View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Springer - Publisher Connector

Tarkhanov and Wallenta *Mathematical Sciences* 2012, **6**:44
http://www.iaumath.com/content/6/1/44

Mathematical Sciences
a SpringerOpen Journal

ORIGINAL RESEARCH Open Access

The Lefschetz number of sequences of trace class curvature

Nikolai Tarkhanov[1*] and Daniel Wallenta[2]

Abstract

For a sequence of Hilbert spaces and continuous linear operators, the curvature is defined to be the composition of any two consecutive operators. This is modeled on the de Rham resolution of a connection on a module over an algebra.

Purpose: We wish to study those sequences for which the curvature is 'small' at each step, e.g., belongs to a fixed operator ideal.

Methods: Our methods are based on combining homological algebra with the theory of Fredholm operators in Hilbert spaces.

Results: We elaborate the theory of Fredholm sequences and show that any Fredholm sequence of trace class curvature can be reduced to a Fredholm complex. This allows one to introduce the Lefschetz number for cochain self-mappings of Fredholm sequences of 'small' curvature.

Conclusion: Our results raise fixed point theory for Fredholm complexes of trace class curvature.

Keywords: Perturbed complexes, Curvature, Lefschetz number

2010 MSC: primary 55U05; secondary 58J10, 19K56

Introduction

Let F be a smooth vector bundle over a compact manifold $\mathcal{X}$ and ∂ a connection on F. This is a first-order differential operator $C^\infty(\mathcal{X},F) \to \Omega^1(\mathcal{X},F)$ on $\mathcal{X}$, satisfying $\partial(\omega f) = d\omega f + \omega\,\partial f$ for all $f \in C^\infty(\mathcal{X},F)$ and $\omega \in C^\infty(\mathcal{X})$. As usual, we denote by $\Omega^i(\mathcal{X},F)$ the space of all smooth differential forms of degree i with coefficients in F on $\mathcal{X}$. On keeping the Leibniz rule, the connection extends to a first-order differential operator $\partial^i : \Omega^i(\mathcal{X},F) \to \Omega^{i+1}(\mathcal{X},F)$ for each $i = 1,\dots,n$, where n is the dimension of $\mathcal{X}$. An easy computation shows that $\partial^{i+1}\partial^i f = \Omega f$, where Ω is a differential form of degree 2 with coefficients in $C^\infty(\mathrm{Hom}(F))$. The form Ω is said to be the curvature of the connection ∂, which generalizes to the curvature homomorphism related to a connection on a module over an algebra (see [1], 2.10).

In a scale of Sobolev spaces on $\mathcal{X}$, if there is any, the operators ∂^i assemble into a sequence

$$0 \to H^s(\mathcal{X},F^0) \xrightarrow{\partial^0} H^{s-1}(\mathcal{X},F^1) \xrightarrow{\partial^1} \dots \xrightarrow{\partial^{n-1}} H^{s-n}(\mathcal{X},F^n) \to 0, \tag{1.1}$$

where $F^i = F \otimes \Lambda^i T^*\mathcal{X}$, $\partial^0 := \partial$, and $s \geq n$ is any fixed number. The compositions $\partial^{i+1} \circ \partial^i$ act through the embeddings $H^{s-i}(\mathcal{X},F^{i+2}) \hookrightarrow H^{s-i-2}(\mathcal{X},F^{i+2})$, which are compact by the Rellich theorem, so $\partial^{i+1} \circ \partial^i$ are compact operators from $H^{s-i}(\mathcal{X},F^i)$ to $H^{s-i-2}(\mathcal{X},F^{i+2})$. On using the scale of Schatten ideals $\mathfrak{S}_p$ with $p > 0$, we can even further specify the 'smallness' of the curvature of sequence (1.1); more precisely, $\partial^{i+1}\partial^i$ is of class $\mathfrak{S}_p$ with any $p > n/2$, see for instance § 15 of [2].

A connection ∂ on the bundle F is said to be flat if its curvature vanishes, that is, $\partial^{i+1}\circ\partial^i = 0$ for all $i = 0,\dots,N-1$. In this case, complex (1.1) possesses a well-defined cohomology, which allows one to define the Euler characteristic for sequence (1.1). In the general case, the cohomology is no longer available, so sequence (1.1) bears no analytical index although the topological index can be easily introduced. In [3], a Fredholm complex is constructed whose differential differs from ∂^i by compact operators. Hence, it follows that the Euler characteristic of this Fredholm complex does not depend on its concrete choice. In this

*Correspondence: tarkhanov@math.uni-potsdam.de
[1]Institut für Mathematik, Universität Potsdam, m Neuen Palais 10, Potsdam, 14469, Germany
Full list of author information is available at the end of the article

© 2012 Tarkhanov and Wallenta; licensee Springer. This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/2.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

way, the analytical index is introduced for sequence (1.1), which has led to a substantial index theory, see [4].

More generally, let $(L^{\cdot}, d)$ stand for a sequence of Hilbert spaces L^i and continuous linear operators $d^i : L^i \to L^{i+1}$. We simply write $df := d^i f$ for $f \in L^i$, if it causes no confusion. When considering bounded sequences, we can certainly assume that $L^i = 0$ for i is different from $0, 1, \ldots, N$; for if not, we shift the indexing. We thus arrive at

$$0 \to L^0 \stackrel{d^0}{\to} L^1 \stackrel{d^1}{\to} \ldots \stackrel{d^{N-1}}{\to} L^N \to 0. \tag{1.2}$$

To adhere to geometric language, we say that the compositions $d^{i+1} \circ d^i$ characterize the curvature of the sequence $(L^{\cdot}, d)$. The sequences of zero curvature are not stable relative to small perturbations of the differential d. Hence, we will be interested in those sequences which have a 'small' curvature. What is meant by 'smallness' is that all the compositions $d^{i+1} \circ d^i$ belong to an operator ideal $\mathcal{I}(L^i, L^{i+2})$. Let $\mathcal{L}$ denote the class of all bounded linear operators acting between arbitrary Banach spaces. Loosely speaking, an operator ideal $\mathcal{I}$ is a subclass of $\mathcal{L}$ such that $\mathcal{I} + \mathcal{I} = \mathcal{I}$ and $\mathcal{L} \circ \mathcal{I} \circ \mathcal{L} = \mathcal{I}$. With a few examples, we recall the ideals of compact, trace class, and absolutely summing operators (see [2]). Formally, the case where $\mathcal{I}$ is the zero ideal is also included.

