



Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra



Pro- p groups with constant generating number on open subgroups

Benjamin Klopsch^a, Ilir Snopce^{b,*}

^a Department of Mathematics, Royal Holloway, University of London, Egham TW20 0EX, UK

^b Department of Mathematics, University of Virginia, Charlottesville, VA 22904-4137, USA

ARTICLE INFO

Article history:

Received 7 June 2010

Available online 7 January 2011

Communicated by Efim Zelmanov

MSC:

20E18

22E20

Keywords:

Pro- p groups

p -adic analytic groups

Saturable groups

p -adic Lie algebras

Minimal number of generators

ABSTRACT

Let p be a prime. We classify finitely generated pro- p groups G which satisfy $d(H) = d(G)$ for all open subgroups H of G . Here $d(H)$ denotes the minimal number of topological generators for the subgroup H . Within the category of p -adic analytic pro- p groups, this answers a question of Iwasawa.

© 2011 Elsevier Inc. All rights reserved.

1. Introduction

Let p be a prime. Motivated by a question of Kenkichi Iwasawa, which we discuss below, we consider profinite groups G satisfying the condition

$$d(H) = d(G) \quad \text{for all open subgroups } H \text{ of } G. \quad (*)$$

Here $d(H)$ denotes the minimal number of topological generators for the subgroup H . For lack of a shorter description, we say that profinite groups G with the property $(*)$ have *constant generating number on open subgroups*. The aim of this paper is to give a complete classification of finitely generated pro- p groups with constant generating number on open subgroups.

* Corresponding author.

E-mail addresses: Benjamin.Klopsch@rhul.ac.uk (B. Klopsch), snopce@math.binghamton.edu (I. Snopce).

Theorem 1.1. *Let G be a finitely generated pro- p group and let $d := d(G)$. Then G has constant generating number on open subgroups if and only if it is isomorphic to one of the groups in the following list:*

- (1) *the abelian group \mathbb{Z}_p^d , for $d \geq 0$;*
- (2) *the metabelian group $\langle y \rangle \rtimes A$, for $d \geq 2$, where $\langle y \rangle \cong \mathbb{Z}_p$, $A \cong \mathbb{Z}_p^{d-1}$ and y acts on A as scalar multiplication by λ , with $\lambda = 1 + p^s$ for some $s \geq 1$, if $p > 2$, and $\lambda = \pm(1 + 2^s)$ for some $s \geq 2$, if $p = 2$;*
- (3) *the group $\langle w \rangle \rtimes B$ of maximal class, for $p = 3$ and $d = 2$, where $\langle w \rangle \cong C_3$, $B = \mathbb{Z}_3 + \mathbb{Z}_3\omega \cong \mathbb{Z}_3^2$ for a primitive 3rd root of unity ω and where w acts on B as multiplication by ω ;*
- (4) *the metabelian group $\langle y \rangle \rtimes A$, for $p = 2$ and $d \geq 2$, where $\langle y \rangle \cong \mathbb{Z}_2$, $A \cong \mathbb{Z}_2^{d-1}$ and y acts on A as scalar multiplication by -1 .*

This generalises the partial results in [10], which were obtained by quite different methods. Theoretically our work is linked to other recent results on generating numbers of pro- p groups; e.g. see [5,7]. The metabelian groups resulting from our classification can easily be realised as subgroups of the affine group $\text{Aff}_d(\mathbb{Z}_p) = \text{GL}_d(\mathbb{Z}_p) \rtimes \mathbb{Z}_p^d$.

In view of our theorem, it is an interesting problem to investigate more general classes of finitely generated profinite groups with respect to the property $(*)$, e.g. compact p -adic analytic groups. We conclude this introduction with a short motivation and a sketch of the proof of Theorem 1.1.

It is well known that the (topological) Schreier index formula can be used to characterise free pro- p groups: a finitely generated pro- p group G is a free pro- p group if and only if $d(H) - 1 = |G : H|(d(G) - 1)$ for every open subgroup H of G ; cf. [8]. For every positive integer n , let \mathcal{E}_n denote the class of all finitely generated pro- p groups G satisfying:

$$d(H) - n = |G : H|(d(G) - n) \quad \text{for all open subgroups } H \text{ of } G.$$

For instance, the class \mathcal{E}_1 consists precisely of the finitely generated free pro- p groups. As described in [2], Iwasawa observed that, for fixed n , pro- p groups belonging to the class \mathcal{E}_n have interesting representation-theoretic properties and he raised the question of determining the groups belonging to \mathcal{E}_n for each $n > 1$. In [2], Dummit and Labute answer Iwasawa's question for $n = 2$ in the case of one-relator groups: G is a Demushkin group if and only if G is a one-relator group, infinite and $G \in \mathcal{E}_2$.

Note that \mathcal{E}_n is non-empty for every n because, clearly, it contains the free abelian pro- p group \mathbb{Z}_p^n of rank n . In [11], Yamagishi remarked that no other examples are known to him when $n \geq 3$. Recently, for $p > 3$, the second author determined all p -adic analytic pro- p groups that belong to the class \mathcal{E}_3 ; see [10]. His approach relied on the classification of 3-dimensional soluble \mathbb{Z}_p -Lie lattices provided in [4].

