega(|S||\bar{S}|/|\mathcal{N}_k| \cdot |\mathcal{M}_k|/|\mathcal{N}_k| \cdot |\mathcal{M}_{k-1}|/|\mathcal{N}_{k-1}|)$ and the theorem follows.

We begin by giving the construction of the paths. Let $M_A \in \mathcal{M}_{k-1}$ and $M_B \in \mathcal{M}_k$. Let $M_{AB} \in \mathcal{M}_k$ be the matching obtained by fixing all the odd length segments contained in the set $M_A \Delta M_B$ with respect to M_A (equivalently, by fixing all the even length cycles and segments in $M_A \Delta M_B$ with respect to M_B). We order all cycles and even length segments in $M_A \Delta M_B$ lexicographically. Let $S_0, \ldots, S_l$ be a random ordering of the set of M_A-augmenting odd length segments in $M_A \Delta M_B$. Similarly, let $S'_1, \ldots, S'_l$ be a random ordering of the set of M_A-deaugmenting odd length segments in $M_A \Delta M_B$. (The number of M_A-augmenting segments is one more than the number of M_A-deaugmenting segments since $M_A \in \mathcal{M}_{k-1}$ and $M_B \in \mathcal{M}_k$.) The path in H_k from M_A to M_B is defined as follows: Start by fixing S_0 and then, for $i = 1, \ldots, l$, first fix S'_i and then fix S_i, all with respect to M_A. At this point the path has reached the matching M_{AB}. The rest of the path to M_B consists of fixing the even length cycles and segments with respect to the current matching in lexicographical order.

Remark. The "fixing" of an even length alternating cycle, segment or odd length segment P with respect to $M \in H_k$ defines a path in H_k as follows: Let $P=\{e_1, f_1, \ldots, e_p, f_p\}$ be an even length alternating cycle or segment in $M \in H_k$ such that $e_1 \notin M$, and in case P is a cycle, f_1 is adjacent to the lexicographically first vertex in the cycle. Fixing P with respect to M defines a path in H_k through the matchings $M_0, M_1, \ldots, M_q$ defined as follows: $M_0 = M$, $M_q = M \Delta P$ and (1) if P is an even length segment then $q=p$ and for $i=1, \ldots, p, M_i \Delta M_{i-1} = \{e_i, f_i\}$; (2) if P is a cycle then $q=p+1$, $M_1 \Delta M_0 = \{f_1\}$, for $i=2, \ldots, p$, $M_i \Delta M_{i-1} = \{e_i, f_i\}$, and $M_{p+1} \Delta M_p = \{e_1\}$. If $P=\{f_1, e_2, f_2, \ldots, e_p, f_p\}$ is an odd length alternating segment which is M-deaugmenting, i.e. $f_1 \in M$, then the fixing of P in M defines a path in H_k through the matchings $M_0, \ldots, M_p$ defined as follows: $M_0 = M$, $M_p = M \Delta P$, $M_1 \Delta M_0 = \{f_1\}$, and for $i=2, \ldots, p$, $M_i \Delta M_{i-1} = \{e_i, f_i\}$. If $P=\{e_1, f_1, \ldots, e_{p-1}, f_{p-1}, e_p\}$ is an odd length alternating segment which is M-augmenting, i.e. $e_1 \notin M$, then the fixing of P in M defines a path in H_k through the matchings $M_0, \ldots, M_p$ defined as follows: $M_0 = M$, $M_p = M \Delta P$, for $i=1, \ldots, p-1$, $M_i \Delta M_{i-1} = \{e_i, f_i\}$, and $M_p \Delta M_{p-1} = \{e_p\}$. (That the sets of matchings $M_0, \ldots, M_q$ defined above do indeed form a path in H_k follows from Definition 2.3.)

Remark. For the remainder of the proof, *a path in H_k between a pair of matchings $M_A \in \mathcal{M}_{k-1}$ and $M_B \in \mathcal{M}_k$* refers to a path constructed as above.

We show that the expected number of the paths between matchings in $\mathcal{M}_{k-1}$ and $\mathcal{M}_k$ that go through an edge $t=(M_1, M_2)$ in H_k is bounded by $O(|\mathcal{M}_{k-2} \cup \mathcal{M}_{k-1}|)$.

Assume t occurs in the path from $M_A \in \mathcal{M}_{k-1}$ to $M_B \in \mathcal{M}_k$. There are two cases to consider: (1) t occurs in the segment of the path going from M_{AB} to M_B; (2) t occurs in the segment of the path going from M_A to M_{AB}. Let $M_t = (M_1 \cup M_2) \Delta (M_A \Delta M_B)$.

Case 1: If $t=(M_1, M_2)$ occurs at the start or end of the fixing of a cycle, then $M_1 \in \mathcal{M}_k$ or $M_2 \in \mathcal{M}_k$, and it is straightforward to show that $M_t \in \mathcal{M}_{k-2} \cup \mathcal{M}_{k-1}$, and moreover, M_A and M_B can be uniquely determined from t together with M_t. If t occurs in a cycle but neither at the start nor the end, then $M_1, M_2 \in \mathcal{M}_{k-1}$, and in $M_1 \cup M_2$ the degree of the lexicographically first vertex of the cycle is 0 and consequently in M_t this vertex has degree 2. All other vertices have degree 0 or 1 in M_t. Letting e be the edge in M_A incident with the lexicographically first vertex, we have in this case $M_t - e \in \mathcal{M}_{k-2}$ and together with t, this uniquely determines the pair M_A and M_B. If t occurs in the fixing of an even length segment, then it is straightforward to show that $M_t \in \mathcal{M}_{k-2}$, and once again, M_A and M_B can be uniquely determined from t together with M_t. This proves that if the transition t occurs in the segment of the path from M_{AB} to M_B then $M_t \in \mathcal{M}_{k-2} \cup \mathcal{M}_{k-1}$ and also, that M_t together with t uniquely specify the pair $M_A \in \mathcal{M}_{k-1}$ and $M_B \in \mathcal{M}_k$. Thus, at most $O(|\mathcal{M}_{k-1} \cup \mathcal{M}_{k-2}|)$ of the paths between matchings $M_A \in \mathcal{M}_{k-1}$ and $M_B \in \mathcal{M}_k$ contain t in the segment going from M_{AB} to M_B.

Case 2: t occurs in the fixing of an odd length segment of the path going from M_A to M_{AB}. In this case, $M_t \in \mathcal{M}_{k-1}$. Assume there exists another path in H_k, say from M'_A to M'_B, that contains t in the fixing of an odd length segment from M'_A to M'_{AB} and that determines the same M_t. This requires that $M'_A \Delta M'_B = M_A \Delta M_B$. We prove that the number of such paths is bounded by $l!(l+1)!$ where $l+1$ is the number of M'_A-augmenting odd length segments and l is the number of M'_A-deaugmenting odd length segments in $M'_A \Delta M'_B$. But each of these paths is chosen with probability $1/((l+1)!l!)$ according to the random ordering given to the l and $l+1$ deaugmenting and augmenting segments, and so the expected number of paths throughout t when M_t is specified is 1. However, $M_t \in \mathscr{M}_{k-1}$, and so the expected number of paths between $\mathscr{M}_{k-1}$ and $\mathscr{M}_k$ matchings that go through t, when t occurs in the fixing of an odd length segment, is $|\mathscr{M}_{k-1}|$.

