The packing of pairs by quadruples

Ahmed M. Assaf

Division of Mathematics, Department of Algebra, Combinatorics and Analysis, Auburn University, Auburn, AL 36849, USA

Received 28 October 1985
Revised 12 September 1989

Abstract


Let $X$ be a finite set of size $v$, further let $\lambda$ be a positive integer and let $\sigma(4, \lambda; v)$ denote the maximum number of quadruples such that each pair of elements of $X$ is contained in at most $\lambda$ of them. The value of $\sigma(4, 1; v)$ has been determined by Brouwer (1979) for all $v \leq 4$. The value of $\sigma(4, \lambda; v)$ has been determined by Billington, Stanton and Stinson (1984) for all $v \equiv 0 \pmod{3}$ and $\lambda > 1$. In this paper we complete the determination of $\sigma(4, \lambda; v)$ for all $v \geq 4$ and $\lambda > 1$.

1. Introduction

Let $X$ be a finite set of points and let $\beta = \{B_i : i \in I\}$ be a family of—not necessarily distinct—subsets $B_i$—called blocks—of $X$. The pair $(X, \beta)$ is called a design.

1.2. Balanced incomplete block design

Let $v \geq k \geq 2$ and $\lambda$ be positive integers. A design $(X, \beta)$ is called a balanced incomplete block design (BIBD) $B[k, \lambda; v]$ if:

(i) $|X| = v$,
(ii) the blocks are of size $k$, and
(iii) every 2-set $\{x, y\} \subset X$ is contained in exactly $\lambda$ blocks of $\beta$.

A well-known result states that necessary conditions for the existence of a BIBD $B[k, \lambda; v]$ are that $\lambda(v - 1) \equiv 0 \pmod{(k - 1)}$ and $\lambda v(v - 1) \equiv 0 \pmod{k(k - 1)}$.

For certain values of $k$ and $\lambda$ the above necessary conditions are also sufficient. For instance, we shall use the following theorem.
Theorem 1.1 (Hanani [5]). Let $\lambda$ and $\nu \geq 4$ be positive integers. Necessary and sufficient conditions for the existence of a BIBD $B[4, \lambda; \nu]$ are that $\lambda(\nu - 1) \equiv 0 \pmod{3}$ and $\lambda \nu(\nu - 1) \equiv 0 \pmod{12}$.

Let $\nu$ and $\lambda$ be positive integers and $K$ a set of positive integers. A design $(X, \beta)$ is a pairwise balanced design $B[K, \lambda; \nu]$ if:

(i) $|X| = \nu$,
(ii) $\{|B_i|: b_i \in \beta\} \subseteq K$ (the block-sizes are from $K$),
(iii) every 2-set $\{x, y\} \subset X$ is contained in exactly $\lambda$ blocks of $\beta$.

We shall consider designs of the form $(X, G, P)$ where $X$ is a finite set of points, $G$ is a partition of $X$ to subsets called groups and $P$ is a family of subsets of $X$ called blocks.

Let $m$, $k$, $\lambda$ and $\nu$ be positive integers. A design $(X, G, P)$ is a group divisible design $GD[k, \lambda, m; \nu]$ if:

(i) $|X| = \nu$,
(ii) $|G_i| = m$ for every $G_i \in G$,
(iii) $|B_j| = k$ for every $B_j \in P$,
(iv) $|G_i \cap B_j| \leq 1$ for every $G_i \in G$ and every $B_j \in P$,
(v) every 2-subset $\{x, y\} \subset X$ such that $x$ and $y$ belong to distinct groups, is contained in exactly $\lambda$ blocks of $P$.

We shall use the following theorem.

Theorem 1.2 (Brouwer–Hanani–Schrijver [4]). Let $m$, $\lambda$ and $\nu$ be positive integers. Necessary and sufficient conditions for the existence of a group divisible design $GD[4, \lambda, m; \nu]$ are that the design is not $GD[4, 1, 2; 8]$ and not $GD[4, 1, 6; 24]$ and that $\nu \equiv 0 \pmod{m}$, $\lambda(\nu - m) \equiv 0 \pmod{3}$, and $\lambda \nu(\nu - m) \equiv 0 \pmod{12}$ and $\nu \geq 4m$ or $\nu = m$.