A cochain self-mapping of $(L^{\cdot}, d)$ means any collection $e = (e^i)_{i\in\mathbb{Z}}$ of operators $e^i \in \mathcal{L}(L^i)$ satisfying $d^i e^i = e^{i+1} d^i$ modulo $\mathcal{I}(L^i, L^{i+1})$ for all $i \in \mathbb{Z}$. If $\mathcal{I}$ is the zero ideal, then any cochain self-mapping e of $(L^{\cdot}, d)$ induces a linear action He on the cohomology of $(L^{\cdot}, d)$. If the cohomology is finite dimensional, then the alternating sum of traces of $(He)^i$ is called the Lefschetz number $L(e)$ of e. The problem of evaluation of $L(e)$ in appropriate terms generalizes the index problem and is usually referred to as Lefschetz theory (see [5,6]). A proper definition of the Lefschetz number for arbitrary sequences of small curvature is expected to initiate an interesting Lefschetz theory for the set of cochain self-mappings becomes more substantial.

The present paper is aimed at specifying the Lefschetz number for sequences of trace class curvature, the corresponding Lefschetz theory being under study. The independence of the Lefschetz number from the choices included presents a more delicate problem than the definition of Euler characteristic (see [3]). It should be noted that the study of geometric operators within the more general framework of elliptic complexes proves to be very successful (see [6-8], etc).

We shall make a standing assumption on the ideals under considerations, namely, that $\mathcal{I}$ is a subclass of compact operators.

A very general idea to reduce a sequence of compact curvature to a complex is traced back to an early paper of Putinar (see [9,10]). When compared with [3], the method falls short of specifying operator algebras in which the reduction is carried out.

Sequences of class $\mathcal{I}$ curvature

As mentioned, from the point of view of analysis, sequences of compact curvature seem to be much more natural objects than complexes. In particular, on perturbing the differential of a complex by operators of the ideal $\mathcal{I}$, we go beyond the framework of complexes. However, the sequences of class $\mathcal{I}$ curvature survive under perturbations of the differential by operators of $\mathcal{I}$. We are thus lead to a class of sequences $(L^{\cdot}, d)$ bearing the property that the compositions $d^i \circ d^{i-1}$ belong to the ideal $\mathcal{I}$ for all $i = 0, 1, \ldots$.

Definition 2.1. A (cochain) sequence $(L^{\cdot}, d)$ of class $\mathcal{I}$ curvature means any sequence of Hilbert spaces L^i, $i \in \mathbb{Z}$ and operators $d^i \in \mathcal{L}(L^i, L^{i+1})$ satisfying $d^i \circ d^{i-1} = 0$ modulo operators of $\mathcal{I}(L^i, L^{i+2})$.

For Hilbert spaces L and M, we write $\mathcal{I}(L, M)$ for the subspace of $\mathcal{L}(L, M)$ consisting of all operators $f \in \mathcal{I}$ which map L to M. This subspace fails to be closed in $\mathcal{L}(L, M)$ in general; the smallest closed nonzero 'ideal' is $\mathcal{K}(L, M)$. For $f_1, f_2 \in \mathcal{L}(L, M)$, we write $f_1 \sim f_2$ if $f_1 - f_2 \in \mathcal{I}(L, M)$. Suppose that $(L^{\cdot}, d_L)$ and $(M^{\cdot}, d_M)$ are two sequences of class $\mathcal{I}$ curvature. A cochain mapping of $(L^{\cdot}, d_L)$ into $(M^{\cdot}, d_M)$ means any collection of operators $f^i \in \mathcal{L}(L^i, M^i)$, $i \in \mathbb{Z}$ such that $d_M^i f^i \sim f^{i+1} d_L^i$ for all $i \in \mathbb{Z}$. In particular, $0 = \left(0_{L^i}\right)_{i\in\mathbb{Z}}$ and $1 = \left(1_{L^i}\right)_{i\in\mathbb{Z}}$ are cochain mappings of (L, d) into itself, and so are all their perturbations by operators of $\mathcal{I}$.

Cochain mappings $(f_0^i)_{i\in\mathbb{Z}}$ and $(f_1^i)_{i\in\mathbb{Z}}$ of $(L^{\cdot}, d_L)$ into $(M^{\cdot}, d_M)$ are said to be homotopic if there is a collection $h^i \in \mathcal{L}(L^i, M^{i-1})$, $i \in \mathbb{Z}$, with the property that $f_1^i - f_0^i \sim d_M^{i-1} h^i + h^{i+1} d_L^i$ for all $i \in \mathbb{Z}$.

The task is now to extend the concept of Fredholm complexes to the more general context of sequences of class $\mathcal{I}$ curvature. Recall that an operator $d \in \mathcal{L}(L, M)$ in Hilbert spaces is Fredholm if and only if its image in the Calkin algebra $\mathcal{L}(L, M)/\mathcal{K}(L, M)$ is invertible. Thus, the idea is to pass in a given sequence to quotients, modulo spaces of operators of $\mathcal{I}$ and require exactness. To this end, we modify correspondingly the functor ϕ_Σ introduced by Putinar [9]. For complexes of pseudodifferential operators, it specifies to what is known as complex of symbols.

For Hilbert spaces K and L, set $\phi_K(L) = \mathcal{L}(K, L)/\mathcal{I}(K, L)$. Moreover, given any $d \in \mathcal{L}(L, M)$, we define $\phi_K(d) \in \mathcal{L}(\phi_K(L), \phi_K(M))$ by the formula

$$\phi_K(d) \left(f + \mathcal{I}(K, L)\right) = d \circ f + \mathcal{I}(K, M)$$

for $f \in \mathcal{L}(K, L)$. Clearly, this operator is well defined. It is easily seen that $\phi_K(d^2 d^1) = \phi_K(d^2)\phi_K(d^1)$ for all $d^1 \in \mathcal{L}(L^1, L^2)$ and $d^2 \in \mathcal{L}(L^2, L^3)$. If 1_L is the identity operator on L, then $\phi_K(1_L)$ is the identity operator on $\phi_K(L)$. These remarks show that ϕ_K is actually a covariant functor in the category of Hilbert spaces.

The crucial fact is that ϕ_K vanishes on operators of ideal $\mathcal{I}$ for every Hilbert space K. Conversely, if $d \in \mathcal{L}(L, M)$ and $\phi_K(d) = 0$ for any Hilbert space K, then $d \in \mathcal{I}(L, M)$. Indeed, taking $K = L$, we deduce from

$$\begin{aligned}\phi_L(d)\,(1_L + \mathcal{I}(L, L)) &= d + \mathcal{I}(L, M)\\ &= \mathcal{I}(L, M)\end{aligned}$$

that $d \in \mathcal{I}(L, M)$.

If $(L^{\cdot}, d)$s is an arbitrary sequence of class $\mathcal{I}$ curvature, then $(\phi_K(L^{\cdot}), \phi_K(d))$ is a complex for each Hilbert space K. Thus, the functor ϕ_K transforms sequences of class $\mathcal{I}$ curvature into ordinary complexes, i.e., ϕ_K 'rectifies' curved sequences. Furthermore, cochain mappings of sequences of class $\mathcal{I}$ curvature transform under ϕ_K into cochain mappings of complexes, and ϕ_K preserves the homotopy classes of cochain mappings.