From $t=(M_1, M_2)$ and $M_t=(M_1 \cup M_2)\Delta(M_A \Delta M_B)$ we can determine the odd length segment in $M_A \Delta M_B$, say P, whose fixing, in the path from M_A to M_B in H_k, contains the edge t. Construct the matching M' satisfying $M' \setminus (M' \cap P) = M_t \setminus (M_t \cap P)$ and, $M' \cap P = M_A \cap P$ if P is M_A-deaugmenting and $M' \cap P = M_B \cap P$ if P is M_A-augmenting (M' is a k matching). Intuitively, if t occurs in the fixing of an odd length segment that is M_1-deaugmenting, then M' is the matching "just before" fixing the odd length segment, and if t occurs in the fixing of an odd length segment which is M_1-augmenting, then M' is the matching "just after" fixing the odd length segment. We make the following observations: (1) all the M_A-augmenting segments (M_A-deaugmenting) fixed by the path going from M_A to M' are now M'-deaugmenting segmets (M'-augmenting), and all the M_A-augmenting segments (M_A-deaugmenting) to be fixed by the path going from M' to M_{AB} remain M'-augmenting (M'-deaugmenting); (2) t and M_t uniquely determine M' (obvious from its construction), and conversely (as can be easily verified) t, P and M' uniquely determine M_t. From the second observation it follows that any other path through t that determines the same M' and P must fix the same set of odd length segments as the path from M_A to M_B. It then follows from the first observation that we can partition the set of odd length segmets in $M_A \Delta M_B$ into l M'-augmenting segments and $l+1$ M'-deaugmenting segments such that all paths through t that specify the same M_t must start by alternately fixing M'-augmenting and M'-deaugmenting segments (starting with an M'-deaugmenting segment) until M' is reached, then fixing P, and finally alternately fixing M'-augmenting and M'-deaugmenting segments (starting with an M'-augmenting segment) until M_{AB} is reached. A simple counting argument shows that there are only $l!(l+1)!$ such paths.

The above results imply that the expected number of paths that go through an edge t in H_k is $O(|\mathscr{M}_{k-2} \cup \mathscr{M}_{k-1}|)$, as claimed. $\square$

We propose the Markov chain $\mathscr{MC}_k$ of Definition 2.4 as a sampling scheme for $\mathscr{N}_k$. It is straightforward to verify that Lemma 3.3 implies Theorem 3.4.

Theorem 3.4. *The conductance of $\mathscr{MC}_k$ is bounded by*

$$\Phi(H_k) = \Omega\left(\frac{1}{|E|} \cdot \frac{|\mathscr{M}_k|}{|\mathscr{N}_k|} \cdot \frac{|\mathscr{M}_{k-1}|}{|\mathscr{N}_{k-1}|}\right).$$

Remark. For $k=n$, Jerrum and Sinclair [11] obtained the same bound on the conductance. Probabilistic arguments similar to those used in the proof of Lemma 3.3 are used in [19] in studying the magnification properties of 0–1 polytopes.

4. An (ε, δ)-approximation algorithm for the permanent

We show how the results of Sections 2 and 3 can be combined to yield an (ε, δ)-approximation algorithm for the permanent that is fully polynomial provided the ratio $|\mathcal{M}_{n-1}|/|\mathcal{M}_n|$ is polynomially bounded. The construction of the algorithm and its analysis is divided into the following three steps.

Step A: *A ζ-sampling scheme for $\mathcal{N}_k$.*

We use the lower bound on the conductance of $\mathcal{MC}_k$ given by Theorem 3.4 and the bound on the rate of convergence of $\mathcal{MC}_k$ given in Proposition 2.5 to prove that $\mathcal{MC}_k$ is a fully polynomial ζ-sampling scheme when $|\mathcal{M}_{n-1}|/|\mathcal{M}_n|$ is polynomially bounded. Since the stationary distribution of $\mathcal{MC}_k$ is uniform, we have that $\min_{i \in \mathcal{N}_k} \pi_i = 1/\mathcal{N}_k > 1/n!$. Furthermore, using Proposition 2.6 the lower bound given in Theorem 3.5 can be expressed as

$$\Phi(H_k) = \Omega\left(\frac{1}{|E|} \frac{|\mathcal{M}_n|^2}{|\mathcal{M}_{n-1}|^2}\right).$$

From this and using Proposition 2.5 it is straightforward to show that after τ steps, where

$$\tau = \Omega\left(|E|^2 \cdot \frac{|\mathcal{M}_{n-1}|^4}{|\mathcal{M}_n|^4} \cdot (n \log n + \log \zeta^{-1})\right),$$

the Markov chain $\mathcal{MC}_k$ is a ζ-sampling scheme. Thus, when $|\mathcal{M}_{n-1}|/|\mathcal{M}_n|$ is polynomially bounded, the sampling scheme is fully polynomial.

Step B: *An (ε', δ)-approximation of $|\mathcal{M}_{k-1}|/|\mathcal{M}_k|$, where $\varepsilon' = \varepsilon/2n$.*

From Proposition 2.2, the stated approximation requires N_k calls to a ζ-sampling scheme of $\mathcal{N}_k$ where

$$N_k = c \cdot \frac{1}{(\varepsilon/2n)^2} \cdot \log \frac{1}{\delta} \cdot \max\left(\frac{|\mathcal{N}_k|}{|\mathcal{M}_k|}, \frac{|\mathcal{N}_k|}{|\mathcal{M}_{k-1}|}\right),$$

and

$$\zeta \leqslant \frac{\varepsilon}{3} \cdot \min\left(\frac{|\mathcal{M}_k|}{|\mathcal{N}_k|}, \frac{|\mathcal{M}_{k-1}|}{|\mathcal{N}_k|}\right).$$

Using Proposition 2.6, an upper bound on N_k is given by N where

$$N = c' \cdot \frac{1}{\varepsilon^2} \cdot \log \frac{1}{\delta} \cdot n^2 \cdot \max\left(\frac{|\mathcal{M}_{n-1}|}{|\mathcal{M}_n|}, |E|\right),$$

c' some constant, and

$$\zeta \leqslant \frac{\varepsilon}{3} \cdot \min\left(\frac{|\mathcal{M}_n|}{|\mathcal{N}_n|}, \frac{1}{|E|}\right).$$

Step C: Running time of algorithm.