1.3. Packing and covering designs

A design $(X, \beta)$ is called a packing design $SD[k, \lambda, \nu, b]$ (respectively a covering design $AD[k, \lambda, \nu, b]$), if:

(i) $|X| = \nu$,
(ii) the blocks are of size $k$,
(iii) $|\beta| = b$,
(iv) every 2-subset $\{x, y\} \subset X$ is included in at most (respectively at least) $\lambda$ blocks.

Naturally, we are interested in packing designs having the maximum number of blocks and conversely, in covering designs having the minimum number of blocks.
Denote by $\sigma(k, \lambda; v)$ the greatest value of $b$ for which $SD[k, \lambda, v, b]$ exists and by $\alpha(k, \lambda; v)$ the smallest number $b$ of blocks for which $AD[k, \lambda, v, b]$ exists, clearly

$$\sigma(k, \lambda; v) \leq \frac{\lambda v(v-1)}{k(k-1)} \leq \alpha(k, \lambda; v)$$

with equality on both sides if and only if a BIBD $B[k, \lambda; v]$ exists.

Schonheim [8] introduced the notation

$$\Psi(k, \lambda; v) = \left\lfloor \frac{v^2 - 1}{k^2 - 1} \right\rfloor, \quad \phi(k, \lambda; v) = \left\lfloor \frac{v^2 - 1}{k^2 - 1} \lambda \right\rfloor,$$

where $\lfloor x \rfloor$ is the smallest and $\lceil x \rceil$ the largest integer satisfying $\lfloor x \rfloor \leq x \leq \lceil x \rceil$ and proved

**Theorem 1.3** (Schonheim [9]). *For all positive integers $k$, $\lambda$ and $v \geq k$*

$$\sigma(k, \lambda; v) \leq \Psi(k, \lambda; v) \leq \frac{\lambda v(v-1)}{k(k-1)} \leq \phi(k, \lambda; v) \leq \alpha(k, \lambda; v).$$

$\Psi SD[k, \lambda, v]$ will denote a packing design $SD[k, \lambda, v, b]$ where $b = \Psi(k, \lambda, v)$. A design $(X, \beta)$ with $X' \subset X$ is called an almost packing design $SD^*[k, \lambda, v(t), b]$ if:

(i) $|X| = v$,
(ii) the blocks are of size $k$,
(iii) $|\beta| = b$,
(iv) $|X'| = t$,
(v) every pairset $\{x, y\} \subset X$ such that $\{x, y\} \notin X'$ is included in at most $\lambda$ blocks of $\beta$,
(vi) no pairset $\{x, y\} \subset X'$ is included in any block of $\beta$.

### 2. Packing design with $k = 4$

The following theorem is the main result of this paper.

**Theorem 2.0.** *For all $v \geq 4$ and $\lambda \geq 1$, $\sigma(4, \lambda; v) = \Psi(4, \lambda; v)$ with the following exceptions:*

1. $\sigma(4, 1; v) = \Psi(4, 1; v) - 1$ for $v \equiv 7$ or $10 \pmod{12}$ with $v \neq 10, 19$.
2. $\sigma(4, 1; v) = \Psi(4, 1; v) - 1$ for $v \in \{9, 17\}$.
3. $\sigma(4, 1; v) = \Psi(4, 1; v) - 2$ for $v \in \{8, 10, 11\}$.
4. $\sigma(4, 1; 19) = \Psi(4, 1; 19) - 3$.
5. $\sigma(4, 2; 9) = \Psi(4, 2; 9) - 1$.
6. $\sigma(4, 3; 6) = \Psi(4, 3; 6) - 1$. 
In the case \( \lambda = 1 \) after results of Schonheim [9] and Hanani –Brouwer [3] completed the determining of \( \sigma(4, 1; v) \) for every \( v \geq 4 \). For \( \lambda > 1 \), Billington, Stanton and Stinson [2] determined \( \sigma(4, \lambda; v) \) for \( v \neq 0 \pmod{3} \) by showing that for \( \lambda > 1, \lambda \neq 0 \pmod{3} \), \( \sigma(4, \lambda; v) \) equals the upper bound given in Theorem 1.3. They suspected that for \( v = 0 \pmod{3} \) a similar result holds with a single exception. We shall prove that result (Theorem 3.1), but it turns out that there are two exceptions. Thus Theorem 3.1 covers all missing cases in the problem of packing pairs by quadruples. In order to establish it, we need the following results of Brouwer, Hanani and Mills.