Definition 2.2. A sequence $(L^{\cdot}, d)$ of class $\mathcal{I}$ curvature is called Fredholm if the associated complex $(\phi_K(L^{\cdot}), \phi_K(d))$ is exact for each Hilbert space K.

Let $(L^{\cdot}, d_1)$ and $(L^{\cdot}, d_2)$ be two sequences of class $\mathcal{I}$ curvature such that $d_1^i \sim d_2^i$ for all $i \in \mathbb{Z}$. Then, the complexes $(\phi_K(L^{\cdot}), \phi_K(d_1))$ and $(\phi_K(L^{\cdot}), \phi_K(d_2))$ obviously coincide for every Hilbert space K. Therefore, $(L^{\cdot}, d_1)$ and $(L^{\cdot}, d_2)$ are simultaneously Fredholm. In other words, any class $\mathcal{I}$ perturbation of a Fredholm sequence of class $\mathcal{I}$ curvature is a Fredholm sequence of class $\mathcal{I}$ curvature.

Theorem 2.3. *A bounded above sequence $(L^{\cdot}, d)$ of class $\mathcal{I}$ curvature is Fredholm if and only if the identity mapping of $(L^{\cdot}, d)$ is homotopic to the zero one.*

This theorem goes back at least as far as [9] where the case $\mathcal{I} = \mathcal{K}$ is treated. The designation 'essential complexes' is used in [9] for what we call 'sequences of compact curvature' here.

Proof. Necessity. Let $(L^{\cdot}, d)$ be Fredholm and bounded above, i.e., $L^i = 0$ for all but $i \leq N$. Our goal is to show that there are operators $\pi^i \in \mathcal{L}(L^i, L^{i-1})$, $i \in \mathbb{Z}$, such that

$$d^{i-1}\pi^i + \pi^{i+1}d^i = 1_{L^i} - c^i \qquad (2.1)$$

for all $i \in \mathbb{Z}$, where $c^i \in \mathcal{I}(L^i)$.

Set $\pi^i = 0$ for all integers $i > N$. If $i = N$, then from the exactness of the complex $(\phi_K(L^{\cdot}), \phi_K(d))$, $K = L^N$, at step N, it follows that there is an operator $\pi^N \in \mathcal{L}(L^N, L^{N-1})$ such that $d^{N-1}\pi^N \sim 1_{L^N}$. Denoting by c^N the difference $1_{L^N} - d^{N-1}\pi^N$, we thus get $c^N \in \mathcal{I}(L^N)$.

We now proceed by induction. Suppose we have already found mappings

$$\begin{array}{l}\pi^i,\ \pi^{i+1},\ \ldots;\\ c^i,\ \ c^{i+1},\ \ \ldots,\end{array}$$

such that the equality (2.1) is satisfied at steps $i, i+1, \ldots$, for some $i \leq N$. Note that

$$\begin{aligned}d^{i-1}\left(1_{L^{i-1}} - \pi^i d^{i-1}\right) &= d^{i-1} - \left(1_{L^i} - c^i - \pi^{i+1}d^i\right) d^{i-1}\\ &= c^i d^{i-1} + \pi^{i+1}d^i d^{i-1}\\ &\sim 0\end{aligned}$$

by equality (2.1). From the exactness of $(\phi_K(L^{\cdot}), \phi_K(d))$, with $K = L^{i-1}$, at step $i-1$, it follows that there is $\pi^{i-1} \in \mathcal{L}(L^{i-1}, L^{i-2})$ such that $d^{i-2}\pi^{i-1} \sim 1_{L^{i-1}} - \pi^i d^{i-1}$. Setting $c^{i-1} = 1_{L^{i-1}} - \pi^i d^{i-1} - d^{i-2}\pi^{i-1}$, we obtain $c^{i-1} \in \mathcal{I}(L^{i-1})$, and equality (2.1) is fulfilled at step $i-1$. This establishes the existence of solutions π^i, c^i to equality (2.1) for each $i \in \mathbb{Z}$, i.e., the homotopy between the identity and zero cochain mappings of (L, d).

Sufficiency. If the identity mapping $1 = \left(1_{L^i}\right)_{i \in \mathbb{Z}}$ is homotopic to the zero mapping $0 = \left(0_{L^i}\right)_{i \in \mathbb{Z}}$ on $(L^{\cdot}, d)$, then the identity mapping on the cohomology $H^i(\phi_K(L^{\cdot}), \phi_K(d))$ vanishes for all $i \in \mathbb{Z}$. Hence, the complex $(\phi_K(L^{\cdot}), \phi_K(d))$ is exact for each Hilbert space K, as required. $\square$

Any solution $\pi^i \in \mathcal{L}(L^i, L^{i-1})$, $i \in \mathbb{Z}$, to equality (2.1) is called a parametrix of sequence $(L^{\cdot}, d)$ modulo class $\mathcal{I}$ operators. Thus, Theorem 2.3 just amounts to saying that a bounded above sequence of class $\mathcal{I}$ curvature is Fredholm if and only if it possesses a parametrix modulo class I operators. Given any Fredholm sequence $(L^{\cdot}, d)$ of class $\mathcal{I}$ curvature, if $f \in L^i$ satisfies $d^i f = 0$, then $f = c^i f + d^{i-1}\pi^i f$, where $(L^{\cdot}, \pi)$ is a parametrix for $(L^{\cdot}, d)$ as in equality (2.1). In other words, the operator d^{i-1} has a right inverse π^i modulo class $\mathcal{I}$ operators on solutions to $d^i u = 0$. However, since the compositions $d^i d^{i-1}$ need not vanish for a curved sequence $(L^{\cdot}, d)$, the range of d^{i-1} no longer lies in solutions of the equation $d^i u = 0$. It follows that the usual cohomology does not make sense for $(L^{\cdot}, d)$. The question on a proper substitute of the cohomology for curved sequences seems to be considerably subtle (see [3,4]).

Reduction to a complex

Let $(L^{\cdot}, d)$ be a Fredholm sequence consisting of Hilbert spaces L^i which are zero for all i but $i = 0, 1, \ldots, N$ and operators $d^i \in \mathcal{L}(L^i, L^{i+1})$ with $d^{i+1} \circ d^i$ of class $\mathcal{I}$.