Using the results given in steps A and B together with Proposition 2.1, the running time of the algorithm is given by

$$\sum_{k=2}^{n} \tau N_k \leqslant \tau N(n-1).$$

Remark. Jerrum and Sinclair, [11, 12] present two distinct (ε, δ)-approximation algorithms for the permanent problem. The first algorithm was originally proposed in [4], and subsequently the running time is analyzed in [11, 12]. The difference between the algorithm presented here and the one in [4, 11, 12] is in the sampling scheme used for $\mathcal{N}_k$ for $k<n$. In particular, simulating $\mathcal{MC}_k$ rather than reducing the problem to sampling $\mathcal{N}_n$ in a modified graph achieves a natural simplification of the algorithm, as well as a speedup in the running time. Broder [4] and Jerrum and Sinclair [11, 12] show that for any bipartite graph G on n vertices and any $1 \leqslant k \leqslant n$, the ratio $|\mathcal{M}_{k-1}|/|\mathcal{M}_k|$ can be obtained by sampling the space of perfect and almost perfect matchings in a modified bipartite graph, G_k on $n+k$ nodes. Specifically, for $1 \leqslant k \leqslant n$, they show a many-to-one reduction from the set of perfect and almost perfect matchings of G_k, $\mathcal{N}(G_k) = \mathcal{M}_{n+k}(G_k) \cup \mathcal{M}_{n+k-1}(G_k)$, to the set $\mathcal{N}' = (2k+1)\mathcal{M}_{n-k} \cup \mathcal{M}_{n-k+1} \cup (k+1)^2 \mathcal{M}_{n-k-1}$, where $\mathcal{M}_{-1}$ is defined to be the empty set. It follows that a uniform generator for elements of $\mathcal{N}(G_k)$ can be viewed as a uniform generator of $\mathcal{N}'$. In a fashion similar to the analysis of the running time of the simple algorithm presented in this section, the running time of the algorithm of [4, 11, 12] can be considered to arise from two contributions. The first comes from the time required for the random walk on $\mathcal{N}(G_k)$ to yield a ζ-sampling scheme. An analysis identical to the one presented in step A of this section shows that the minimum number of steps τ for which the Markov chain of [4, 11, 12] is a ζ-sampling scheme for all k is

$$\tau = \Omega\left(\max_{1 \leqslant k \leqslant n}\left[(|E|+2k^2)^2 \cdot \frac{|\mathcal{M}_{n+k-1}(G_k)|^4}{|\mathcal{M}_{n+k}(G_k)|^4}\right] \cdot (n \log n + \log \zeta^{-1})\right).$$

However, Broder [4] shows that

$$\frac{|\mathcal{M}_{n+k-1}(G_k)|}{|\mathcal{M}_{n+k}(G_k)|} = 2k + \frac{|\mathcal{M}_{n-k+1}|}{|\mathcal{M}_{n-k}|} + (k+1)^2 \cdot \frac{|\mathcal{M}_{n-k-1}|}{|\mathcal{M}_{n-k}|},$$

and we need to find an upper bound for all k on the product of this term with $(|E|+2k^2)^{1/2}$ in order to obtain a value for τ. Using Proposition 2.6 one can show that the dominant term in the expression for

$$\frac{|\mathcal{M}_{n+k-1}(G_k)|}{|\mathcal{M}_{n+k}(G_k)|}$$

is

$$(k+1)^2 \cdot \frac{|\mathcal{M}_{n-k-1}|}{|\mathcal{M}_{n-k}|}.$$

Thus it suffices to have an upper bound for all k on the product of this term with $(|E|+2k^2)^{1/2}$ in order to obtain a value for τ. However, Jerrum and Sinclair do not give this bound.

The second contribution to the running time comes from the number N_k of calls to a ζ-sampling scheme of $\mathcal{N}'$ that is required to obtain an (ε', δ)-approximation of the ratio $|\mathcal{M}_{n-k-1}|/|\mathcal{M}_{n-k}|$, where $\varepsilon' = \varepsilon/2n$. By an argument similar to step B of this section, we get that

$$N_k = c \cdot \frac{1}{(\varepsilon/2n)^2} \cdot \log\frac{1}{\delta} \cdot \max\left(\frac{|\mathcal{N}'|}{(2k+1)|\mathcal{M}_{n-k}|}, \frac{|\mathcal{N}'|}{(k+1)^2|\mathcal{M}_{n-k-1}|} \right).$$

One can show that an upper bound on N_k is given by N where

$$N = c' \cdot \frac{1}{\varepsilon^2} \cdot \log\frac{1}{\delta} \cdot n^2 \cdot \max\left(\max_{0 \leqslant k \leqslant n-1} \left[\frac{(k+1)^2}{2k+1} \cdot \frac{|\mathcal{M}_{n-k-1}|}{|\mathcal{M}_{n-k}|} \right], |E| \right).$$

Once again, we need an upper bound for all k on

$$\frac{(k+1)^2}{2k+1} \cdot \frac{|\mathcal{M}_{n-k-1}|}{|\mathcal{M}_{n-k}|},$$

but this is not provided in [11, 12].

The second (ε, δ)-approximation algorithm presented in [11, 12] for the permanent uses a Markov chain on the space of all matchings to generate matchings of a graph G uniformly. The estimate of the ratios $|\mathcal{M}_{k-1}|/|\mathcal{M}_k|$ requires observing the relative numbers of $(k-1)$- and k-matchings. Since these may be an exponentially small fraction of the size of the space of all matchings, weights are assigned to these matchings so that the probability of being at a size k- or $(k-1)$-matching is at least $1/(n+1)$ and $|\mathcal{M}_n|/(|E| \cdot (n+1)|\mathcal{M}_{n-1}|)$ respectively. In the simple algorithm presented in this paper, sampling of the space of $(k-1)$- and k-matchings is done directly via the Markov chain $\mathcal{MC}_k$. Now, the probability of being at a size k- or $(k-1)$-matching is $|\mathcal{M}_k|/|\mathcal{N}_k|$ and $|\mathcal{M}_{k-1}|/|\mathcal{N}_k|$ respectively. Using Propositions 2.2 and 2.6 one can readily conclude that the simple algorithm achieves an $\mathrm{O}(n \cdot \min(|\mathcal{M}_{n-1}|/|\mathcal{M}_n|, |E|))$ running time improvement over the latter algorithm of Jerrum and Sinclair.

5. Approximating the permanent of graphs with large factors

Let $\mathscr{L}_f^{2n}$ be the set of bipartite graphs on $2n$ vertices that have factor size at least f.

Definition 5.1. For every constant $\alpha>0$, $\mathscr{L}_\alpha=\bigcup_{n\geqslant 1}\mathscr{L}_{\alpha n}^{2n}$ is a *large factor class.*

The main result in this section, Theorem 5.2, states that if $G\in\mathscr{L}_f^{2n}$, then every almost perfect matching has an augmenting path of length at most $6n/f$ to a perfect matching. Theorem 5.3 shows that the bound on the length of the minimum augmenting path is tight: for infinitely many f and infinitely many n, there exists graphs in $\mathscr{L}_f^{2n}$ that contain minimum augmenting paths of length $6n/f\cdot(1-\mathrm{O}(1/f))$.

By Theorem 5.2 and Proposition 5.4, $|\mathscr{M}_{n-1}|/|\mathscr{M}_n|\leqslant n^{3n/f}$. Proposition 5.5 shows that the factor size of G can be computed in polynomial time. Combining this, there is an easily computable upper bound on the running time of the simple algorithm. Furthermore, this upper bound is polynomial for any large factor class of bipartite graphs, and is the smallest easily computable upper bound for bipartite graphs that have factor size at least $c\log n$ for some small constant $c>0$.

Theorem 5.2. *Let $G=(U,V,E)$, $|U|=|V|=n$, be a bipartite graph that has factor size at least f. Then, for every $M\in\mathscr{M}_{n-1}$ there is an augmenting path to a perfect matching of length at most $6n/f$.*

Proof. Let $N(X)=\{y\colon (x,y)\in E,\ x\in X\}$ and let $e(X,Y)$ be the number of edges in G with one endpoint in X and the other endpoint in Y.