**Theorem 2.1** (Brouwer [3]). A B\( \{(4, 7^*), 1; v\} \) that is, a pairwise balanced design on \( v \) points with blocks of size 4 and exactly one block of size 7 (and \( \lambda = 1 \)), exists iff \( v = 7 \) or \( v = 10 \pmod{12} \) and \( v \neq 10, 19 \).

**Theorem 2.2** (Brouwer [3]). A GD\( \{4, 1, \{2, 5^*\}; v\} \), that is, a group divisible design on \( v \) points with blocks of size 4 and groups of size 2 and exactly one group of size 5 (and \( \lambda = 1 \)), exists iff \( v = 5 \pmod{6} \), \( v \neq 11, 17 \).

**Lemma 2.1** (Hanani [5]). For given integers \( k, \lambda \) and \( m \), let GD\( \{k, \lambda, m; mn\} \) exists for every integer \( n \geq k \). If in addition for \( u = m + t \) (\( 0 \leq t \leq m \)) both designs

\[
\text{SD}\left[k, \lambda, u, \frac{\lambda(u^2 + a_1u + a_0)}{k(k-1)}\right]
\]

and

\[
\text{SD}^*\left[k, \lambda, u(t), \frac{m(\lambda(u + t) + a_1)}{k(k-1)}\right]
\]

exist then \( \sigma(k, \lambda; v) = (\lambda v^2 + a_1v + a_0)/k(k-1) \) for every \( v = mn + t \).

**Proof.** We have to prove the existence of a packing design

\[
\text{SD}\left[k, \lambda, v, \frac{\lambda(v^2 + a_1v + a_0)}{k(k-1)}\right]
\]

for every \( v = mn + t \) (\( n \geq k \)). Let \( X = (I(m) \times I(n)) \cup I(t) \) where \( I(m) \) denotes the set of nonnegative integers \( I(m) = \{0, 1, 2, \ldots, m-1\} \). Form a group divisible design GD\( \{k, \lambda, m; mn\} \) on \( I(m) \times I(n) \) with groups \( G_i = I(m) \times \{i\} \). This design has \( m^2n(n-1)\lambda/k(k-1) \) blocks. Further form

\[
\text{SD}\left[k, \lambda, u, \frac{\lambda(u^2 + a_1u + a_0)}{k(k-1)}\right]
\]

on \( (I(m) \times \{0\}) \cup I(t) \), and

\[
\text{SD}^*\left[k, \lambda, u(t), \frac{m(\lambda(u + t) + a_1)}{k(k-1)}\right]
\]
The packing of pairs by quadruples

on \((I(m) \times \{i\}) \cup I(t)\), where \(i = 1, 2, \ldots, n - 1\) and \(X' = I(t)\). It is easily checked that the total number of blocks is \((\lambda v^2 + a_1 v + a_0)/k(k - 1)\).

Lemma 2.2 (Mills [7]). Let \(X\) be a set of order \(4w + u\) where \(w = 0\) or \(1\) (mod 4) and \(0 < u < w\). Then there exists a collection \(D\) of \(w^2 + 5\) subsets of \(X\) such that:

(i) the collection \(D\) covers all pairs of \(X\) exactly once.

(ii) \(D\) consists of \(w(w - u)\) sets of order 4, \(wu\) sets of order 5, four sets of order \(w\) and one set of order \(u\).

3. Main result and its proof

We proceed to prove the following theorem.

Theorem 3.1. For all positive integers \(\lambda > 1\) and \(v \equiv 0\) (mod 3), \(v \neq 3\), \(\sigma(4, \lambda; v) = \Psi(4, \lambda; v)\) with the exception of \(\sigma(4, 2; 9) = \Psi(4, 2; 9) - 1\) and \(\sigma(4, 3; 6) = \Psi(4, 3; 6) - 1\).