These spaces and operators are fit together to form a sequence of Hilbert spaces of class $\mathcal{I}$ curvature, namely,

$$0 \to L^0 \stackrel{d^0}{\to} L^1 \stackrel{d^1}{\to} \dots \stackrel{d^{N-1}}{\to} L^N \to 0. \tag{3.1}$$

Theorem 3.1. *For every sequence (3.1) of class $\mathcal{I}$ curvature, there exist bounded operators $D^i \in \mathcal{L}(L^i, L^{i+1})$ satisfying $D^i = d^i$ modulo operators of $\mathcal{I}(L^i, L^{i+1})$ and $D^{i+1}D^i = 0$ for all i.*

Proof. Set $D^{N-1} = d^{N-1}$. The Laplacian

$$\Delta^N = D^{N-1}D^{N-1*}$$

is a self-adjoint operator on L^N, and its kernel just amounts to the kernel of D^{N-1*}. By Theorem 2.3, the latter operator D^{N-1*} has a left parametrix modulo compact operators. In fact, the equality

$$\pi^{N*}D^{N-1*} = 1_{L^N} - c^{N*}$$

holds on L^N. Hence, the identity operator on $\ker D^{N-1*}$ is compact. It follows that the kernel of D^{N-1*} is finite dimensional, so Δ^N is Fredholm.

By the abstract Hodge theory, there is a self-adjoint operator $G^N \in \mathcal{L}(L^N)$ mapping into the orthogonal complement of $\ker \Delta^N$ such that $1_{L^N} = H^N + \Delta^N G^N$ on L^N, where H^N is the orthogonal projection onto the finite-dimensional space $\ker \Delta^N = \ker D^{N-1*}$.

The space $\ker D^{N-1*}$ is, thus, an obstruction to the existence of a right inverse operator for D^{N-1}. The operator $\Phi^N = D^{N-1*}G^N$ is a special right parametrix for D^{N-1} in $\mathcal{L}(L^N, L^{N-1})$.

We now show that $P^{N-1} = 1_{L^{N-1}} - \Phi^N D^{N-1}$ is an orthogonal projection onto the kernel of D^{N-1}. To this end, we note that P^{N-1} is the identity operator on the kernel of D^{N-1}, and

$$\begin{aligned} D^{N-1}P^{N-1} &= D^{N-1} - \Delta^N G^N D^{N-1} \\ &= D^{N-1} - (1_{L^N} - H^N)D^{N-1} \\ &= 0 \end{aligned}$$

for $H^N D^{N-1} = (D^{N-1*}H^N)^* = 0$. From this, the desired conclusion follows.

In order to construct D^{N-2}, we consider the last fragment of sequence (3.1), namely,

$$L^{N-2} \stackrel{d^{N-2}}{\to} L^{N-1} \stackrel{D^{N-1}}{\to} L^N.$$

Set

$$D^{N-2} = P^{N-1}d^{N-2},$$

then $D^{N-2} \in \mathcal{L}(L^{N-2}, L^{N-1})$ satisfies

$$\begin{aligned} D^{N-1}D^{N-2} &= D^{N-1}P^{N-1}d^{N-2} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} D^{N-2} &= \left(1_{L^{N-1}} - \Phi^N d^{N-1}\right) d^{N-2} \\ &= d^{N-2}, \end{aligned}$$

modulo operators in $\mathcal{I}(L^{N-2}, L^{N-1})$, as desired.

We now restrict ourselves to the suitably modified preceding fragment of sequence (3.1), i.e.,

$$L^{N-3} \stackrel{d^{N-3}}{\to} L^{N-2} \stackrel{D^{N-2}}{\to} L^{N-1}.$$

The Laplacian $\Delta^{N-1} = D^{N-1*}D^{N-1} + D^{N-2}D^{N-2*}$ is a self-adjoint operator in $\mathcal{L}(L^{N-1})$, whose kernel is obviously $\ker D^{N-1} \cap \ker D^{N-2*}$. Our next goal is to prove that this kernel is of finite dimension. To this end, we observe that the equality

$$D^{N-2}\pi^{N-1} + \pi^N D^{N-1} = 1_{L^{N-1}} - C^{N-1}$$

holds for some compact operator $C^{N-1} \in \mathcal{I}(L^{N-1})$ since both $D^{N-1} - d^{N-1}$ and $D^{N-2} - d^{N-2}$ are of class $\mathcal{I}$. Hence, the identity operator on the cohomology $H^{N-1}(L^\cdot, D)$ is compact, and so the dimension of $H^{N-1}(L^\cdot, D)$ is finite. Since the natural embedding $\ker \Delta^{N-1} \hookrightarrow H^{N-1}(L^\cdot, D)$ is injective, we immediately deduce that the kernel of Δ^{N-1} is finite dimensional, too. This shows that the Laplacian Δ^{N-1} is Fredholm.

By the abstract Hodge theory, there is a self-adjoint operator $G^{N-1} \in \mathcal{L}(L^{N-1})$ which maps into the orthogonal complement of $\ker \Delta^{N-1}$ and fulfills

$$1_{L^{N-1}} = H^{N-1} + \Delta^{N-1}G^{N-1}$$

on L^{N-1}, where H^{N-1} is the orthogonal projection onto the finite-dimensional space $\ker \Delta^{N-1}$.

We claim that $D^{N-1}G^{N-1} = G^N D^{N-1}$. To prove this, pick an arbitrary element $u \in L^{N-1}$. Then,

$$D^{N-1}u = D^{N-1}D^{N-1*}D^{N-1}G^{N-1}u$$

on one hand, and

$$D^{N-1}u = D^{N-1}D^{N-1*}G^N D^{N-1}u$$

on the other hand. Hence, it follows that $\Delta^N \left(D^{N-1}G^{N-1}u - G^N D^{N-1}u\right) = 0$, and since $D^{N-1}G^{N-1}u - G^N D^{N-1}u$ is orthogonal to $\ker \Delta^N$, we conclude that $D^{N-1}G^{N-1}u - G^N D^{N-1}u = 0$, as desired.

The composition $\Phi^{N-1} = D^{N-2*}G^{N-1}$ is, thus, an operator in $\mathcal{L}(L^{N-1}, L^{N-2})$ satisfying the homotopy equation

$$\Phi^N D^{N-1} + D^{N-2}\Phi^{N-1} = 1_{L^{N-1}} - H^{N-1}.$$

In other words, the pair $\{\Phi^{N-1}, \Phi^N\}$ is a special parametrix at steps $N-1$ and N for sequence (3.1).

To construct D^{N-3}, we can now argue in the same way as in the construction of D^{N-2}. Namely, let us show that $P^{N-2} = 1_{L^{N-2}} - \Phi^{N-1}D^{N-2}$ is an orthogonal projection onto the kernel of D^{N-2}. To this end, we note that P^{N-2} is the identity operator on the kernel of D^{N-2}, and

$$\begin{aligned} D^{N-2}P^{N-2} &= D^{N-2} - D^{N-2}\Phi^{N-1}D^{N-2} \\ &= D^{N-2} - (1_{L^{N-1}} - H^{N-1} - \Phi^N D^{N-1})D^{N-2} \\ &= 0 \end{aligned}$$

for $H^{N-1}D^{N-2} = (D^{N-2*}H^{N-1})^* = 0$. From this, the desired conclusion readily follows.