Theorem 1.1 answers completely the question of Iwasawa for pro- p groups of finite rank, or equivalently p -adic analytic pro- p groups. Indeed, it is easy to see that such a group G belongs to \mathcal{E}_n if and only if (i) $G \cong C_2$ and $n = 2$ or (ii) G has constant generating number on open subgroups and $n = \dim(G)$. Here $\dim(G)$ denotes the dimension of G as a p -adic Lie group. We record this consequence as

Corollary 1.2. *Let G be a p -adic analytic pro- p group and let $n \in \mathbb{N}$. Then G belongs to the class \mathcal{E}_n if and only if (i) $G \cong C_2$ and $n = 2$ or (ii) G is isomorphic to one of the groups listed in Theorem 1.1 and $n = \dim(G)$.*

Our proof of Theorem 1.1 proceeds as follows. By a Lie theoretic argument, we first deal with saturable pro- p groups: in this special situation we only obtain groups listed in (1) and (2) of the theorem; see Corollary 2.4. A general p -adic analytic pro- p group G contains a saturable normal subgroup of finite index. By induction on the index, it suffices to study the case where G contains an open normal subgroup H of 'known type' – e.g. saturable, and hence listed in (1) or (2) of the theorem – such that $G/H \cong C_p$. If H is abelian, we consider H as a $\mathbb{Z}_p[G/H]$ -module, and the minimal number of generators $d(\mathcal{M})$ for the $\mathbb{Q}_p[G/H]$ -module $\mathcal{M} := \mathbb{Q}_p \otimes H$ becomes a key invariant. If H is not abelian, we consider the actions of G/H and H/A on an abelian characteristic subgroup A of

co-dimension 1 in H . While the details of the argument work out smoothly for $p \geq 5$, a more careful analysis is required to deal the small primes $p = 2$ and $p = 3$.

Notation. Throughout the paper, p denotes a prime. We write \mathbb{N} for the set of natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The p -adic integers and p -adic numbers are denoted by \mathbb{Z}_p and \mathbb{Q}_p . The minimal number of generators of a group G is denoted by $d(G)$, and a similar notation is employed for other algebraic structures, such as Lie algebras, Lie lattices and modules. Moreover, we tacitly interpret generators as topological generators as appropriate.

2. Saturable pro- p groups and preliminaries

Let G be a finitely generated pro- p group with constant generating number on open subgroups. Then G has finite rank and admits the structure of a p -adic analytic group (in a unique way); cf. [1, Chapters 8 and 9]. There is a normal open subgroup H of G which is uniformly powerful or, more generally, saturable. This group H corresponds via Lie theory to a \mathbb{Z}_p -Lie lattice L_H which is powerful (respectively saturable); see [1,6,4]. At the level of saturable pro- p groups the group theoretic property we are interested in translates readily to an equivalent condition on the associated Lie lattice.

Proposition 2.1. *Let G be a saturable pro- p group with associated \mathbb{Z}_p -Lie lattice $L = L_G$. Then G has constant generating number on open subgroups if and only if the \mathbb{Q}_p -Lie algebra $\mathcal{L} := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} L$ satisfies $d(\mathcal{L}) = \dim_{\mathbb{Q}_p} \mathcal{L}$.*

The proof of this plausible result is not completely straightforward, because subgroups of saturable pro- p groups are not necessarily saturable. We begin by proving an auxiliary lemma.

Lemma 2.2. *Let \mathcal{L} be a finite-dimensional Lie algebra over a field F such that $d(\mathcal{L}) = \dim_F(\mathcal{L})$. Then either \mathcal{L} is abelian or \mathcal{L} is metabelian of the form $\mathcal{L} = Fx \oplus \mathcal{A}$, where \mathcal{A} is an abelian ideal of co-dimension 1 in \mathcal{L} and $\text{ad}(x)|_{\mathcal{A}} = \text{id}_{\mathcal{A}}$.*

Proof. Suppose that \mathcal{L} is not abelian. By [9, Lemma 4.3], the Lie algebra \mathcal{L} is metabelian, i.e. $[\mathcal{L}, \mathcal{L}]$ is abelian. Let \mathcal{A} be a maximal abelian Lie subalgebra of \mathcal{L} containing $[\mathcal{L}, \mathcal{L}]$. Then \mathcal{A} is an ideal of \mathcal{L} and equal to its centraliser $C_{\mathcal{L}}(\mathcal{A})$ in \mathcal{L} . Because $d(\mathcal{L}) = \dim_F(\mathcal{L})$, it is clear that for every $x \in \mathcal{L}$ the action of $\text{ad}(x)$ on \mathcal{A} is scalar, and we obtain a representation $\rho : \mathcal{L} \rightarrow \mathfrak{gl}_1(F)$. Since $C_{\mathcal{L}}(\mathcal{A}) = \mathcal{A} \not\subseteq \mathcal{L}$, the sequence

$$0 \longrightarrow \mathcal{A} \xrightarrow{\text{incl.}} \mathcal{L} \xrightarrow{\rho} \mathfrak{gl}_1(F) \longrightarrow 0$$

is exact, and we find $x \in \mathcal{L} \setminus \mathcal{A}$ such that $\mathcal{L} = Fx \oplus \mathcal{A}$ and $\text{ad}(x)|_{\mathcal{A}} = \text{id}_{\mathcal{A}}$. \square