If there exists a \(B[4, \lambda; v]\) and \(\sigma(4, \mu; v) = \Psi(4, \mu; v)\) then it follows that \(\sigma(4, \lambda + \mu; v) = \Psi(4, \lambda + \mu; v)\) by taking the union of the blocks of the BIBD and the blocks of the maximal packing of index \(\mu\). Hence, by Theorem 1.1, and Brouwer's result on \(\sigma(4, 1; v)\) it suffices to prove Theorem 3.1 in only the following cases:

1. \(v \equiv 0\) or \(9\) (mod 12), \(v \geq 9\) and \(\lambda = 2\).
2. \(v = 9\) and \(\lambda = 4, 5\).
3. \(v \equiv 3\) or \(6\) (mod 12), \(v \equiv 6\) and \(2 \leq \lambda \leq 5\).
4. \(v = 6\) and \(\lambda = 9\).

First we need the following two lemmas.

Lemma 3.1. For \(v = 7\) and \(\lambda = 3\) there exists \(SD[4, 3, 7, 10]\).

Proof. Let \(X = Z_5 \cup \{a, b\}\), then the required blocks are

\[\langle 0, 1, 3, a \rangle \pmod{5}, \quad \langle 0, 1, 2, b \rangle \pmod{5}\]

Lemma 3.2 (Hartman [6]). For \(v = 10\) and \(\lambda = 3\) there exists \(SD[4, 3, 10, 22]\).

Proof. Let \(X = Z_8 \cup \{a, b\}\) then the required blocks are

\[\langle 0, 1, 3, a \rangle \pmod{8}, \quad \langle 0, 1, 3, b \rangle \pmod{8}, \quad \langle 0, 2, 4, 6 \rangle + i, \quad i \in Z_2, \quad \langle 0, 1, 4, 5 \rangle + i, \quad i \in Z_4\]

In order to prove Theorem 3.1 we divide it into several cases.
3.1. \( \nu = 0 \) (mod 12)

In this case \( \sigma(4, \lambda; \nu) = \Psi(4, \lambda; \nu) \). To prove that we need the following lemma.

**Lemma 3.3.** Let \((I(12n + 1), \beta)\) be a \(B[4, 1; 12n + 1]\) containing a set of \( n \) mutually disjoint blocks \( B_0, B_1, \ldots, B_{n-1} \) none of which contain the point \( p \) and with \( |B_i \cap C| = 1 \) for all \( i \) and for all blocks \( C \in \beta \) which contain the point \( p \). Then \( \sigma(4, 2; 12n) = \Psi(4, 2; 12n) \).

**Proof.** Without loss of generality we can label the points of the design so that \( B_i = \{12i, 12i + 3, 12i + 6, 12i + 9\} \), \( p = 12n \), and the blocks \( C_j \) \((0 \leq j < 4n)\), containing the point \( p \) are \( C_j = \{12n, 3j, 3j + 1, 3j + 2\} \). We now take another copy of the design with points labelled so that the blocks \( D_j \) \((0 \leq j < 4n)\) containing the point \( 12n \) are given by \( D_j = \{12n, 3j + 1, 3j + 2, 3j + 3\} \), when \( j \neq 3 \) (mod 4) and \( D_j = \{12n, 3j + 1, 3j + 2, 3j - 9\} \), when \( j = 3 \) (mod 4). Now remove all the blocks \( B_i, C_j, \) and \( D_j \) and replace them by blocks \( E_j \) \((0 \leq j < 4n)\) given by \( E_j = \{3j, 3j + 1, 3j + 2, 3j + 3\} \), when \( j \neq 3 \) (mod 4) and \( E_j = \{3j, 3j + 1, 3j + 2, 3j - 9\} \), when \( j = 3 \) (mod 4). Note that all blocks containing the point \( p = 12n \) have been deleted, and the number of blocks in the new configuration is precisely \( \psi(4, 2; 12n) \). \( \square \)

**Corollary.** \( \sigma(4, 2; 12n) = \Psi(4, 2; 12n) \) for \( n = 1, 2, 3 \).

**Proof.** The design \( B[4, 1; 13] \) is a projective plane of order 3, and thus any two blocks intersect in a point, hence the configuration required in the Lemma exists.