We first prove the theorem for f-regular bipartite graphs and then generalize to bipartite graphs that have factor size at least f. (Notice that G is f-regular if and only if the edges can be partitioned into exactly f edge disjoint perfect matchings.) Let $M\in\mathscr{M}_{n-1}$ and let $u\in U$, $v\in V$ be the unmatched vertices in M. We define $\{U_i\subseteq V\mid i\in[n]\}$ and $\{V_i\subseteq V\mid i\in[n]\}$ as follows: $U_1=N(\{v\})$ and V_1 is the set of vertices matched to U_1 in M. For all $i>1$, $U_i=N(V_{i-1})$ and V_i is the set of vertices matched to U_i in M. It is straightforward to verify that for all $i\geqslant 1$, $u\in U_i$ implies that there exists an augmenting path for M of length at most $2i-1$. The heart of the proof (which we show below) is that for all $i\geqslant 1$, $u\notin U_{i+3}$ implies that $|U_{i+3}-U_i|\geqslant f$. From this fact and because $|U_3|\geqslant|U_2|\geqslant|U_1|\geqslant f$, it follows that for all $i\geqslant 1$, $u\notin U_i$ implies that $|U_i|\geqslant\lceil i/3\rceil f$. Therefore for $i>3\lfloor n/f\rfloor$, $u\in U_i$ and the theorem follows.

Fix $i\geqslant 1$ and assume $u\notin U_{i+3}$. Let $A_1=U_{i+1}-U_i$, $A_2=U_{i+2}-U_{i+1}$, $A_3=U_{i+3}-U_{i+2}$, and similarly let $B_1=V_{i+1}-V_i$, $B_2=V_{i+2}-V_{i+1}$, $B_3=V_{i+3}-V_{i+2}$. Because $u\notin U_{i+3}$, all vertices in U_{i+3} are matched in M and thus $|U_i|=|V_i|$, $|A_1|=|B_1|$, $|A_2|=|B_2|$ and $|A_3|=|B_3|$. Since G is f-regular, the total number of edges out of A_2 is $f|A_2|$. We now count the edges out of A_2 in a different way. By construction, no edge out of A_2 goes into V_i. Thus, the total number of edges out of A_2 is $e(A_2,B_1)+e(A_2,B_2)+e(A_2,V-V_{i+2})$. We derive an upper bound on this quantity as follows. Clearly, $e(A_2,V-V_{i+2})\leqslant e(U_{i+2},V-V_{i+2})$ because $A_2\subseteq U_{i+2}$. But

$e(U_{i+2}, V-V_{i+2})=e(U-U_{i+2}, V_{i+2})$ because G is f-regular. On the other hand, $e(U-U_{i+2}, V_{i+2})=e(A_3, B_2)$ by construction. From this we conclude that

$$f|A_2| \leqslant e(A_2, B_1)+e(A_2, B_2)+e(A_3, B_2).$$

Using the upper bounds $e(A_2, B_1) \leqslant |A_2||B_1|$, $e(A_2, B_2) \leqslant |A_2||B_2|$ and $e(A_3, B_2) \leqslant |A_3||B_2|=|A_2||B_3|$, we get that $f|A_2| \leqslant |A_2|(|B_1|+|B_2|+|B_3|)$ and thus $|U_{i+3}-U_i| \geqslant f$.

Now consider the case when G has factor size at least f. Let F be the set of edges in the f edge disjoint perfect matchings and as before we let $M \in \mathscr{M}_{n-1}$, u be the unmatched vertex in U and v be the unmatched vertex in V. Let $G'=(U, V, F \cup M)$. The only additional observation needed in the proof is the following. For any $X \subseteq U-\{u\}$ let Y be the vertices matched to X in M. Then, with respect to the graph G', $e(X, V-Y)=e(U-X, Y)$. This is because for any $e=(x,y) \in M$, either $e \in F$ in which case both x and y are of degree f in G', or $e \notin F$ and both x and y are of degree $f+1$ in G'. After this, the outline of the proof for the case when G is f-regular can be used to show that there is an augmenting path of length at most $6n/f$ in G'. □

Theorem 5.3. *For each of infinitely many values of f there are infinitely many values of n for which there exist $G \in \mathscr{L}_f^{2n}$ that contain minimum augmenting paths of length $6n/f \cdot (1-\mathrm{O}(1/f))$.*

Proof. Given any two positive integers j and k, $f=9j-3$ and $n=6jk+18j+1$, we construct f-regular bipartite graphs $G=(U, V, E)$ on $2n$ vertices that contain minimum augmenting paths of length $4k+5=6(n-1)/(f+3)-7$.

For any $X \subset U$, $Y \subset V$, let $G(X, Y)$ be the subgraph of G induced by X and Y. $K_{3j,3j}$ is the complete bipartite graph on $6j$ vertices, 3-FACTOR refers to some fixed 3-factor of $K_{3j,3j}$ and 3-EDGES refers to some fixed 3 edges in 3-FACTOR (it does not matter which 3-factor or which 3 edges from the 3-factor we choose). Partition U (V) into the disjoint sets $U_0, \ldots, U_{2k+2}$ ($V_0, \ldots, V_{2k+2}$), where U_0 and U_{2k+1} (V_0 and V_{2k+1}) contain $9j$ vertices, U_i (V_i) for $i=1, \ldots, 2k$ contain $3j$ vertices each, and $U_{2k+2}=\{u\}$ ($V_{2k+2}=\{v\}$). The vertices are connected as follows: For $i=1, \ldots, 2k$ we have $G(U_i, V_i)=K_{3j,3j}$. For odd $i=1,3, \ldots, 2k-1$ we have $G(U_i, V_{i+1})=K_{3j,3j}-$ 3-FACTOR, and $G(U_{i+1}, V_i)=K_{3j,3j}-$3-EDGES. For odd $i=3,5, \ldots, 2k-1$ we have $G(U_i, V_{i-1})=K_{3j,3j}$, and $G(U_{i-1}, V_i)=K_{3j,3j}-$3-FACTOR$+$3-EDGES. (See Fig. 1.) For $i=2, \ldots, 2k-1$, the vertices in U_i and V_i are not adjacent to any other vertices, and thus their degree is $9j-3$. We connect the $3j$ vertices in U_1 (V_{2k}) to the $9j$ vertices in V_0 (U_{2k+1}) with $(3j)^2$ edges such that we add degree $3j$ to each vertex in U_1 (V_{2k}) and each vertex in V_0 (U_{2k+1}) has degree j. We connect the $3j$ vertices in V_1 (U_{2k}) to the $9j$ vertices in U_0 (V_{2k+1}) with $(3j)^2-(9j-3)$ edges as follows. First we add a set S of $(3j)^2$ edges such that each vertex in V_1 (U_{2k}) has additional degree $3j$ and each vertex in U_0 (V_{2k+1}) has degree j. Next we delete $9j-3$ edges from S chosen to satisfy the