Corollary 2.3. *Let L be a \mathbb{Z}_p -Lie lattice of dimension d such that $d(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} L) = d$. Then one of the following holds:*

- (i) $L \cong \mathbb{Z}_p^d$ is abelian;
- (ii) $L = \mathbb{Z}_p x \oplus A$ where $A \cong \mathbb{Z}_p^{d-1}$ is an abelian ideal of L and $\text{ad}(x)$ acts on A as multiplication by p^s for suitable $s \in \mathbb{N}_0$.

Proof of Proposition 2.1. Put $d := \dim_{\mathbb{Q}_p} \mathcal{L}$, and suppose that G has constant generating number on open subgroups. For a contradiction assume that $d(\mathcal{L}) < d$. Then L admits an open Lie sublattice M which requires less than d generators, say a_1, \dots, a_r where $r < d$. The saturable group G is recovered from the \mathbb{Z}_p -Lie lattice L by defining a group multiplication on the set L via the Hausdorff series. We may thus consider the subgroup $H := \langle a_1, \dots, a_r \rangle$ of G . To arrive at the required contradiction, it suffices to show that H is open in G . We find an open saturable subgroup K of H and $k \in \mathbb{N}$ such

that $a_1^{p^k}, \dots, a_r^{p^k} \in K$. The \mathbb{Z}_p -Lie lattice associated to K is naturally a Lie sublattice of L and contains $p^k a_1, \dots, p^k a_r$. It is therefore of finite index in M and hence open in L . Consequently, K is open in G .

Conversely, suppose that $d(\mathcal{L}) = d$. Then Corollary 2.3 shows that L is of a rather restricted form. Clearly, if L is abelian, so is G and G has constant generating number on open subgroups. Suppose that L is not abelian so that we have $L = \mathbb{Z}_p x \oplus A$ where $A \cong \mathbb{Z}_p^{d-1}$ is an abelian ideal of L and $\text{ad}(x)$ acts on A as multiplication by p^s for suitable $s \in \mathbb{N}_0$. Since L is saturable, we have $s \geq 1$ if $p > 2$ and $s \geq 2$ if $p = 2$; cf. [6, Section 2]. Thus G is isomorphic to the group $\langle y \rangle \times A$, where $\langle y \rangle \cong \mathbb{Z}_p$ and y acts on A as scalar multiplication by $1 + p^s$. One checks readily that the open subgroups of G are essentially of the same form, with the parameter s possibly taking larger values. This explicit description shows that G has constant generating number on open subgroups. \square

Corollary 2.4. *Let G be a saturable pro- p group of dimension d . Then G has constant generating number on open subgroups if and only if one of the following holds:*

- (i) $G \cong \mathbb{Z}_p^d$ is abelian;
- (ii) $G \cong \langle y \rangle \times A$, where $d \geq 2$, $\langle y \rangle \cong \mathbb{Z}_p$, $A \cong \mathbb{Z}_p^{d-1}$ and y acts on A as scalar multiplication by $1 + p^s$ for some $s \geq 1$, if $p > 2$, and $s \geq 2$, if $p = 2$.

We conclude this section with a technical lemma which plays a central role in our proof of Theorem 1.1.

Lemma 2.5. *Let $\langle z \rangle \cong C_p$ be a cyclic group of order p , and let M be a finitely generated $\mathbb{Z}_p\langle z \rangle$ -module which is free as a \mathbb{Z}_p -module. Then M decomposes as*

$$M \cong n_1 \cdot I_1 \oplus n_2 \cdot I_2 \oplus n_3 \cdot I_3,$$

where $n_1, n_2, n_3 \in \mathbb{N}_0$ and I_1, I_2, I_3 are indecomposable $\mathbb{Z}_p\langle z \rangle$ -modules of \mathbb{Z}_p -dimensions 1, $p - 1$, p so that $n_1 + (p - 1)n_2 + pn_3 = \dim_{\mathbb{Z}_p} M$.

If the $\mathbb{Q}_p\langle z \rangle$ -module $\mathcal{M} := \mathbb{Q}_p \otimes M$ satisfies $1 + d(\mathcal{M}) \geq \dim_{\mathbb{Q}_p} \mathcal{M}$, then

$$1 + \max\{0, n_2 - n_1\} \geq (p - 1)(n_2 + n_3). \tag{2.1}$$

Proof. According to [3], there are three types of indecomposable $\mathbb{Z}_p\langle z \rangle$ -modules which are free as \mathbb{Z}_p -modules:

- (i) the trivial module $I_1 = \mathbb{Z}_p$ of \mathbb{Z}_p -dimension 1,
- (ii) the module $I_2 = \mathbb{Z}_p\langle z \rangle / (\Phi(z))$ of \mathbb{Z}_p -dimension $p - 1$, where Φ denotes the p th cyclotomic polynomial,
- (iii) the free module $I_3 = \mathbb{Z}_p\langle z \rangle$ of \mathbb{Z}_p -dimension p .