(1) For \( \lambda = 2 \), the blocks of \( \Psi SD[4, 2, \nu] \) can be constructed in the following way. (a) Take the blocks \( \Psi SD[4, 1, \nu - 1] \). Note that a \( \Psi SD[4, 1, \nu - 1] \) exists by Brouwer's result [3]. (b) Take the blocks of \( B[4, 1; \nu + 1] \). We can assume that we have the block \( \langle \nu - 2, \nu - 1, \nu, \nu + 1 \rangle \), drop this block and in the remaining blocks of \( B[4, 1; \nu + 1] \) change \( \nu + 1 \) to \( \nu \).

(2) For \( \lambda = 3 \) apply Lemma 2.1 with \( m = 12 \) and \( t = 3 \). According to this lemma it is sufficient to prove the existence of \( SD^*[4, 3, 15(3), 51] \), \( SD[4, 3, 15, 52] \), \( SD[4, 3, 27, 175] \) and \( SD[4, 3, 39, 370] \).
The packing of pairs by quadruples

The blocks of \( \text{SD}^*[4, 3, 15(3), 51] \) can be constructed in the following way. (a) Take the blocks of \( B[4, 1; 13] \). (b) Take the blocks of \( B[4, 1; 16] \) and drop the block \( \langle 13, 14, 15, 16 \rangle \). In the remaining blocks change 16 to 14. (c) Again take the blocks of \( B[4, 1; 16] \), drop the block \( \langle 13, 14, 15, 16 \rangle \) and change the point 16 to 15.

For \( \text{SD}[4, 3, 15, 52] \) let \( X = \mathbb{Z}_{13} \cup \{a, b\} \), then the required blocks are:

- \( \langle 0, 1, 4, a \rangle \mod 13 \)
- \( \langle 0, 1, 3, 7 \rangle \mod 13 \)
- \( \langle 0, 4, 5, b \rangle \mod 13 \)
- \( \langle 0, 2, 5, 7 \rangle \mod 13 \).

For \( \text{SD}[4, 3, 27, 175] \) let \( X = \mathbb{Z}_{25} \cup \{a, b\} \) then the required blocks are:

- \( \langle 0, 8, 9, a \rangle \mod 25 \)
- \( \langle 0, 4, 9, 12 \rangle \mod 25 \)
- \( \langle 0, 2, 5, b \rangle \mod 25 \)
- \( \langle 0, 3, 7, 13 \rangle \mod 25 \)
- \( \langle 0, 2, 7, 13 \rangle \mod 25 \)
- \( \langle 0, 1, 7, 11 \rangle \mod 25 \).

For \( \text{SD}[4, 3, 39, 370] \) let \( X = \mathbb{Z}_{37} \cup \{a, b\} \) then the blocks are:

- \( \langle 0, 6, 10, a \rangle \mod 37 \)
- \( \langle 0, 2, 15, 16 \rangle \mod 37 \)
- \( \langle 0, 14, 15, b \rangle \mod 37 \)
- \( \langle 0, 3, 13, 17 \rangle \mod 37 \)
- \( \langle 0, 2, 8, 13 \rangle \mod 37 \)
- \( \langle 0, 2, 11, 19 \rangle \mod 37 \)
- \( \langle 0, 7, 16, 19 \rangle \mod 37 \)
- \( \langle 0, 7, 12, 16 \rangle \mod 37 \).

Note that for all \( v = 3 \mod 12 \) with \( v \geq 15 \) the \( \Psi\text{SD}[4, 3, v] \) constructed above contains two points \( a, b \) such that the pair \( (a, b) \) appears in no block.

(3) For \( \lambda = 4 \) the blocks of \( \Psi\text{SD}[4, 4, v] \) are the blocks of \( \Psi\text{SD}[4, 3, v] \) and the blocks of \( \Psi\text{SD}[4, 1, v] \).

(4) For \( \lambda = 5 \) the blocks of \( \Psi\text{SD}[4, 5, v] \) can be constructed in the following way. (a) Take the blocks of \( \Psi\text{SD}[4, 3, v] \) such that the pair \( (v - 1, v) \) does not appear. (b) Take the blocks of \( B[4, 1; v - 2] \). (c) Take the blocks of \( \Psi\text{SD}[4, 1, v + 2] \), which were given by Brouwer [2, p. 280]. According to this construction we can assume that the two pairs \( (v + 2, v), (v + 1, v - 1) \) do not occur in the blocks of \( \Psi\text{SD}[4, 1, v + 2] \). Now change the point \( v + 2 \) to \( v \) and \( v + 1 \) to \( v - 1 \).