Set

$$D^{N-3} = P^{N-2}d^{N-3},$$

then $D^{N-3} \in \mathcal{L}(L^{N-3}, L^{N-2})$ satisfies

$$\begin{aligned} D^{N-2}D^{N-3} &= D^{N-2}P^{N-2}d^{N-3} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} D^{N-3} &= \left(1_{L^{N-2}} - \Phi^{N-1}D^{N-2}\right)d^{N-3} \\ &= \left(1_{L^{N-2}} - \Phi^{N-1}d^{N-2}\right)d^{N-3} \\ &= d^{N-3} \end{aligned}$$

modulo operators in $\mathcal{I}(L^{N-3}, L^{N-2})$, as desired.

We now proceed by induction, thus completing the proof, for sequence (3.1) terminates. □

Lefschetz number

Consider a Fredholm sequence (3.1) of trace class curvature, with L^i being Hilbert spaces. By Theorem 3.1, there are operators $D^i \in \mathcal{L}(L^i, L^{i+1})$ such that $D^i = d^i$ modulo trace class operators and $D^{i+1}D^i = 0$ for all i. We thus arrive at a Fredholm complex

$$0 \to L^0 \xrightarrow{D^0} L^1 \xrightarrow{D^1} \dots \xrightarrow{D^{N-1}} L^N \to 0, \tag{4.1}$$

the latter being a consequence of the fact that Fredholm sequences of trace class curvature are stable under perturbations of trace class.

Suppose $e = (e^i)_{i\in\mathbb{Z}}$ is a cochain mapping of complex $(L^\cdot, D)$ into itself, i.e., $e^i \in \mathcal{L}(L^i)$ satisfies $D^i e^i = e^{i+1}D^i$ for all i. Such a mapping preserves the spaces of cocycles and coboundaries of complex $(L^\cdot, D)$. On passing to quotient spaces, it induces the homomorphisms $(He)^i$ of cohomology $H^i(L^\cdot, D)$ for each i. Since the cohomology is finite dimensional at each step, the traces $\mathrm{tr}\,(He)^i$ of the linear mappings are well defined.

Definition 4.1. The Lefschetz number of a cochain mapping e of $(L^\cdot, D)$ is defined to be the (possibly, complex) number

$$L(e) = \sum_{i=0}^{N} (-1)^i \,\mathrm{tr}\,(He)^i.$$

In particular, if e is the identity mapping of $(L^\cdot, D)$, then $L(e)$ is the Euler characteristic of this complex. In [3], the Euler characteristic of $(L^\cdot, D)$ is proved to depend on the sequence $(L^\cdot, d)$ solely. In this way, the Euler characteristic is defined not only for sequences of zero curvature but also for those of compact curvature. The question arises whether the Lefschetz number is actually independent of the complex $(L^\cdot, D)$ and is determined by $(L^\cdot, d)$. The following theorem gives a partial evidence of this fact.

Theorem 4.2. *As defined above, the Lefschetz number of the endomorphism e is given by the formula*

$$L(e) = \sum_{i=0}^{N} (-1)^i \, tr\left(e^i - \left(e^i\pi^{i+1}\right)d^i - d^{i-1}\left(e^{i-1}\pi^i\right)\right), \tag{4.2}$$

where $\{\pi^i\}$ *is a parametrix of* $(L^\cdot, d)$ *modulo trace class operators.*

Proof. Since $D^i - d^i$ is of trace class for each $i = 0, 1, \dots, N$, it follows that $\{\pi^i\}$ is a parametrix of complex $(L^\cdot, D)$ modulo trace class operators. By the homotopy formula,

$$\pi^{i+1}D^i - D^{i-1}\pi^i = 1_{L^i} - r^i$$

for all i, where r^i is a trace class operator on L^i. On applying the cochain mapping e^i to both sides of this equality, we obtain

$$(e^i\pi^{i+1})D^i - D^{i-1}(e^{i-1}\pi^i) = e^i - e^i \circ r^i,$$

i.e., the cochain self-mappings $(e^i)_{i\in\mathbb{Z}}$ and $(e^i \circ r^i)_{i\in\mathbb{Z}}$ of complex $(L^\cdot, D)$ are homotopic. Hence, they induce the same action on the cohomology of $(L^\cdot, D)$, which gives

$$\begin{aligned} L(e) &= L(e \circ r) \\ &= \sum_{i=0}^{N} (-1)^i \,\mathrm{tr}\, e^i \circ r^i, \end{aligned}$$

the latter equality being a consequence of the Euler identity (see for instance Theorem 19.1.15 in [11]). To complete the proof, it remains to use an argument of [12] (p. 203), namely,

$$\begin{aligned}\sum_{i=0}^{N}(-1)^i \operatorname{tr} e^i \circ r^i &= \sum_{i=0}^{N}(-1)^i \operatorname{tr}\left(e^i - (e^i\pi^{i+1})D^i\right.\\ &\quad\left. -D^{i-1}\left(e^{i-1}\pi^i\right)\right)\\ &= \sum_{i=0}^{N}(-1)^i \operatorname{tr}\left(e^i - \left(e^i\pi^{i+1}\right)d^i\right.\\ &\quad\left. -d^{i-1}\left(e^{i-1}\pi^i\right)\right),\end{aligned}$$

which is due to a familiar theorem of Lidskii, for D^i and d^i differ by trace class operators. □

Obviously, $(e^i)_{i\in\mathbb{Z}}$ is a cochain mapping of sequence $(L^\cdot, d)$ for $d^ie^i = e^{i+1}d^i$ modulo trace class operators. However, we are able to introduce the Lefschetz number only for those cochain mappings of $(L^\cdot, d)$ which are cochain mappings of some Fredholm complex $(L^\cdot, D)$ which is a perturbation of $(L^\cdot, d)$ by trace class operators. It would be desirable to show that, given any cochain mapping e of $(L^\cdot, d)$, there is a Fredholm complex $(L^\cdot, D)$, whose differential D differs from d by trace class operators and commutes with e, but we have not been able to do this. In any case, we can define the Lefschetz number of arbitrary cochain mapping e of $(L^\cdot, d)$ by formula (4.2). As already mentioned, this definition will depend on the particular choice of neither parametrix π nor the differential d up to trace class operators.

Relative de Rham cohomology

In this section, we indicate how formula (4.2) may be used to derive an explicit formula for the Lefschetz number in relative de Rham cohomology.

Let $\mathcal{X}$ be a smooth compact closed manifold of dimension n, and $\mathcal{Y}$ be a submanifold of $\mathcal{X}$ of dimension q. For simplicity, we assume that $\mathcal{X}$ is orientable.

For $i \in \mathbb{Z}$, we denote by $\Lambda^iT^*\mathcal{X}$ the bundle of exterior forms of degree i over$\mathcal{X}$. These bundles are nonzero only for $i = 0, 1, \ldots, n$. They fit together to form a complex $\Omega^\cdot(\mathcal{X})$ on $\mathcal{X}$ whose differential is given by the exterior derivative on differential forms. This complex is referred to as the de Rham complex on $\mathcal{X}$ and is known to be elliptic.