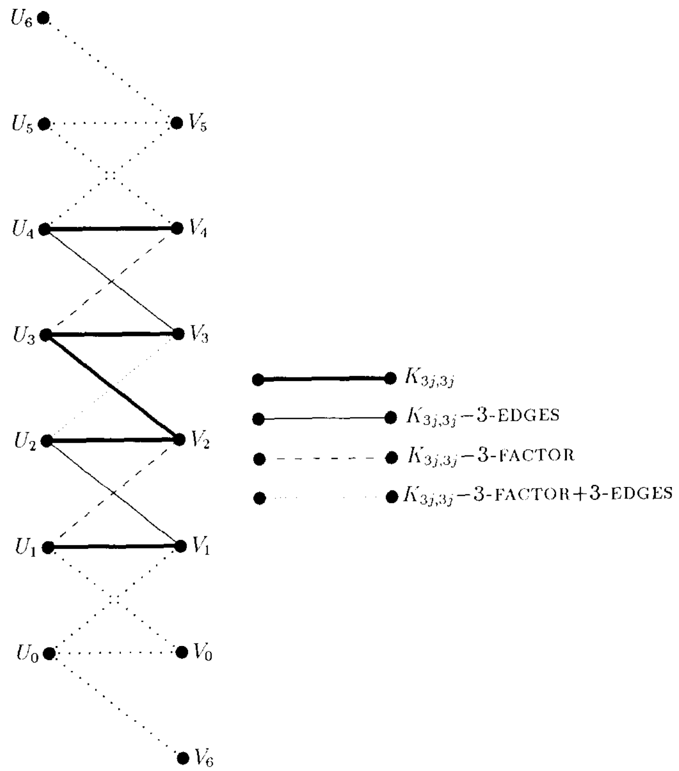

Fig. 1. Graph G with $k=2$ and $n=30j+1$.

following two conditions: (i) the $9j-3$ edges cover $9j-3$ vertices in U_0 (V_{2k+1}), and (ii) the $9j-3$ edges together with the 3-EDGES deleted from $G(U_2, V_1)$ ($G(U_{2k}, V_{2k+1})$) cover each vertex in V_1 (U_{2k}) three times, i.e. the deletion of the $9j-3$ edges from S and 3-EDGES from $G(U_2, V_1)$ ($G(U_{2k}, V_{2k+1})$) reduces the degree of every vertex in V_1 (U_{2k}) by three. The resulting degree of the vertices in U_1, V_1, U_{2k} and V_{2k} is $9j-3$ and the degree of the vertices in V_0 and U_{2k+1} is j. From condition (i) above, there are $9j-3$ vertices in U_0 (V_{2k+1}) that have degree $j-1$ and 3 vertices having degree j. Connect these $9j-3$ vertices to $v \in V_{2k+2}$ ($u \in U_{2k+2}$), making the degree of these $9j-3$ vertices in U_0 (V_{2k+1}) j, and the degree of u and v $9j-3$. To complete the construction, connect the vertices in U_0 to V_0 and U_{2k+1} to V_{2k+1} with $9j(8j-3)$ edges so that they each have total degree $9j-3$. The theorem follows because the almost perfect matchings where u and v are unmatched, and where the vertices in each U_i match exclusively with the vertices in V_i, have minimum augmenting paths of the required length. □

Proposition 5.4. *Let $G=(U, V, E)$ be a bipartite graph with $|U|=|V|=n$. Suppose that for every $M \in \mathcal{M}_{n-1}$ there is an augmenting path to a perfect matching of length at most $2l-1$. Then, $|\mathcal{M}_{n-1}|/|\mathcal{M}_n| < n^l$.*

Proof. Let $M \in \mathcal{M}_n$ and $M' \in \mathcal{M}_{n-1}$ be such that there is an augmenting path of length $2k-1$ from M' to M. Any such M' is uniquely defined by M, the k edges of M contained in the augmenting path and the order of the k edges in the path, and thus there are at most $\binom{n}{k}k!$ matchings M' in $\mathcal{M}_{n-1}$ with an augmenting path of length $2k-1$ to M. Thus, there are at most $\sum_{k=0}^{l}\binom{n}{k}k! < n^l$ matchings in $\mathcal{M}_{n-1}$ with an augmenting path to a perfect matching of length at most $2l-1$. $\square$

Proposition 5.5. *Let $G=(U, V, E)$ be a bipartite graph with $|U|=|V|=n$ and $|E|=m$. The largest factor of G can be found in time* $O(nm \log n \log\log n)$.

Proof. Construct the bipartite graph G' with bipartitions $U \cup \{t\}$ and $V \cup \{s\}$, and edge set $E \cup \{s\} \times U \cup \{t\} \times V$. We make G' into a network with source s, sink t, and edge capacity 1 for every edge in E and capacity f for every other edge. It is straightforward to verify that the network on G' has a maximum flow of fn if and only if G has factor size at least f. But maximum network flow for an input graph with $2n$ vertices, m edges and maximum capacity f can be solved in $O(nm \log(n/m\sqrt{\log f}+2))$, [1], which is $O(nm \log\log n)$ for the above network since $f \leqslant n$. Using binary search on f, the result follows. $\square$

Definition 5.6. Let $G=(U, V, E)$ be a bipartite graph, $|U|=|V|=n$, and $1 \leqslant f \leqslant g \leqslant n$ any two integers. An *(f, g)-factor* of G is a subgraph of G such that for every vertex $v \in V$, $f \leqslant \deg(v) \leqslant g$, where $\deg(v)$ is the degree of v *in the subgraph.*

Let $\mathcal{L}^{2n}_{[f,g]}$ be the set of bipartite graphs on $2n$ vertices that have an (f, g)-factor.

Theorem 5.7 generalizes Theorem 5.2 to the class of bipartite graphs that contain an (f, g)-factor (see Definition 5.6), and Theorem 5.8 is the analog of Theorem 5.3 for this class. The proofs of these theorems are similar in nature to the proofs of Theorems 5.2 and 5.3 and can be found in [7].

For any bipartite graph G, determining whether G contains an (f, g)-factor can be done in polynomial time: G has an (f, g)-factor iff the network in the proof of Proposition 5.5 has a feasible flow when the capacity of the edges adjacent to the source and the sink have a lower bound of f and an upper bound of g. This problem is reducible to finding a circulation in a network with lower and upper bounds on the capacities of the edges, and is solvable in polynomial time (see [16, pp. 139 and problem 9.1]).

Because the condition that a bipartite graph contain an (f, g)-factor can be verified in polynomial time, there is a polynomially computable upper bound on the running time of the simple algorithm based on finding the (f, g)-factor minimizing the function $L(f, g, n)$ of Theorem 5.7. Furthermore this upper bound is polynomial for the class of graphs containing an $(\alpha n, \alpha n+c)$-factor, for some constants $0<\alpha \leqslant 1$ and $c \geqslant 0$.