The above construction does not work for \( v = 15 \) and \( \lambda = 5 \) since no \( \Psi\text{SD}[4, 1, 17] \) exists. For a construction of \( \text{SD}[4, 5, 15, 86] \) see [6].

3.3. \( v = 6 \mod 12 \)

(1) For \( \lambda = 2 \) the blocks of \( \Psi\text{SD}[4, 2, v] \) can be constructed in the following way. (a) Take the blocks of \( B[4, 1; v - 2] \). (b) Take the blocks of \( \Psi\text{SD}[4, 1, v + 2] \). Without loss of generality we can assume that we have the block \( \langle v - 1, v, v + 1, v + 2 \rangle \), drop this block and in the remaining blocks of \( \Psi\text{SD}[4, 1, v + 2] \) change \( v + 1 \) to \( v - 1 \) and \( v + 2 \) to \( v \).
The above construction doesn’t work for \( v = 6 \). Hence for \( v = 6 \) and \( \lambda = 2 \) let \( X = Z_6 \), then the required blocks are

\[
(0, 1, 3, 5) + i, \quad i \in Z_4
\]

(2) For \( \lambda = 3 \) we distinguish two cases:

Case (a): \( v = 6 \) or \( 18 \mod 48 \), \( v \neq 6 \), then let \( v - 4w + 2 \) where \( w = 0 \) or \( 1 \mod 4 \), then by Lemma 2.2 there exists a collection of \( w^2 + 4 \) blocks of sizes 4, 5, \( w \) and one block of size 2. We drop the block of size 2 and on the blocks of sizes 4, 5 and \( w \) we construct BIBD’s with \( \lambda = 3 \).

Case (b): \( v = 30 \) or \( 42 \mod 48 \), \( v \neq 30, 42 \), let \( v = 4w + 10 \), then for \( v \geq 78 \), \( w = 0 \) or \( 1 \mod 4 \) and \( w \geq 10 \). Again apply Lemma 2.2 and the proof of case (b) is exactly the same as case (a), using Lemma 3.2 for the block of size 10.

For SD[4, 3, 30, 217] let \( X = Z_{23} \cup \{a, b, c, d, e, f, g\} \), then the required blocks are:

\[
\begin{align*}
&\langle 0, 2, 8, 11 \rangle \mod 23 & \langle 0, 10, 11, d \rangle \\
&\langle 0, 3, 8, 10 \rangle \mod 23 & \langle 0, 8, 11, e \rangle \\
&\langle 0, 4, 9, a \rangle \mod 23 & \langle 0, 7, 9, f \rangle \\
&\langle 0, 6, 7, b \rangle \mod 23 & \langle 0, 4, 5, g \rangle \\
&\langle 0, 4, 10, c \rangle \mod 23 & \langle a, b, c, d, e, f, g \rangle.
\end{align*}
\]

For the last block apply Lemma 3.1.

For \( v = 42 \) and \( \lambda = 3 \) let \( X = Z_{35} \cup \{a, b, c, d, e, f, g\} \) then the required blocks are:

\[
\begin{align*}
&\langle 0, 3, 11, 17 \rangle \mod 35 & \langle 0, 5, 15, 21 \rangle \mod 35 & \langle 0, 12, 13, d \rangle \mod 35 \\
&\langle 0, 5, 12, 14 \rangle \mod 35 & \langle 0, 3, 4, a \rangle \mod 35 & \langle 0, 5, 11, e \rangle \mod 35 \\
&\langle 0, 7, 8, 20 \rangle \mod 35 & \langle 0, 15, 16, b \rangle \mod 35 & \langle 0, 2, 10, f \rangle \mod 35 \\
&\langle 0, 4, 13, 23 \rangle \mod 35 & \langle 0, 18, 17, c \rangle \mod 35 & \langle 0, 3, 7, g \rangle \mod 35 \\
&\langle a, b, c, d, e, f, g \rangle.
\end{align*}
\]

For the last block apply Lemma 3.1.