Similarly, we have the de Rham complex $\Omega^\cdot(\mathcal{Y})$ on $\mathcal{Y}$. The length of this latter is actually equal to $q < n$. However, we may complete it by the zero bundles $\Lambda^{q+1}T^*\mathcal{Y}, \ldots, \Lambda^nT^*\mathcal{Y}$, thus arriving at a complex of length n.

Let ι stand for the embedding $\mathcal{Y} \hookrightarrow \mathcal{X}$. Thus, ι is a differentiable mapping, and we denote by ι^* the corresponding 'pull-back' operator on differential forms. Then, ι^* is well known to be a cochain mapping of the complexes $\Omega^\cdot(\mathcal{X}) \to \Omega^\cdot(\mathcal{Y})$. The cone of this mapping is

$$\begin{aligned}\mathcal{C}^\cdot:\ 0 \longrightarrow \begin{matrix} C^\infty(\mathcal{X}) \\ \oplus \\ 0 \end{matrix} \xrightarrow{\mathcal{A}^0} \begin{matrix} \Omega^1(\mathcal{X}) \\ \oplus \\ C^\infty(\mathcal{Y}) \end{matrix} \xrightarrow{\mathcal{A}^1} \ldots\\ \xrightarrow{\mathcal{A}^{n-1}} \begin{matrix} \Omega^n(\mathcal{X}) \\ \oplus \\ \Omega^{n-1}(\mathcal{Y}) \end{matrix} \longrightarrow 0,\end{aligned} \tag{5.1}$$

where

$$\mathcal{A}^i = \begin{pmatrix} -d^i & 0 \\ \iota^* & d^{i-1} \end{pmatrix},$$

d^i meaning the exterior derivative restricted to differential forms of degree i.

The key result on complex (5.1) is that it actually bears an information on the relative singular cohomology of the pair $(\mathcal{X}, \mathcal{Y})$. The following result can be certainly attributed to the mathematical *folk lore* (*cf.* for instance Lemma 6.1 in [13]).

Lemma 5.1. *For each $i = 0, 1, \ldots, n$, there is natural isomorphism*

$$H^i(\mathcal{C}^\cdot) \cong H^i((\mathcal{X}, \mathcal{Y}), \mathbb{R}),$$

where $H^i((\mathcal{X}, \mathcal{Y}), \mathbb{R})$ is the relative cohomology of the pair $(\mathcal{X}, \mathcal{Y})$ with real coefficients.

Proof. By the de Rham theorem, we have natural isomorphisms

$$\begin{aligned}H^i(\Omega^\cdot(\mathcal{X})) &\cong H^i(\mathcal{X}, \mathbb{R}),\\ H^i(\Omega^\cdot(\mathcal{Y})) &\cong H^i(\mathcal{Y}, \mathbb{R})\end{aligned} \tag{5.2}$$

for each i. We are going to make use of these to derive the desired isomorphisms in the relative cohomology.

To this end, we invoke a standard exact long sequence of singular homology with coefficients in $\mathbb{R}$,

$$\begin{aligned}0 \longleftarrow H_0((\mathcal{X}, \mathcal{Y}), \mathbb{R}) \xleftarrow{i} H_0(\mathcal{X}, \mathbb{R}) \xleftarrow{i} H_0(\mathcal{Y}, \mathbb{R}) \xleftarrow{\partial}\\ \longleftarrow H_1((\mathcal{X}, \mathcal{Y}), \mathbb{R}) \xleftarrow{i} H_1(\mathcal{X}, \mathbb{R}) \xleftarrow{i} H_1(\mathcal{Y}, \mathbb{R}) \xleftarrow{\partial} \ldots\\ \longleftarrow H_i((\mathcal{X}, \mathcal{Y}), \mathbb{R}) \xleftarrow{i} H_i(\mathcal{X}, \mathbb{R}) \xleftarrow{i} H_i(\mathcal{Y}, \mathbb{R}) \xleftarrow{\partial} \ldots,\end{aligned} \tag{5.3}$$

i being induced by the inclusion of cycles and ∂ being induced by the boundary operator. Dual to this, we have a standard exact long cohomological sequence

$$\begin{aligned}0 \longrightarrow H^0((\mathcal{X}, \mathcal{Y}), \mathbb{R}) \xrightarrow{i'} H^0(\mathcal{X}, \mathbb{R}) \xrightarrow{i'} H^0(\mathcal{Y}, \mathbb{R}) \xrightarrow{\delta}\\ \longrightarrow H^1((\mathcal{X}, \mathcal{Y}), \mathbb{R}) \xrightarrow{i'} H^1(\mathcal{X}, \mathbb{R}) \xrightarrow{i'} H^1(\mathcal{Y}, \mathbb{R}) \xrightarrow{\delta} \ldots\\ \longrightarrow H^i((\mathcal{X}, \mathcal{Y}), \mathbb{R}) \xrightarrow{i'} H^i(\mathcal{X}, \mathbb{R}) \xrightarrow{i'} H^i(\mathcal{Y}, \mathbb{R}) \xrightarrow{\delta} \ldots,\end{aligned} \tag{5.4}$$

δ being known as the coboundary operator. Recall that this latter sequence is obtained from sequence (5.3) by applying the functor $\mathrm{Hom}_{\mathbb{R}}(\cdot, \mathbb{R})$

The task is now to construct a sequence of the de Rham cohomology analogous to (5.4), i.e.,

$$\begin{aligned} 0 &\longrightarrow H^0(\mathcal{C}^\cdot) \xrightarrow{\pi} H^0(\Omega^\cdot(\mathcal{X})) \xrightarrow{\iota^*} H^0(\Omega^\cdot(\mathcal{Y})) \xrightarrow{\delta} \\ &\longrightarrow H^1(\mathcal{C}^\cdot) \xrightarrow{\pi} H^1(\Omega^\cdot(\mathcal{X})) \xrightarrow{\iota^*} H^1(\Omega^\cdot(\mathcal{Y})) \xrightarrow{\delta} \dots \\ &\longrightarrow H^i(\mathcal{C}^\cdot) \xrightarrow{\pi} H^i(\Omega^\cdot(\mathcal{X})) \xrightarrow{\iota^*} H^i(\Omega^\cdot(\mathcal{Y})) \xrightarrow{\delta} \dots . \end{aligned} \tag{5.5}$$

Namely, we define π, ι^*, and δ by

$$\begin{array}{llllll} \pi: & \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} & \mathrm{mod}\ B^i(\mathcal{C}^\cdot) & \mapsto & u_1 & \mathrm{mod}\ B^i(\Omega^\cdot(\mathcal{X})) \\ \iota^*: & u & \mathrm{mod}\ B^i(\Omega^\cdot(\mathcal{X})) & \mapsto & \iota^* u & \mathrm{mod}\ B^i(\Omega^\cdot(\mathcal{Y})) \\ \delta: & f & \mathrm{mod}\ B^i(\Omega^\cdot(\mathcal{Y})) & \mapsto & \begin{pmatrix} 0 \\ f \end{pmatrix} & \mathrm{mod}\ B^{i+1}(\mathcal{C}^\cdot) \end{array}$$

for $\binom{u_1}{u_2} \in Z^i(\mathcal{C}^\cdot)$, $u \in Z^i(\Omega^\cdot(\mathcal{X}))$, and $f \in Z^i(\Omega^\cdot(\mathcal{Y}))$. By a familiar result of homological algebra, sequence (5.5) is exact.