Theorem 5.7. *Let $G=(U, V, E)$, $|U|=|V|=n$, be a bipartite graph that contains an (f, g)-factor. Then, for every $M \in \mathscr{M}_{n-1}$ there is an augmenting path to a perfect matching of length at most*

$$L(f, g, n)=12 \cdot \frac{n}{f+g}+4 \cdot (g-f)\left(\frac{n}{g+f}\right)^2.$$

Theorem 5.8. *For any constant $0<\alpha<\frac{1}{2}$ and for all sufficiently large values of n, there are values of f and g for which there exist $G \in \mathscr{L}^{2n}_{[f,g]}$ that contain minimum augmenting paths of length at least $\alpha \cdot L(f, g, n)$.*

6. Completeness results for the permanent problem

One of the main implications of the preceding section is that, for any large factor class, there is a fully polynomial approximation algorithm for perfect matchings. This section is devoted to giving complementary completeness results to this implication. We prove four types of results for classes of bipartite graphs: (1) the exact counting problem is $\sharp$**P**-complete; (2) the approximate counting problem is approximation complete for the permanent; (3) the ratio $|\mathscr{M}_{n-1}|/|\mathscr{M}_n|$ is not polynomially bounded; (4) the ratio $n!/|\mathscr{M}_n|$ is not polynomially bounded. The first type of result is used to justify developing efficient approximation algorithms for classes of graphs, since it is unlikely that there is any fast algorithm for exact counting. The second type of result is used to pinpoint classes of graphs for which the approximation problem is as hard as the general problem (which is still an open problem!). The third type of result is used to identify classes of graphs for which the simple algorithm does not run in polynomial time. The fourth type of result is used to describe classes of graphs for which the naive algorithm does not run in polynomial time, thus justifying the development of a different approximation algorithm. In all cases considered below, a type (2) result implies a type (3) result and a type (3) result implies a type (4) result.

Definition 6.1. Let $\mathscr{C}$ be a class of bipartite graphs. We say that $\mathscr{C}$ is *approximation complete for the permanent* if we can show that there is a fully polynomial approximation algorithm B for all bipartite graphs given a fully polynomial approximation algorithm A for $\mathscr{C}$. On input bipartite graph G, B typically constructs a graph $C \in \mathscr{C}$ and runs A on input C. From the approximation of the number of perfect matchings in C produced by A, B produces an approximation of the number of perfect matchings in G.

In this section we let $|\mathscr{M}(G)|$ represent the number of perfect matchings in G.

Theorem 6.2. (1) *Exact counting of perfect matchings in* 3*-regular bipartite graphs is* $\sharp$**P**-*complete.*

(2) *The class of 3-regular bipartite graphs is approximation complete for the permanent.*

Proof. We prove (1) by reducing exact counting in general bipartite graphs (proved to be $\sharp$**P**-complete by Valiant [22]) to counting in 3-regular bipartite graphs. (2) follows because the reduction is parsimonious.

Let $G=(U,V,E)$, $|U|=|V|=n$, $|E|=m$ and without loss of generality no vertex has degree 0 or 1 in G (and thus $m\geqslant 2n$). We first use a construction of Dahlhaus and Karpinski [9] to construct from G a graph $G'=(X,Y,F)$, $|X|=|Y|=2m-n$, with $m+n$ vertices of degree 2 and $m-2n$ vertices of degree 3 in each bipartition such that there is a 1–1 correspondence between perfect matchings in G' and perfect matchings in G. G' is constructed as follows:

For each $u\in U$ and $v\in V$ let $A_u=\{a_1^u,\dots,a_{d(u)}^u\}$, $S_u=\{s_1^u,\dots,s_{d(u)-1}^u\}$ and $B_v=\{b_1^v,\dots,b_{d(v)}^v\}$, $T_v=\{t_1^v,\dots,t_{d(v)-1}^v\}$, where $d(u),d(v)$ are the degrees of u and v in G. Let $X=\bigcup_{u\in U}A_u\cup\bigcup_{v\in V}T_v$ and $B=\bigcup_{v\in V}B_v\cup\bigcup_{u\in U}S_u$. For each $u\in U$ ($v\in V$) F contains the edges (a_i^u,s_i^u) and (s_i^u,a_{i+1}^u) for $i=1,\dots,d(u)-1$ ((b_i^v,t_i^v) and (t_i^v,b_{i+1}^v) for $i=1,\dots,d(v)-1$). For each edge $(u,v)\in E$, connect a vertex $a_i^u\in A_u$ to a vertex $b_j^v\in B_v$ where a distinct a_i^u (b_j^v) is chosen for each edge in E with endpoint u (v). (See Fig. 2.)

From G' we construct a 3-regular graph G'' as follows. Let K be the graph constructed from $K_{3,3}$ by deleting some edge, and shown in Fig. 3. ($K_{3,3}$ is the complete bipartite graph with three vertices in each partition.) Let $K_1,\dots,K_p$ be p disjoint copies of K, and let $u(K_i),v(K_i)$ be the vertices of degree 2 in each K_i. (The value of p is appropriately chosen below.) Let the graph H have vertex set the union of the vertices in $K_1,\dots,K_p$, and edge set the union of the edges in $K_1,\dots,K_p$ together with the edges $(v(K_i),u(K_{i+1}))$ $i=1,\dots,p-1$. Observe that H has two vertices of degree 2, and all other vertices are of degree 3. (See Fig. 3.)

Let $H_1,\dots,H_{m+n}$ be $m+n$ disjoint copies of H, and let $u(H_i),v(H_i)$ be the vertices of degree 2 in each H_i. Let $x_1,\dots,x_{m+n}$ and $y_1,\dots,y_{m+n}$ be the vertices of degree 2 of G' in

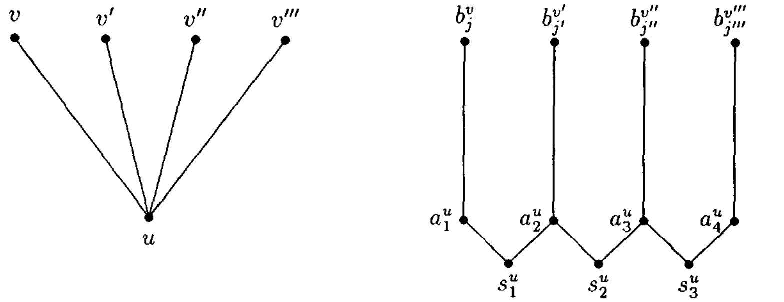

Fig. 2. A vertex u and its neighbors in G, and the sets of vertices A_u and S_u and their neighbors in G'.

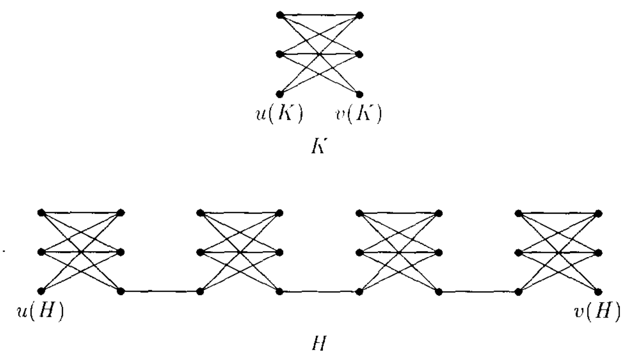

Fig. 3. Graph K and graph H when $p=4$.

X and Y, respectively. The graph G'' has vertex set the union of the vertices in G' and $H_1, \ldots, H_{m+n}$, and edge set the union of the edges in $G', H_1, \ldots, H_{m+n}$ together with the edges $(x_i, u(H_i))$ and $(y_i, v(H_i))$ for $i=1, \ldots, m+n$. G'' is bipartite.

