The smallest ideal of $\beta S$ is not closed

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Let $S$ be an infinite discrete semigroup which can be embedded algebraically into a compact topological group and let $\beta S$ be the Stone–Čech compactification of $S$. We show that the smallest ideal of $\beta S$ is not closed.

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Given a discrete semigroup $S$, the operation can be naturally extended to the Stone–Čech compactification $\beta S$ of $S$ making $\beta S$ a compact right topological semigroup with $S$ contained in its topological center. That is, for each $p \in \beta S$, the right translation $\beta S \ni x \mapsto xp \in \beta S$ is continuous, and for each $a \in S$, the left translation $\beta S \ni x \mapsto ax \in \beta S$ is continuous.

We take the points of $\beta S$ to be the ultrafilters on $S$, the principal ultrafilters being identified with the points of $S$. Given $A \subseteq S$,

$$\overline{A} = \{ p \in \beta S : A \in p \}$$

and we write $A^*$ and $U(A)$ for the sets of nonprincipal and uniform ultrafilters from $\overline{A}$, respectively. The family $\{ \overline{A} : A \subseteq S \}$ is a base for the topology of $\beta S$. For $p, q \in \beta S$, the ultrafilter $pq$ has a base consisting of subsets

$$\bigcup \{ xB_x : x \in A \}$$

where $A \in p$ and $B_x \in q$.

As any compact Hausdorff right topological semigroup, $\beta S$ has a smallest two sided ideal $K(\beta S)$ which is a disjoint union of minimal right ideals and a disjoint union of minimal left ideals. The intersection of a minimal right ideal and a minimal left ideal is a group, and all these groups are isomorphic. The closure $\overline{cE(K(\beta S))}$ of $K(\beta S)$ is also a two sided ideal. An elementary introduction to $\beta S$ can be found in [2].

It is known that for every countably infinite discrete cancellative semigroup $S$, $K(\beta S)$ is not closed. Also the set $E(\overline{cE(K(\beta S))})$ of idempotents from $cE(K(\beta S))$ is not closed. These are consequences of the fact that there are elements in $cE(K(\beta S))$ which are not in $S^*S^*$ [2, Theorem 8.22].

In this note we prove the following result.

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Theorem 1. Let $S$ be an infinite discrete semigroup which can be embedded algebraically into a compact group. Then both $K(\beta S)$ and $E(\text{cl} \ K(\beta S))$ are not closed.

By a compact group one means a compact Hausdorff topological group. Note that the class of semigroups which can be embedded algebraically into a compact group is big enough. It includes, in particular, the commutative cancellative semigroups (= the subsemigroups of Abelian groups) and the free semigroups.

The proof of Theorem 1 involves the following concept.

Let $(S, T)$ be a left topological semigroup with identity $1$. Suppose that $T$ satisfies the $T_1$ separation axiom and let $\mathcal{F}$ denote the neighborhood filter at $1$. Define $\text{Ult}(T) \subseteq S^*$ by

$$\text{Ult}(T) = \bigcap_{V \in \mathcal{F}} V \setminus \{1\}.$$ 

Equivalently, $\text{Ult}(T)$ consists of all nonprincipal ultrafilters on $S$ containing $\mathcal{F}$, that is, converging to $1$ in $T$.

Lemma 2. $\text{Ult}(T)$ is a closed subsemigroup of $\beta S$.

Proof. Being the intersection of closed sets, $\text{Ult}(T)$ is closed. To see that it is a subsemigroup, let $p, q \in \text{Ult}(T)$ and let $V \in \mathcal{F}$. We have to show that $V \setminus \{1\} \in p q$. Clearly, one may suppose that $V$ is open, and so is $V \setminus \{1\}$. For every $x \in V \setminus \{1\}$, there is $W_x \in \mathcal{F}$ such that $x W_x \subseteq V \setminus \{1\}$. Then $\bigcup_{x \in V \setminus \{1\}} x W_x \in p q$. Since $V \setminus \{1\} \in p$ and $W_x \in q$, one has $\bigcup_{x \in V \setminus \{1\}} x W_x \in p q$. Hence $V \setminus \{1\} \in p q$. □

The semigroup $\text{Ult}(T)$ is called the ultrafilter semigroup of $T$.

Lemma 3. If $(S, T)$ can be topologically and algebraically embedded into a compact group, then $\text{Ult}(T)$ contains all the idempotents of $S^*$.