The module M decomposes as described as a direct sum of indecomposable submodules.

Note that $\mathcal{I}_1 := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} I_1$ and $\mathcal{I}_2 := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} I_2$ are irreducible $\mathbb{Q}_p\langle z \rangle$ -modules, while $\mathcal{I}_3 := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} I_3$ decomposes as $\mathcal{I}_3 \cong \mathcal{I}_1 \oplus \mathcal{I}_2$. From this it follows that \mathcal{M} decomposes as $\mathcal{M} \cong (n_1 + n_3) \cdot \mathcal{I}_1 \oplus (n_2 + n_3) \cdot \mathcal{I}_2$, and consequently $d(\mathcal{M}) = \max\{n_1, n_2\} + n_3$. Since $\dim_{\mathbb{Q}_p} \mathcal{M} = \dim_{\mathbb{Z}_p} M = n_1 + (p - 1)n_2 + pn_3$, the assumption $1 + d(\mathcal{M}) \geq \dim_{\mathbb{Q}_p} \mathcal{M}$ leads to the inequality (2.1). \square

For later use we record in Table 1 the numerical consequences of this lemma.

3. The classification – proof of Theorem 1.1

In this section we prove Theorem 1.1. First we present an argument applying to the ‘generic’ case $p \geq 5$. The exceptional primes 2 and 3 can be treated in a similar way, but lead to extra complications.

Table 1

Values of $n_1, n_2, n_3 \in \mathbb{N}_0$ satisfying the inequality (2.1) and the condition $n_1 + (p - 1)n_2 + pn_3 = \dim_{\mathbb{Q}_p} \mathcal{M} \geq 2$.

Range of p	Conditions	n_1	n_2	n_3	Case label
$p = 2$	$n_1 > n_2$	≥ 2	0	0	(T 2.1)
		≥ 2	1	0	(T 2.2)
		≥ 1	0	1	(T 2.3)
	$n_1 \leq n_2$	0	≥ 2	0	(T 2.4)
		1	≥ 1	0	(T 2.5)
		0	≥ 0	1	(T 2.6)
$p = 3$	$n_1 > n_2$	≥ 2	0	0	(T 3.1)
	$n_1 \leq n_2$	0	1	0	(T 3.2)
$p \geq 5$	-	≥ 2	0	0	-

We leave it as an easy exercise to show that the groups listed in Theorem 1.1 have indeed constant generating number on open subgroups, apart from a brief comment on the case (3) in the appropriate subsection below. Clearly, the assertions of Theorem 1.1 for virtually pro-cyclic pro- p groups are correct, hence in the proofs below we only consider groups of dimension at least 2.

3.1. First suppose that $p \geq 5$. Let G be a finitely generated pro- p group with constant generating number $d = d(G) \geq 2$ on open subgroups. Then G has finite rank and thus admits an open normal subgroup H which is saturable. We need to show that G is isomorphic to one of the groups listed in (1) and (2) of Theorem 1.1, which are saturable.

We argue by induction on $|G : H|$. If $G = H$, the claim follows from Corollary 2.4. Hence suppose that $|G : H| \geq p$. Running along a subnormal series from H to G , we may assume by induction that $|G : H| = p$. This means that $G = \langle z \rangle H$ with $G/H = \langle \bar{z} \rangle \cong C_p$. Moreover, by Corollary 2.4, there are essentially two possibilities for the group H .

Case 1: H is abelian. We regard H as a $\mathbb{Z}_p\langle \bar{z} \rangle$ -module and put $\mathcal{M} := \mathbb{Q}_p \otimes H$. It is easily seen that G admits an open subgroup K , containing z , which requires at most $1 + d(\mathcal{M})$ generators. This implies $1 + d(\mathcal{M}) \geq d(K) = d = \dim_{\mathbb{Q}_p} \mathcal{M}$. Thus Lemma 2.5 and Table 1, for $p = 5$, show that z acts trivially on H . Hence G is abelian. Clearly, this implies that $G \cong \mathbb{Z}_p^d$.

Case 2: H is not abelian. Then Corollary 2.4 shows that H is of the form $\langle y \rangle \rtimes A$, where $\langle y \rangle \cong \mathbb{Z}_p$, $A \cong \mathbb{Z}_p^{d-1}$ is abelian, normal in H and conjugation by y on A corresponds to scalar multiplication by $1 + p^s$ for some $s \geq 1$. Note that A is characteristic in H , hence normal in G . (Indeed, A is the isolator of the commutator subgroup $[H, H]$ in H .) We consider two separate cases, $z^p \in A$ and $z^p \notin A$.

Case 2.1: $z^p \in A$. We contend that the group $\langle z \rangle A$ has constant generating number $d - 1$ on open subgroups. Indeed, given an open subgroup B of $\langle z \rangle A$, we choose a normal open subgroup N of G and $r \in \mathbb{N}$ such that $d(B) = d(BN/N)$ and $y^{p^r} \in N$. Then $K := \langle B \cup \{y^{p^r}\} \rangle$ is an open subgroup of G with $d(K) \leq d(B) + 1$. But $KN/N = BN/N$ shows that $d(K) \geq d(B)$ and, moreover, that any minimal generating set of B extends to a minimal generating set of K . Since B is properly contained in K , we deduce that $d(K) = d(B) + 1$. From this we get $d(B) = d(K) - 1 = d - 1$, as claimed.