We make the following observations: (1) each H_i has 4^p perfect matchings; (2) the graph H_i with the vertices $u(H_i), v(H_i)$ deleted has only 2^p perfect matchings; (3) for each i, every perfect matching of G'' either contains both edges $(x_i, u(H_i))$ and $(y_i, v(H_i))$ or neither edge. Partition the perfect matchings in G'' as follows: let $\mathscr{P}$ be the set of perfect matchings that, for all i, do not contain edges $(x_i, u(H_i))$ or $(y_i, v(H_i))$; let $\mathscr{Q}$ be the set of perfect matchings that, for some i, contains both edges $(x_i, u(H_i))$ and $(y_i, v(H_i))$. For every $M \in \mathscr{P}$, $M \cap E(G')$ is a perfect matching in G'. Thus, from observation (1), $|\mathscr{P}| = 4^{p(m+n)} |\mathscr{M}(G')|$. For every matching $M \in \mathscr{Q}$, $M \cap E(G')$ is a matching that is not perfect in G', and there is an $x_i, y_i \in V(G')$ such that $(x_i, u(H_i)), (y_i, v(H_i)) \in M$. Thus, M defines a perfect matching on the subgraph $H_i - u(H_i) - v(H_i)$, i.e. the graph H_i with the vertices $u(H_i), v(H_i)$ deleted. There are at most $5m$ edges in G' and, therefore, there are at most 2^{5m} matchings in G'. Thus, it follows from observation (2) that $|\mathscr{Q}| \leqslant (4^{p(m+n)}/2^p) \cdot 2^{5m}$. But then $4^{p(m+n)} |\mathscr{M}(G')| \leqslant |\mathscr{M}(G'')| \leqslant 4^{p(m+n)} |\mathscr{M}(G')| + 4^{p(m+n)} \cdot 2^{5m-p}$, and thus for $p = 5m+2$, $|\mathscr{M}(G')| = \lfloor \mathscr{M}(G'')|/|4^{p(m+n)} \rfloor$. $\square$

Remark. By suitably modifying the construction given in the proof of Theorem 6.2, it can be shown that Theorem 6.2 holds for any f-regular bipartite graph on $2N$ vertices with $3 \leqslant f \leqslant N^{1/(2+\varepsilon)}$, for any $\varepsilon > 0$. The basic idea is to construct K from $K_{f,f}$ by deleting one edge, and proceed as before, choosing the values of f and p appropriately.

Lemma 6.3. *Let $G=(U, V, E)$ be a 3-regular bipartite graph with $|U| = |V| = l$. For any m divisible by 3, we can construct an m-regular bipartite graph $G' = (U', V', E')$ with $|U'| = |V'| = lm$ such that the ratio of the number of perfect matchings in G' to perfect matchings in G is an easily computable function of m and l.*

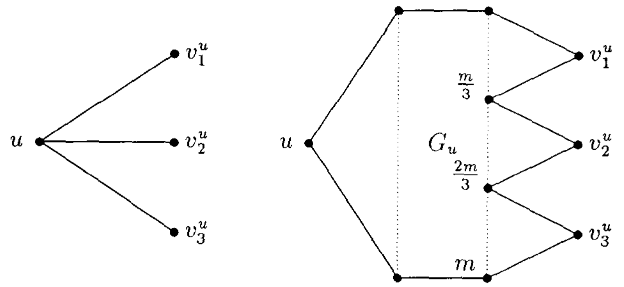

Fig. 4. The vertex u and its neighbors in the 3-regular graph G, and the same vertices in the m-regular graph G'.

Proof. For each $u\in U$, let $G_u=(X_u, Y_u, E_u)$ be an $(m-1)$-regular bipartite graph with $|X_u|=|Y_u|=m$. Let $U'=U\cup\{Y_u\colon u\in U\}$ and $V'=V\cup\{X_u\colon u\in U\}$. For all $u\in U$ let v_1^u, v_2^u, v_3^u be the neighbors of u in G. The edges in G' are, for all $u\in U$, E_u, an edge from u to each $x\in X_u$, an edge from each of the first $m/3$ vertices of Y_u to v_1^u, an edge from each of the next $m/3$ vertices of Y_u to v_2^u and an edge from each of the last $m/3$ vertices of Y_u to v_3^u. (See Fig. 4.) Let D_k be the number of perfect matchings in a $(k-1)$-regular bipartite graph on $2k$ vertices (D_k is the number of *derangements* on k objects). It can be shown that each perfect matching M of G gives rise to $[m/3(D_m+D_{m-1})]^l$ perfect matchings M' in G' such that if $(u,v)\in M$ then in M' v is matched to some vertex in Y_u. Moreover, it is easy to verify that all perfect matchings in G' are of this form. □

Theorem 6.4. *For any* $\alpha<1$:

(1) *Exact counting of perfect matchings in* $n^{1-\alpha}$*-regular bipartite graphs is* ♯**P**-*complete.*

(2) *The class of* $n^{1-\alpha}$*-regular bipartite graphs is approximation complete for the permanent.*

Proof. Use Theorem 6.2 and Lemma 6.3. Setting $m=l^{(1-\alpha)/\alpha}$ and $n=lm$ yields $m=n^{1-\alpha}$. □

It is not hard to show that, for all $\alpha>0$ and for infinitely many n, there are $n^{1-\alpha}$-regular bipartite graphs for which $|M_{n-1}|/|M_n|>n^c$ for all $c>0$.

Proposition 6.5. *Let G be a bipartite graph on $2n$ vertices and $\bar{G}$ its complement. Let $\mathcal{M}_k$ be the matchings of size k in G and $\bar{\mathcal{M}}_k$, the matchings of size k in $\bar{G}$. There is an algorithm that on inputs $|\bar{\mathcal{M}}_j|$, $j=0,\ldots,n$, computes $|\mathcal{M}_k|$, $k=0,\ldots,n$, in polynomial time.*

Proof. The proof is a straightforward application of inclusion–exclusion. Let S be the set of $n!$ perfect matchings in the complete bipartite graph on $2n$ vertices. For any bipartite graph G on $2n$ vertices, let $1,\ldots,m$ be an indexing of the edges in G. Define the *G-properties* $\mathscr{P}_G = \{P_1, \ldots, P_m\}$ on S, where P_i holds for $M \in S$ iff M contains the edge in G with index i. Let $W_G(P_{i_1}, \ldots, P_{i_k})$ be the number of elements in S that satisfy all of the G-properties $P_{i_1}, \ldots, P_{i_k}$. Clearly, $W_G(P_{i_1}, \ldots, P_{i_k}) = (n-k)!$ if the set of edges with indices $i_1, \ldots, i_k$ is a k matching in G, and 0 otherwise. For all $k = 0, \ldots, n$, let $W_G(k) = \sum_{i_1, \ldots, i_k} W_G(P_{i_1}, \ldots, P_{i_k}) = (n-k)!\,|\mathscr{M}_k|$. To prove the proposition, it suffices to show that for $k = 0, \ldots, n$, $W_G(k)$ can be computed in polynomial time given the inputs $W_{\bar{G}}(j)$, $j = 0, \ldots, n$, where $\bar{G}$ is the complement of G. For all $k = 0, \ldots, n$, let $E_G(k)$ ($E_{\bar{G}}(k)$), be the number of elements in S that satisfy *exactly* k G-properties (k $\bar{G}$-properties). By definition,

$$W_G(k) = E_G(k) + \binom{k+1}{k} E_G(k+1) + \binom{k+2}{k} E_G(k+2) + \cdots + \binom{n}{k} E_G(n). \qquad (*)$$