(3) For \( \lambda = 4 \) the blocks of \( \Psi SD[4, 4, v] \) can be constructed in the following way. (a) Take the blocks of B[4, 2; \( v - 2 \)]. (b) Take the blocks of \( \Psi SD[4, 1, v + 2] \) and assume we have the block \( \langle v - 1, v, v + 1, v + 2 \rangle \), drop this block and in the remaining blocks of \( \Psi SD[4, 1, v + 2] \) change \( v + 2 \) to \( v \) and \( v + 1 \) to \( v - 1 \). (c) Again take the blocks of \( \Psi SD[4, 1, v + 2] \). According to the construction given by Brouwer [2, p. 280] we can assume that the two pairs \( \langle v, v + 2 \rangle \) and \( \langle v - 1, v + 1 \rangle \) do not occur in the blocks of \( \Psi SD[4, 1, v + 2] \). Now change \( v + 2 \) to \( v \) and \( v + 1 \) to \( v - 1 \).

The above construction does not work for \( v = 6 \) and \( \lambda = 4 \), hence for SD[4, 4, 6, 9] let \( X = Z_6 \), then the required blocks are:

\[
\begin{align*}
&\langle 0, 1, 3, 5 \rangle \mod 6, & \langle 0, 1, 2, 5 \rangle + 2i, & i \in Z_3.
\end{align*}
\]
(4) For $\lambda = 5$ the blocks of $\Psi SD[4, 5, v]$ can be constructed in the following way. (a) Take the blocks of $\Psi SD[4, 3, v]$. According to the construction given above, there is exactly one pair say $(v - 1, v)$ which does not occur in the blocks of $\Psi SD[4, 3, v]$. (b) Take the blocks of $B[4, 1; v - 2]$. (c) Take the blocks of $\Psi SD[4, 1, v + 2]$ and assume that the two pairs $(v, v + 2)$ and $(v - 1, v + 1)$ do not occur in the blocks of $\Psi SD[4, 1, v + 2]$. Now change $v + 2$ to $v$ and $v + 1$ to $v - 1$.

For $SD[4, 5, 6, 12]$, let $X = Z_6$ then the required blocks are:

\[
\langle 0, 1, 2, 4 \rangle \pmod{6} \quad \text{and} \quad \langle 0, 1, 2, 5 \rangle \pmod{6}.
\]

3.4. $v = 9 \pmod{12}$

(1) For $\lambda = 2$ the blocks of $\Psi SD[4, 2, v]$ can be constructed in the following way. (a) Take the blocks of $\Psi SD[4, 1, v - 1]$. According to the construction given by Brouwer [2, p. 281] we can assume that the pair $(v - 2, v - 1)$ does not occur in the blocks of $\Psi SD[4, 1, v - 1]$. (b) Take the blocks of a $B[\{4, 7^*\}, 1; v + 1]$ (which exists by Theorem 2.1). Suppose the block of size 7 is $\langle v - 5, v - 4, v - 3, v - 2, v - 1, v, v + 1 \rangle$, replace this block by the three blocks

\[
\langle v - 5, v - 4, v - 3, v \rangle, \quad \langle v - 4, v - 2, v - 1, v \rangle, \quad \langle v - 3, v - 2, v - 1, v \rangle,
\]

then, in all the remaining blocks of $B[\{4, 7^*\}, 1; v + 1]$, change $v + 1$ to $v$.

The above construction does not work for $v = 9$ since no $B[\{4, 7^*\}, 1, 10]$ exists. For a construction of $SD[4, 2, 9, 21]$ see [6].

Now by the comment given at the beginning of Section 3, the only cases necessary to complete the proof of Theorem 3.1 are $(v, \lambda) = (9, 4), (9, 5)$ and $(6, 9)$.