By Case 1, we conclude that $\langle z \rangle A \cong \mathbb{Z}_p^{d-1}$ is abelian. Since y acts as multiplication by $1 + p^s$ on A it must also act in the same way on $\langle z \rangle A$, and hence $G \cong H$ is of the required form.

Case 2.2: $z^p \notin A$. Without loss of generality we may assume that $z^p = y^{p^k} a$ with $k \in \mathbb{N}_0$ and $a \in A$. First consider the case $k \geq 1$. Since A is characteristic in H , z acts on $H/A \cong \mathbb{Z}_p$. As $p > 2$, this action of z is trivial, i.e. $[y, z] \in A$. Put $z_1 := z^{-1} y^{p^{k-1}}$. Then $z_1^p = (z^{-1} y^{p^{k-1}})^p \equiv z^{-p} y^{p^k} \equiv 1$ modulo A , i.e. $z_1^p \in A$. Replacing z by z_1 we return to Case 2.1.

Finally suppose that $k = 0$. Replacing y by ya , we may assume that $z^p = y$. Since A is characteristic in H , we conclude that $G = \langle z \rangle \rtimes A$. Regarding A as a $\mathbb{Z}_p\langle z \rangle$ -module, we put $\mathcal{M} := \mathbb{Q}_p \otimes A$. It is easily seen that G admits an open subgroup K , containing z , which requires at most $1 + d(\mathcal{M})$ generators. Since G has constant generating number d on open subgroups, it follows that $d(\mathcal{M}) =$

$d - 1 = \dim_{\mathbb{Q}_p} \mathcal{M}$. This implies that z must operate as multiplication by a scalar $\lambda \in \mathbb{Z}_p^*$ on A . Since $p > 2$, we can replace z by a suitable power of itself which acts as multiplication by $1 + p^s$ for some $s \in \mathbb{N}$.

3.2. Next we deal with the ‘exceptional’ case $p = 3$. A short argument shows that the pro-3 group $G_0 = \langle w \rangle \rtimes B$ of maximal class, defined in (3) of Theorem 1.1, has constant generating number 2 on open subgroups. Indeed, every open subgroup of G_0 requires at least 2 generators, because G_0 is not virtually pro-cyclic. Moreover, putting $\pi := \omega - 1$, the filtration $\pi^k B$, $k \in \mathbb{N}_0$, of the subgroup B is tight: the index $|\pi^k B : \pi^{k+1} B|$ between any two consecutive terms is equal to 3. Hence the open subgroups of G_0 not contained in B are of the form $\langle w_1 \rangle \rtimes \pi^k B$ for some $w_1 \in wB$ of order 3 and $k \in \mathbb{N}_0$. These groups are all isomorphic to G_0 and require no more than 2 generators. Open subgroups contained in $B \cong \mathbb{Z}_3^2$ obviously require no more than 2 generators.

Now let G be a finitely generated pro-3 group with constant generating number $d = d(G) \geq 2$ on open subgroups. Similarly as in the generic case (i.e. $p \geq 5$), Corollary 2.4 provides the starting point for an induction argument. We consider an open normal subgroup H of index 3 in G , and we write $G = \langle z \rangle H$. Inductively, we assume that H is isomorphic to one of the groups listed in (1), (2) and (3) of Theorem 1.1, and we need to prove that the same holds true for G . As before, there are several cases to consider.

Case 1: H is abelian. Similarly as in the generic case, we regard H as a $\mathbb{Z}_3\langle \bar{z} \rangle$ -module and put $\mathcal{M} := \mathbb{Q}_3 \otimes H$. Since G admits an open subgroup K , containing z , which requires at most $1 + d(\mathcal{M})$ generators, we have $1 + d(\mathcal{M}) \geq d(K) = d = \dim_{\mathbb{Q}_3} \mathcal{M}$. Thus Lemma 2.5 and Table 1 for $p = 3$ show that there are two possibilities. If z acts trivially on H , corresponding to case (T 3.1) in Table 1, then $G \cong \mathbb{Z}_3^d$ is abelian as in the generic case. Now suppose that z does not act trivially on H . Then H is indecomposable as a $\mathbb{Z}_3\langle \bar{z} \rangle$ -module and of \mathbb{Z}_3 -dimension 2, corresponding to case (T 3.2) in Table 1. Since H contains no non-trivial elements which are fixed under conjugation by z , we must have $z^3 = 1$. Consequently, G is isomorphic to the pro-3 group of maximal class described in (3) of Theorem 1.1.