Clearly, an analogous relation holds between $W_{\bar{G}}(k)$ and $E_{\bar{G}}(k)$, and from the principle of inclusion–exclusion,

$$E_{\bar{G}}(k) = W_{\bar{G}}(k) - \binom{k+1}{k} W_{\bar{G}}(k+1) + \binom{k+2}{k} W_{\bar{G}}(k+2) - \cdots + (-1)^{n-k} \binom{n}{k} W_{\bar{G}}(n). \qquad (**)$$

Thus, given the inputs $W_{\bar{G}}(k)$, $k = 0, \ldots, n$, from (**) we can efficiently compute $E_{\bar{G}}(k)$, $k = 0, \ldots, n$. However, observe that $E_{\bar{G}}(k) = E_G(n-k)$, and therefore from (*) it follows that $W_G(k)$, $k = 0, \ldots, n$, can also be efficiently computed. □

Theorem 6.6. *For any constant f, the exact counting of perfect matchings in $(n-f)$-regular bipartite graphs on $2n$ vertices is* $\sharp$**P**-*complete.*

Proof. Theorem 6.2 shows that the exact counting of perfect matchings in f-regular bipartite graphs is $\sharp$**P**-complete. From Proposition 6.5, computing the number of perfect matchings of f-regular bipartite graphs on $2n$ vertices is reducible to computing the number of k-matchings of the complement $(n-f)$-regular bipartite graph $G = (U, V, E)$, for $k = 1, \ldots, n$. We show that for $k = 1, \ldots, n$ and f constant, computing the number of k matchings of any $(n-f)$-regular bipartite graph G on $2n$ vertices, $|\mathscr{M}_k(G)|$, is reducible to computing the number of perfect matchings of $(N-f)$-regular bipartite graphs on $2N$ vertices, where $N \in n^{O(1)}$.

Let $H = (X, Y, F)$ be a $(m-f)$-regular bipartite graph on $2m$ vertices. We define the *join*, $G + H$, of G and H, where $(U \cup V) \cap (X \cup Y) = \emptyset$, as the bipartite graph with bipartitions $U \cup X, V \cup Y$ and edge set $E \cup F \cup \{(x,y)\colon x \in U, y \in Y\} \cup \{(x,y)\colon x \in X, y \in V\}$. Note that the join $G+H$ is $(n+m-f)$-regular. It is straightforward to show that

$$|\mathcal{M}_{n+m}(G+H)| = \sum_{k=0}^{\min\{m,n\}} |\mathcal{M}_{n-k}(G)| \cdot |\mathcal{M}_{m-k}(H)| \cdot (k!)^2. \qquad (*)$$

We show that we can construct $n+1$ bipartite graphs H_i, $i=0,\ldots,n$ that satisfy the following properties: (1) for $i=0,\ldots,n, H_i$ is $(fi-f)$-regular on $2fi$ vertices; (2) for all $i=0,\ldots,n$, f and $k=0,\ldots,fi$, $|\mathcal{M}_k(H_i)|$ can be computed in time polynomial in n. Suppose for $i=0,\ldots,n$ we can compute $|\mathcal{M}_{n+fi}(G+H_i)|$, i.e. the number of perfect matchings in $G+H_i$. Then, for each i, $(*)$ gives us a linear equation in $n+1$ variables $|\mathcal{M}_k(G)|$, $k=0,\ldots,n$, with easily computable coefficients. From $n+1$ linearly independent such equations (it can be easily checked that the equations we get are linearly independent) we can solve for the $n+1$ variables, and from Proposition 6.5 this yields a way to count the number of perfect matchings in the original f-regular graph.

Let $\bar{H}_i$ be the bipartite graph on $2fi$ vertices constructed from the disjoint union of i copies of $K_{f,f}$. The graphs $H_0,\ldots,H_n$ are defined as the complement of the graphs $\bar{H}_0,\ldots,\bar{H}_n$. $H_0,\ldots,H_n$ are $(fi-f)$-regular on $2fi$ vertices as required. From Proposition 6.5, for all $i=0,\ldots,n$, a polynomial time computation of $|\mathcal{M}_k(\bar{H}_i)|$, for all $k=0,\ldots,fi$ implies a polynomial time computation of $|\mathcal{M}_k(H_i)|$, for all $k=0,\ldots,fi$. Let $F(k,i)=|\mathcal{M}_k(\bar{H}_i)|$. Since $\bar{H}_0$ is the empty graph, we define $F(0,0)=1$ and for all $k\geqslant 1$, $F(k,0)=0$. Then, $F(k,i)=\sum_{j=0}^{\min\{f,k\}}\binom{f}{j}^2 j!\,F(k-j,i-1)$. Thus, $F(k,i)$ can be computed in time polynomial in k and i using dynamic programming. $\square$

The reduction of Theorem 6.6 is rather indirect and does not show in any way that for constant f a fully polynomial approximation algorithm for $(n-f)$-regular bipartite graphs implies a fully polynomial approximation algorithm for the permanent. In fact, the following naive algorithm is a fully polynomial approximation algorithm for this class of bipartite graphs.

Definition 6.7. The *naive algorithm* for approximating the number of perfect matchings in a graph G is the (ε,δ)-approximation algorithm that generates uniformly at random perfect matchings of $K_{n,n}$ and estimates the fraction of these perfect matchings that are also perfect matchings of G.

A matrix is called doubly stochastic if it is nonnegative and all its row sums and column sums are 1. If A is a doubly stochastic $n\times n$ matrix, then the proof of the van der Waerden conjecture [10] asserts that the permanent of A is at least $n!/n^n$.

Proposition 6.8. *Let* $G\in\mathcal{L}_d^{2n}$. *Then*

$$|\mathcal{M}_n| \geqslant n!\left(\frac{d}{n}\right)^n.$$

Proof. Follows from the proof of the van der Waerden conjecture on the permanent of doubly stochastic matrices. □

From Proposition 2.2, the naive algorithm runs in time polynomial in $1/\varepsilon, \log 1/\delta$ and $n!/|\mathscr{M}_n|$. Thus, from Proposition 6.8, for any constant $c>0$, the naive algorithm is a fully polynomial (ε, δ)-approximation algorithm on inputs from the class of graphs $\bigcup_{n=1,\ldots,\infty} \mathscr{L}^{2n}_{n-c\log n}$. This is in contrast to Theorem 6.6 which shows that exact counting of perfect matchings in these graphs is ♯**P**-complete.

The following is a corollary of Theorem 6.6. This corollary shows that the simple algorithm is a fully polynomial (ε, δ)-approximation algorithm for classes of graphs for which the exact counting problem is hard.

Corollary 6.9. *For $\alpha = 1/i$ for any fixed integer $i \geqslant 1$, exact counting of perfect matchings in $\alpha(N-3)$-regular bipartite graphs on $2N$ vertices is ♯**P**-complete. (Thus, exact counting for $\mathscr{L}_\alpha$ is ♯**P**-complete.)*

Proof. Let G be an $(n-3)$-regular bipartite graph on $2n$ vertices. Let G' be the disjoint union of G and $i-1$ copies of $K_{n-3,n-3}$. G' is $(n-3)$-regular on $2(i(n-3)+3)$ vertices, and $|\mathscr{M}(G')| = |\mathscr{M}(G)| \cdot (n-3)!^{i-1}$. Letting $N = i(n-3)+3$ yields the reduction. □

Acknowledgment

We thank Milena Mihail and Umesh Vazirani for numerous enlightening discussions.

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