For $v = 9$ and $\lambda = 4$ let $X = Z_2 \times Z_4 \cup \{a\}$, then the required blocks are:

\[
\langle 0, 0 \rangle, \langle 0, 1 \rangle, \langle 1, 1 \rangle, \langle 1, 3 \rangle \pmod{(-, 4)}, \quad \langle 0, 0 \rangle, \langle 1, 0 \rangle, \langle 1, 3 \rangle, a \pmod{(-, 4)},
\]

\[
\langle 0, 0 \rangle, \langle 0, 2 \rangle, \langle 1, 0 \rangle, \langle 1, 1 \rangle \pmod{(-, 4)}, \quad \langle 0, 0 \rangle, \langle 0, 1 \rangle, \langle 0, 2 \rangle, a,
\]

\[
\langle 0, 0 \rangle, \langle 0, 3 \rangle, \langle 1, 0 \rangle, \langle 1, 1 \rangle \pmod{(-, 4)}, \quad \langle 1, 0 \rangle, \langle 1, 1 \rangle, \langle 1, 2 \rangle, a,
\]

\[
\langle 0, 0 \rangle, \langle 0, 1 \rangle, \langle 1, 3 \rangle, a \pmod{(-, 4)}.
\]

For $v = 9$ and $\lambda = 5$ see [6].

For $v = 6$ and $\lambda = 9$ let $X = Z_4 \cup \{a, b\}$ then the blocks are:

\[
\langle a, b, 0, 1 \rangle \quad \langle a, 0, 1, 2 \rangle \pmod{4},
\]

\[
\langle a, b, 1, 2 \rangle \quad \langle b, 0, 1, 2 \rangle \pmod{4},
\]

\[
\langle a, b, 1, 3 \rangle \quad \langle a, b, 2, 3 \rangle,
\]

\[
\langle a, b, 0, 3 \rangle \quad \langle a, b, 0, 2 \rangle.
\]

3.5. The exceptional cases

**Lemma 3.4.** For $v = 6$ and $\lambda = 3$, $\sigma(4, 3, 6) \neq \Psi(4, 3, 6)$
Proof. If a packing of $3K_6$ with 7 blocks exists, then the graph of edges not covered, contains 3 edges, and every vertex of the graph must have degree congruent to 0 modulo 3. Hence in this graph there are two vertices which are not joined. Let $X = \{B_0, B_1, B_2, B_3, A_0, A_1\}$, with no blocks contain both $A_0$ and $A_1$. Let $x$ be the number of blocks containing either $A_0$ or $A_1$ and let $y$ be the number of blocks containing neither $A_0$ nor $A_1$. It is easy to see that $y = 0$ and $3x = 24$, i.e. $x = 8$ which is impossible.

Let $X = Z_6$ then the following 6 blocks are packing for $\nu = 6$ and $\lambda = 3$:

\[\langle 0, 1, 3, 5 \rangle \pmod{6}.\]

Hence $\sigma(4, \lambda, 6) = \Psi(4, \lambda, 6) - 1$. □

For $\nu = 9$ and $\lambda = 2$ see [6].

Corollary (Assaf [1]). For every $\nu \geq 4$, $\nu \neq 6$, $\nu = 2, 3, 5, 6, 7, 10, 11 \pmod{12}$ a design $B[4, 6; \nu]$ may be constructed by taking the blocks of $\Psi SD[4, 3; \nu]$ and the blocks of $AD[4, 3, \nu, b]$ with $b = \phi(4, 3, \nu)$.

Proof. Using the argument of Lemma 3.4 it gives us that in the designs $\Psi SD[4, 3, \nu]$, $\nu = 2, 3, 5, 6, 7, 10, 11 \pmod{12}$, there is exactly one pair, say $(\nu - 1, \nu)$, which does not occur in the blocks of $\Psi SD[4, 3, \nu]$ and each other pair occurs exactly three times.

On the other side by the construction of $AD[4, 3, \nu, b]$, $\nu = 2, 3, 5, 6, 7, 10, 11 \pmod{12}$, given by Assaf [1] there is one pair, say $(\nu - 1, \nu)$, which occurs exactly 6 times in the blocks of $AD[4, 3, \nu, b]$ and each other pair occurs exactly three times. Hence the blocks of $\Psi SD[4, 3, \nu]$ and the blocks of $AD[4, 3, \nu, b]$ give the blocks of $B[4, 6; \nu]$. □

This finishes the proof of Theorem 3.1 and the complete solution of the announced problem.

Acknowledgement

I am very grateful to A. Hartman for reading the manuscript and for his important and most helpful remarks.

References

The packing of pairs by quadruples