Case 2: H is not abelian. First we consider the ‘new’ case that $H = \langle w \rangle \rtimes B$ is a pro-3 group of maximal class, as described in (3) of Theorem 1.1. Being the unique maximal torsion-free subgroup of H , the subgroup B is characteristic in H and hence normal in G . Because $H/B \cong C_3$ does not admit automorphisms of order 3, the element z acts trivially on H/B and thus conjugation by z preserves the $\mathbb{Z}_3[\omega]$ -structure of B . But B is a free $\mathbb{Z}_3[\omega]$ -module and its automorphisms are given by multiplication by units of the ring $\mathbb{Z}_3[\omega]$. The torsion subgroup of the unit group $\mathbb{Z}_3[\omega]^*$ is $\langle \omega \rangle \cong C_3$. Replacing z by $z\omega$ or $z\omega^2$, if necessary, we may assume that z acts trivially on B . But then $\langle z \rangle B = C_G(B)$ is an abelian normal subgroup of index 3 in G and, returning to Case 1, we find out that G is, in fact, isomorphic to H and covered by (3) of Theorem 1.1.

It remains to consider, similarly as in the generic case (i.e. $p \geq 5$), the situation for H of the form $\langle y \rangle \rtimes A$, where $\langle y \rangle \cong \mathbb{Z}_3$, $A \cong \mathbb{Z}_3^{d-1}$ is abelian and conjugation by y on A corresponds to scalar multiplication by $1 + 3^s$ for some $s \geq 1$. As before, A is characteristic in H , hence normal in G , and we consider two cases.

Case 2.1: $z^3 \in A$. As explained in the generic case, the group $\langle z \rangle A$ has constant generating number $d - 1$ on open subgroups. By Case 1, one of the following holds: $\langle z \rangle A \cong \mathbb{Z}_3^{d-1}$ is abelian, which can be dealt with as before, or $\langle z \rangle A$ is a pro-3 group of maximal class. We now show that the latter option leads to a contradiction and hence does not occur. Without loss of generality, we may assume that z acts on $A = \mathbb{Z}_3 + \mathbb{Z}_3\omega$ as multiplication by a third root of unity ω . Then yz operates on A as multiplication by $\omega(1 + 3^s)$. Thus $\langle yz, a \rangle$, for any non-trivial element $a \in A$ is an open subgroup of G which requires less than 3 generators, a contradiction.

Case 2.2: $z^3 \notin A$. The argument goes through as in the generic case (i.e. $p \geq 5$).

3.3. Finally, we consider the remaining ‘exceptional’ case $p = 2$. Let G be a finitely generated pro-2 group with constant generating number $d = d(G) \geq 2$ on open subgroups. Similarly as before, suppose that H is an open normal subgroup of index 2 in $G = \langle z \rangle H$ and that we inductively understand H . There are, once more, several cases to consider. In several places we will use the basic

equality $\log_2 |G : G^2| = d(G) = d$, which relies on the fact that $G^2 = \langle g^2 \mid g \in G \rangle$ is equal to the Frattini subgroup $\Phi(G)$ of G .

Case 1: $H \cong \mathbb{Z}_2^d$ is abelian. Similarly as in the situations considered previously (i.e. for $p \geq 5$ and $p = 3$), we regard H as a $\mathbb{Z}_2\langle \bar{z} \rangle$ -module and put $\mathcal{M} := \mathbb{Q}_2 \otimes H$. As before, one shows that $1 + d(\mathcal{M}) \geq \dim_{\mathbb{Q}_2} \mathcal{M}$. Thus Lemma 2.5 and Table 1 for $p = 2$ imply that there are six possibilities, labelled (T 2.1) up to (T 2.6).

Clearly, the case (T 2.1) can be dealt with as before: G is seen to be abelian. Next we show that the case (T 2.2) leads to a contradiction. Indeed, if z acts on $H = \langle x_1, \dots, x_d \rangle$ according to $x_i^z = x_i$ for $1 \leq i \leq d - 1$ and $x_d^z = x_d^{-1}$, where $d \geq 3$, then $\log_2 |G : G^2| = d(G) = d$ implies that $z^2 \notin H^2$. Thus we may assume without loss of generality that $z^2 = x_1$. But then $K := \langle z, x_2, \dots, x_{d-2}, x_{d-1}x_d \rangle$ is an open subgroup of G with $d(K) = d - 1$, a contradiction.

Now we show that the case (T 2.3) also leads to a contradiction. Indeed, suppose that z acts on $H = \langle x_1, \dots, x_d \rangle$ according to $x_1^z = x_2, x_2^z = x_1$ and $x_i^z = x_i$ for $3 \leq i \leq d$, where $d \geq 3$. Since z^2 commutes with z , we have $z^2 \in \langle x_1x_2, x_3, \dots, x_d \rangle$. If $z^2 \notin \langle x_1x_2 \rangle$, then G admits an open subgroup K generated by z, x_1 and $d - 3$ of the elements x_3, \dots, x_d , in contradiction to the requirement $d(K) = d$. Now suppose that $z^2 \in \langle x_1x_2 \rangle$. Then $z^2 = (x_1x_2)^\lambda$ for $\lambda \in \mathbb{Z}_2$, and $z_1 := zx_1^{-\lambda}$ satisfies $z_1^2 = z^2x_2^{-\lambda}x_1^{-\lambda} = 1$. Replacing z by z_1 , if necessary, we may assume that $z^2 = 1$. But then G admits the open subgroup $K = \langle z, x_1x_2, x_1^2, x_3, \dots, x_d \rangle$ with $d(K) = d + 1$, a contradiction.