From isomorphisms (5.2), sequence (5.4), and sequence (5.5), it follows that the spaces $H^i(\mathcal{C}^\cdot)$ and $H^i((\mathcal{X}, \mathcal{Y}), \mathbb{R})$ are of the same dimension for each i. However, the lemma states more, namely, there is a natural isomorphism of these spaces. The existence of such an isomorphism is a consequence of the fact that there is a duality between sequences (5.5) and (5.3). This duality is given on

$$\begin{aligned} &H^i(\Omega^\cdot(\mathcal{X})) \times H_i(\mathcal{X}, \mathbb{R}), \\ &H^i(\Omega^\cdot(\mathcal{Y})) \times H_i(\mathcal{Y}, \mathbb{R}) \end{aligned}$$

by integrating differential forms over singular cycles, just as in the classical de Rham theorem (*cf.* [14]). On $H^i(\mathcal{C}^\cdot) \times H_i((\mathcal{X}, \mathcal{Y}), \mathbb{R})$, the duality is defined by

$$\begin{aligned} &\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \mathrm{mod}\ B^i(\mathcal{C}^\cdot), \sum_{\nu=1}^{N} c_\nu \Delta_\nu \quad \mathrm{mod}\ B_i((\mathcal{X}, \mathcal{Y}), \mathbb{R}) \right) \mapsto \\ &\qquad \times \sum_{\nu=1}^{N} c_\nu \left(\int_{\Delta_\nu} u_1 + \int_{\partial \Delta_\nu} u_2 \right) \end{aligned} \tag{5.6}$$

for $\binom{u_1}{u_2} \in Z^i(\mathcal{C}^\cdot)$ and $\sum_{\nu=1}^{N} c_\nu \Delta_\nu \in Z_i((\mathcal{X}, \mathcal{Y}), \mathbb{R})$, where Δ_ν are singular simplexes and $c_\nu \in \mathbb{R}$. It is immediate that (5.6) is well-defined.

Thus, both sequences (5.4) and (5.5) are dual to sequence (5.3). This gives natural homomorphisms of the spaces in sequence (5.5) to the corresponding spaces in sequence (5.4) Hence, we arrive at the commutative diagram

$$\begin{array}{ccccccc} H^{i-1}(\Omega^\cdot(\mathcal{Y})) & \xrightarrow{\delta} & H^i(\mathcal{C}^\cdot) & \xrightarrow{\pi} & H^i(\Omega^\cdot(\mathcal{X})) & \xrightarrow{\iota^*} & H^i(\Omega^\cdot(\mathcal{Y})) \\ \downarrow \cong & & \downarrow & & \downarrow \cong & & \downarrow \cong \\ H^{i-1}(\mathcal{Y}, \mathbb{R}) & \xrightarrow{\delta} & H^i((\mathcal{X}, \mathcal{Y}), \mathbb{R}) & \xrightarrow{\iota'} & H^i(\mathcal{X}, \mathbb{R}) & \xrightarrow{\iota'} & H^i(\mathcal{Y}, \mathbb{R}) \end{array}$$

with exact rows for each $i = 0, 1, \dots, n$. The homomorphisms marked by the vertical arrows in the diagram are actually isomorphisms, with the exception of $H^i(\mathcal{C}^\cdot) \to H^i((\mathcal{X}, \mathcal{Y}), \mathbb{R})$. Applying the 'lemma on five isomorphisms,' we can therefore assert that this latter homomorphism is also an isomorphism. This is our assertion. □

Having disposed of this preliminary step, we can now return to the Lefschetz fixed point formula.

Let f be a differentiable mapping of the manifold $\mathcal{X}$ with the property that $f(\mathcal{Y}) \subset \mathcal{Y}$. Then, f induces a mapping $f = (f_\mathcal{X}, f_\mathcal{Y})$ of the pair $(\mathcal{X}, \mathcal{Y})$ via $f_\mathcal{X} = f|_\mathcal{X}, f_\mathcal{Y} = f|_\mathcal{Y}$. The 'pull-back' operator f^* under f commutes with the exterior derivative on both $\mathcal{X}$ and $\mathcal{Y}$. Moreover, we have

$$\begin{aligned} f_\mathcal{Y}^* \iota^* &= (\iota \circ f_\mathcal{Y})^* \\ &= (f_\mathcal{X} \circ \iota)^* \\ &= \iota^* f_\mathcal{X}^* \end{aligned}$$

the second equality being due to the fact that $f(\mathcal{Y}) \subset \mathcal{Y}$. Hence, it follows that f has a lift to the complex $\mathcal{C}^\cdot$, the lift being given by f^*. We write $L(f^*)$ for the corresponding Lefschetz number. Lemma 5.1 allows one to conclude that $L(f^*)$ is just the classical Lefschetz number of f with respect to the relative cohomology of the pair $(\mathcal{X}, \mathcal{Y})$.

Suppose $p \in \mathcal{Y}$ is a fixed point of f. Then, the tangent mappings to $f_\mathcal{X}$ and $f_\mathcal{Y}$ induce linear transformations

$$\begin{aligned} &f'_\mathcal{X}(p) : T_p\mathcal{X} \to T_p\mathcal{X}, \\ &f'_\mathcal{Y}(p) : T_p\mathcal{Y} \to T_p\mathcal{Y} \end{aligned}$$

of the tangent spaces to $\mathcal{X}$ and $\mathcal{Y}$ at the point p, respectively. We thus arrive at a linear transformation of the quotient space $T_p\mathcal{X}/T_p\mathcal{Y}$, namely,

$$f'_{\mathcal{X}/\mathcal{Y}}(p) : \frac{T_p\mathcal{X}}{T_p\mathcal{Y}} \to \frac{T_p\mathcal{X}}{T_p\mathcal{Y}}.$$

Moreover,

$$\det\left(I - f'_\mathcal{X}(p)\right) = \det\left(I - f'_\mathcal{Y}(p)\right) \det\left(I - f_{\mathcal{X}/\mathcal{Y}}(p)\right),$$

as is easy to see by using local coordinates at p.