Next we show that the case (T 2.4) again leads to a contradiction. Indeed, if z acts on $H = \langle x_1, \dots, x_d \rangle$ according to $x_i^z = x_i^{-1}$ for $1 \leq i \leq d$, where $d \geq 2$, then $\log_2 |G : G^2| = d(G) = d$ implies that $z^2 \notin H^2$. But z commutes with $z^2 \in H$ so that $z^2 = 1$, a contradiction.

Now we consider the case (T 2.5): z acts on $H = \langle x_1, \dots, x_d \rangle$ according to $x_1^z = x_1$ and $x_i^z = x_i^{-1}$ for $2 \leq i \leq d$, where $d \geq 2$. Since $\log_2 |G : G^2| = d(G) = d$, we have $z^2 \notin H^2$ and we may assume that $z^2 = x_1$. This shows that G is isomorphic to the group in (4) of Theorem 1.1.

Finally, we treat the case (T 2.6): z acts on $H = \langle x_1, \dots, x_d \rangle$ according to $x_1^z = x_2, x_2^z = x_1$ and $x_i^z = x_i^{-1}$ for $3 \leq i \leq d$, where $d \geq 2$. Since z^2 commutes with z , we have $z^2 \in \langle x_1x_2 \rangle$. We can now argue similarly as in the case (T 2.3): $z^2 = (x_1x_2)^\lambda$ for $\lambda \in \mathbb{Z}_2$, and $z_1 := zx_1^{-\lambda}$ satisfies $z_1^2 = z^2x_2^{-\lambda}x_1^{-\lambda} = 1$. Replacing z by z_1 , if necessary, we may assume that $z^2 = 1$. But then G admits the open subgroup $K = \langle z, x_1x_2, x_1^2, x_3, \dots, x_d \rangle$ with $d(K) = d + 1$, a contradiction.

Case 2: H is not abelian. First we deal with the case that H (and hence G) is virtually abelian; this means that H is of the form described in (4) of Theorem 1.1. In this case we show that G has an abelian normal subgroup of index 2 so that we can return to Case 1. Indeed, let $A \cong \mathbb{Z}_2^d$ be a maximal open abelian normal subgroup of G . For a contradiction, we may assume that $|G : A| = 4$. There are two possibilities: G/A is either cyclic or a Klein 4-group.

If $G/A = \langle \bar{z} \rangle$ is cyclic of order 4, then \bar{z} acts on A as an element of order 4. Consequently, the $\mathbb{Q}_2\langle \bar{z} \rangle$ -module $\mathcal{M} := \mathbb{Q}_2 \otimes A$ admits at least one irreducible submodule \mathcal{J} of dimension 2 featuring the eigenvalues $\pm\sqrt{-1}$ for \bar{z} . This means that \mathcal{M} requires at most $d - 1$ generators. Moreover, by Case 1, the open subgroup $\langle z^2 \rangle A$ must be isomorphic to the group described in (4) of Theorem 1.1. The structure of this group shows that $z^4 \in \mathcal{M} \setminus \mathcal{J}$, thus providing one extra module generator. Consequently, G admits an open subgroup K , containing z , which requires at most $1 + (d - 1) - 1 = d - 1$ generators, a contradiction.

Now suppose that $G/A = \langle \bar{w}, \bar{z} \rangle \cong C_2 \times C_2$. Since A is a maximal open abelian normal subgroup of G , the isomorphism types of the three larger subgroups $\langle w \rangle A$, $\langle z \rangle A$ and $\langle wz \rangle A$ of G are limited by Case 1: each of these three groups is isomorphic to the group described in (4) of Theorem 1.1. This shows that the eigenvalues associated to the action of each of the three elements $\bar{w}, \bar{z}, \bar{w}\bar{z}$ on the \mathbb{Q}_2 -vector space $\mathbb{Q}_2 \otimes A$ are: 1, with multiplicity 1, and -1 , with multiplicity $d - 1$. Moreover, $w^2, z^2, (wz)^2$ each give a generator of A , i.e. an element of $A \setminus A^2$. The action of the abelian group $\langle \bar{w}, \bar{z} \rangle$ on $\mathbb{Q}_2 \otimes A$ can, of course, be represented by diagonal matrices. The described eigenvalue spectra of $\bar{w}, \bar{z}, \bar{w}\bar{z}$ thus show that $\dim_{\mathbb{Q}_2} \mathbb{Q}_2 \otimes A = 3$ and that the action of \bar{w}, \bar{z} and $\bar{w}\bar{z}$ on $\mathbb{Q}_2 \otimes A$ with respect to a suitable \mathbb{Q}_2 -basis is given by the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

But then w^2 , z^2 and $(wz)^2$, which as explained each give rise to a generator of A , actually form a basis for $\mathbb{Q}_2 \otimes A$. This shows that $G = \langle w, z \rangle$ requires only 2 generators, a contradiction.

It remains to consider the situation where H is not virtually abelian, and thus isomorphic to one of the groups in (2) of Theorem 1.1: H is of the form $\langle y \rangle \rtimes A$, where $\langle y \rangle \cong \mathbb{Z}_2$, $A \cong \mathbb{Z}_2^{d-1}$ is abelian, normal in H and conjugation by y on A corresponds to scalar multiplication by $\pm(1 + 2^s)$ for some $s \geq 2$. As in the previous cases (i.e. $p \geq 5$ and $p = 3$), A is characteristic in H , hence normal in G . We consider two separate cases.

Case 2.1: $z^2 \in A$. As before, the group $\langle z \rangle A$ has constant generating number $d - 1$ on its open subgroups. If the group $\langle z \rangle A$ is not abelian, then it is of the form described in (4) of Theorem 1.1, with z acting as \bar{z} of order 2 on A . The $\mathbb{Q}_2(\bar{z})$ -module $\mathcal{M} := \mathbb{Q}_2 \otimes A$ is isomorphic to $\mathcal{I}_1 \oplus (d - 2)\mathcal{I}_2$, where we are using the notation introduced in the proof of Lemma 2.5. Thus $d(\mathcal{M}) = d - 2$. We observe that this number does not change, if we regard \mathcal{M} as a $\mathbb{Q}_2\langle yz \rangle$ -module rather than a $\mathbb{Q}_2\langle z \rangle$ - or $\mathbb{Q}_2\langle \bar{z} \rangle$ -module. This shows that G contains an open subgroup K , containing yz , such that $d(K) = d - 1$, a contradiction. Thus $\langle z \rangle A \cong \mathbb{Z}_2^{d-1}$ is abelian. Since y acts as multiplication by $\pm(1 + 2^s)$ on A it must also act in the same way on $\langle z \rangle A$, and hence $G \cong H$ is of the required form.

Case 2.2: $z^2 \notin A$. Without loss of generality we may assume that $z^2 = y^{2k}a$ with $k \in \mathbb{N}_0$ and $a \in A$. First consider the case $k \geq 1$. Since A is characteristic in H , z acts on $H/A \cong \mathbb{Z}_2$ and there are two possibilities: either $y^z \equiv y$ or $y^z \equiv y^{-1}$ modulo A . If z acts trivially on H/A , we return to Case 2.1, just as in the generic case (i.e. $p = 5$). We claim that the second possibility does not occur. Indeed, suppose that $y^z \equiv y^{-1}$ modulo A . Then, for every $a \in A$, we have

$$\pm(1 + 2^s)a^z = (a^y)^z = (a^z)^{y^z} = (a^z)^{y^{-1}} = \pm(1 + 2^s)^{-1}a^z.$$

This implies that $(1 + 2^s)^2 = 1$, in contradiction to $s \geq 2$.

Finally suppose that $k = 0$. Similarly as in the generic case (i.e. $p = 5$), we may assume without loss of generality that $z^2 = y$ and we subsequently reduce to $G = \langle z \rangle \rtimes A$, with z operating as multiplication by a scalar $\lambda \in \mathbb{Z}_2^*$ on $A \cong \mathbb{Z}_2^{d-1}$. This implies that $z^2 = y$ acts on A as multiplication by $\lambda^2 = 1 + 2^s$ where $s \geq 3$. Hence, in a final step we can replace z by a suitable power of itself which acts as multiplication by $\pm(1 + 2^{s-1})$.

Acknowledgment

The second author started his investigation of Iwasawa’s question as a PhD student at Binghamton University. He thanks his adviser Prof. Marcin Mazur for continuous support and helpful discussions.

References

[1] J.D. Dixon, M.P.F. du Sautoy, A. Mann, D. Segal, *Analytic Pro- p Groups*, second ed., Cambridge Stud. Adv. Math., vol. 61, Cambridge University Press, Cambridge, 1999.
 [2] D. Dummit, J. Labute, On a new characterization of Demuskin groups, *Invent. Math.* 73 (1983) 413–418.
 [3] A. Heller, I. Reiner, Representations of cyclic groups in rings of integers I, *Ann. of Math.* 76 (1962) 73–92.
 [4] J. González-Sánchez, B. Klopsch, Analytic pro- p groups of small dimensions, *J. Group Theory* 12 (2009) 711–734.
 [5] J. González-Sánchez, B. Klopsch, On w -maximal groups, *J. Algebra* 328 (1) (2011) 155–166.
 [6] B. Klopsch, On the Lie theory of p -adic analytic groups, *Math. Z.* 249 (2005) 713–730.
 [7] B. Klopsch, On the rank of compact p -adic Lie groups, preprint.
 [8] A. Lubotzky, Combinatorial group theory for pro- p groups, *J. Pure Appl. Algebra* 25 (1982) 311–325.
 [9] A. Lubotzky, A. Mann, Powerful p -groups. II. p -adic analytic groups, *J. Algebra* 105 (1987) 506–515.
 [10] I. Snopce, Pro- p groups of rank 3 and the question of Iwasawa, *Arch. Math. (Basel)* 92 (2009) 19–25.
 [11] M. Yamagishi, An analogue of the Nielsen–Schreier formula for pro- p groups, *Arch. Math. (Basel)* 88 (2007) 304–315.