In particular, we deduce that if $p \in \mathcal{Y}$ is a simple fixed point of $f_\mathcal{X}$, then the determinant of $I - f'_{\mathcal{X}/\mathcal{Y}}(p)$ is different from zero. Denote by $\mathrm{Fix}^{(\pm)}(f, \mathcal{Y})$ the set of all simple fixed points of f on $\mathcal{Y}$ with the property that $\pm\det(I - f'_{\mathcal{X}/\mathcal{Y}}(p)) > 0$ is valid.

Theorem 5.2. *Let f be a differentiable mapping of the pair $(\mathcal{X}, \mathcal{Y})$ with simple fixed points. Then,*

$$L(f^*) = \sum_{p \in \mathrm{Fix}(f, \mathcal{X} \setminus \mathcal{Y})} \mathrm{sgn}\ \det(I - f'(p)) + 2 \times \sum_{p \in \mathrm{Fix}^{(-)}(f, \mathcal{Y})} \mathrm{sgn}\ \det(I - f'(p)).$$

Proof. Indeed, if $p \in \mathcal{Y}$ is a simple fixed point of f, then

$$\mathrm{sgn}\ \det\left(I - f'_{\mathcal{X}}(p)\right) = \pm\, \mathrm{sgn}\ \det\left(I - f'_{\mathcal{Y}}(p)\right)$$

where '+' is taken for $p \in \mathrm{Fix}^{(+)}(f, \mathcal{Y})$ and '−' for $p \in \mathrm{Fix}^{(-)}(f, \mathcal{Y})$. Thus, the contributions of the points $p \in \mathrm{Fix}^{(+)}(f, \mathcal{Y})$ in the Lefschetz number cancel, while the contributions of the points $p \in \mathrm{Fix}^{(-)}(f, \mathcal{Y})$ duplicate. This establishes the formula. □

In contrast to [15], the specification of simple fixed points of f on $\mathcal{Y}$ by those in $\mathrm{Fix}^{(+)}(f, \mathcal{Y})$ and $\mathrm{Fix}^{(-)}(f, \mathcal{Y})$ is much more complicated than the specification by being attracting or repulsing ones. We emphasize that $\mathrm{Fix}^{(+)}(f, \mathcal{Y})$ contains all attracting fixed points of f on $\mathcal{Y}$ along with a part of repulsing fixed points.

Note that Theorem 5.2 can be also obtained from the algebraic alternating sum formula applied to sequence (5.4). Indeed, this sequence is exact, and f induces an endomorphism f^* of the sequence, which results immediately in the equality $L(f^*) = L(f^*_M) - L(f^*_S)$.

When combined with Theorem 4.2, Theorem 5.2 gives much more. Namely, perturbations of the differentials of $\Omega^{\cdot}(\mathcal{X})$ and $\Omega^{\cdot}(\mathcal{Y})$ by trace class terms do not affect the alternating sum of the traces. Moreover, the 'pull-back' operator f^* need not satisfy the commutativity relations with perturbed differentials of $\Omega^{\cdot}(\mathcal{X})$ and $\Omega^{\cdot}(\mathcal{Y})$ precisely, but only up to trace class operators. The fixed point formula of Theorem 5.2 remains still valid under reasonably regularized definition of the Lefschetz number.

Conclusion

We introduced the Lefschetz number of cochain mappings for arbitrary Fredholm sequences of trace class curvature. Our definition applies in particular to geometric cochain self-mappings of elliptic sequences of pseudodifferential operators on a compact closed manifold. This initiates a substantial fixed point theory for elliptic sequences which extends the Lefschetz theory for elliptic complexes by Atiayh and Bott. As but one example, we evaluated the Lefschetz number for the relative de Rham cohomology.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
NT elaborated the techniques of reducing a Fredholm sequence of compact curvature to a Fredholm complex and introduced the Euler characteristic for Fredholm complexes. DW applied the techniques to elliptic sequences of pseudodifferential operators on a compact closed manifold and generalized the Euler characteristic to the Lefschetz number. Both authors read and approved the final manuscript.

Acknowledgements
This research was supported by Russian Foundation for Basic Research, grant 11-01-91330, and by the German Research Society (DFG), grant TA 289/4-2.

Author details
[1]Institut für Mathematik, Universität Potsdam, m Neuen Palais 10, Potsdam, 14469, Germany. [2]Institut für Mathematik, Universität Potsdam, Am Neuen Palais 10, Potsdam, 14469, Germany.

Received: 10 May 2012 Accepted: 5 June 2012
Published: 4 October 2012

References

1. Bourbaki, N: Algèbre Homologique. Masson, Paris (1980)
2. Pietsch, A: Operator Ideals. VEB Deutscher Verlag der Wissenschaften, Berlin (1978)
3. Tarkhanov, N: Euler characteristic of Fredholm quasicomplexes. Funct. Anal. Appl. **41**(4), 87–93 (2007)
4. Wallenta, D: Elliptic quasicomplexes on compact closed manifolds. Integr. Equ. Oper. Theory. **73**, 517–536 (2012)
5. Atiyah, MF, Bott, R: A Lefschetz fixed point formula for elliptic complexes: I. Ann. Math. **86**(2), 374–407 (1967)
6. Tarkhanov, N: Complexes of Differential Operators. Kluwer Academic, Dordrecht (1995)
7. Kuz'minov, VI, Shvedov, IA: On the compact solvability of differentials of an elliptic differential complex. Siberian Math. J. **44**(6), 1003–1014 (2003)
8. Kuz'minov, VI, Shvedov, IA: The exterior differentiation operator on Riemannian manifolds with cylindrical ends. Siberian Math. J. **48**(3), 500–507 (2007)
9. Putinar, M: Some invariants for semi-Fredholm systems of essentially commuting operators. J. Operator Theory. **8**, 65–90 (1982)
10. Eschmeier, J, Putinar, M: Spectral Decompositions and Analytic Sheaves. The Clarendon Press, New York (1996)
11. Hörmander, L: The Analysis of Linear Partial Differential Operators III. Springer, Berlin (1985)
12. Fedosov, BV: Index theorems. In: Modern Problems of Mathematics. Fundamental Directions, vol. 65, pp. 165–268. VINITI, Moscow, (1991)
13. Tarkhanov, N, Schulze, B-W: A Lefschetz fixed point formula in the relative elliptic theory. Comm. Applied Anal. **4**(1), 99–119 (2000)
14. De Rham, G: Variétés différentiables. Formes, courants, formes harmoniques (Actualites scientifiques et industrielles), vol 1222. Hermann, Paris (1955)
15. Brenner, AV, Shubin, MA: The Atiyah-Bott-Lefschetz theorem for manifolds with boundary. Funkts. Analiz. **15**(4), 67–68 (1981)

doi:10.1186/2251-7456-6-44
Cite this article as: Tarkhanov and Wallenta: **The Lefschetz number of sequences of trace class curvature.** *Mathematical Sciences* 2012 **6**:44.