Proof. Let $(S, T)$ be a subsemigroup of a compact group $G$ and let $G_d$ denote the group $G$ reendowed with the discrete topology. Then $\beta S$ can be identified with the subset $\mathfrak{S}$ of $\beta G_d$. By [3, Lemma 3], every idempotent from $G_d$ converges to the identity of $G$. It follows that every idempotent from $S^*$ converges to the identity of $(S, T)$. □

Note that if $(S, T)$ can be topologically and algebraically embedded into a compact group, then it can be embedded also into a compact group of weight $\leq |S|$. (Recall that the weight of a space is the minimum of cardinalities of its bases.) Indeed, without loss of generality one may assume that $(S, T)$ is a subsemigroup of a compact group $G$. For every $s \in S \setminus \{1\}$, there is a unitary group $G_s$ (a compact group of countable weight) and a continuous homomorphism $h_s : G \to G_s$ such that $h_s(s) \neq h_s(1)$. Then $G^s = \prod_{s \in S \setminus \{1\}} G_s$. Define $h : G \to G'$ by $h(x) = (h_s(x))_{s \in S \setminus \{1\}}$. Then the restriction of $h$ to $(S, T)$ is the required embedding.

Now let $(S, T)$ be a subsemigroup of a compact group $G$ of weight $\leq |S|$ and let $T = \text{Ult}(T)$. By Lemma 3, $T$ contains all the idempotents of $S^*$, in particular, the idempotents of $K(\beta S)$. Consequently, $T \cap K(\beta S) \neq \emptyset$. Then by [2, Theorem 1.65],

$$K(T) = K(\beta S) \cap T.$$ 

Hence, the idempotents of $K(T)$ are the same as those of $K(\beta S)$. It follows that in order to prove Theorem 1, it suffices to show the following.

Theorem 4. There are elements in $\text{cl} \ E(\text{cl} \ K(T))$ which are not in $\text{cl} \ E(\text{cl} \ K(T)) T$.

Indeed, let $p \in (\text{cl} \ E(\text{cl} \ K(T))) \setminus ((\text{cl} \ E(\text{cl} \ K(T))) T)$. Then, since $K(T) \subseteq K(\beta S)$, one has $p \in \text{cl} \ E(K(\beta S)) \subseteq (\text{cl} \ K(\beta S)) T$. To see that $p \notin K(\beta S)$, assume the contrary. Let $q$ be the identity of the maximal group in $K(\beta S)$ containing $p$. Then $p = p q$. Since $T$ contains all the idempotents of $S^*$, one has $q \in T$. Hence, $p = p q \in (\text{cl} \ E(\text{cl} \ K(T))) T$ — a contradiction.

To see that $p \notin E(\text{cl} \ K(\beta S))$, assume the contrary. Then $p \in T$. Consequently, $p = p p \in (\text{cl} \ E(\text{cl} \ K(T))) T$ — a contradiction.

In the rest of the note we prove Theorem 4. Let $H$ be the subgroup of $G$ algebraically generated by $S$ and let $|H| = \kappa$.

Lemma 5. There is a function $\phi : H \to \kappa$ such that

(a) for every $X \subseteq H$ with $|X| = \kappa$, $|\phi(X)| = \kappa$,
(b) if $\kappa = \omega$, then for every $x \in H$, there is a neighborhood $V$ of $1 \in H$ such that $\phi(xy) = \phi(y)$ for all $y \in V \setminus \{1\}$, and
(c) if $\kappa > \omega$, then for every $x \in H$, there is a subset $Y \subseteq H$ with $|H \setminus Y| < \kappa$ such that $\phi(xy) = \phi(x y) = \phi(y)$ for all $y \in Y$.

Proof. Consider two cases.

Case 1: $\kappa = \omega$. Let $\mathbb{B}$ denote the countably infinite Boolean group $\bigoplus_\omega \mathbb{Z}_2$ endowed with the topology induced by the product topology on $\prod_\omega \mathbb{Z}_2$. By [4, Theorem 3.1], there is a homeomorphism $h : H \to \mathbb{B}$ such that $h(1) = 0$ and $h(xy) = \phi(xy)$ for all $x, y \in H$. □
\( h(x) + h(y) \) whenever \( \max \text{supp}(h(x)) + 2 \leq \min \text{supp}(h(y)) \). Define \( \phi : H \to \omega \) by

\[
\phi(x) = \begin{cases} 
\max \text{supp}(h(x)) & \text{if } x \neq 0, \\
0 & \text{if } x = 0.
\end{cases}
\]

Since \( \phi \) is finite-to-one, (a) is satisfied. To check (b), let \( x \in H \). If \( x = 1 \), then \( \phi(x) = \phi(y) \) for all \( y \in H \). Therefore one may assume that \( x \neq 1 \). Let

\[ V = \{ y \in H : \max \text{supp}(h(x)) + 2 \leq \min \text{supp}(h(y)) \} \cup \{ 1 \}. \]

Then \( V \) is a neighborhood of \( 1 \in H \) and for every \( y \in V \setminus \{ 1 \} \), one has

\[
\phi(x + y) = \max \text{supp}(h(x + y)) = \max \text{supp}(h(x) + h(y)) = \max \text{supp}(h(y)) = \phi(y).
\]

Case 2: \( \kappa > \omega \). Construct inductively a \( \kappa \)-sequence \( (H_\alpha)_{\alpha < \kappa} \) of subgroups of \( H \) with \( H_0 = \{ 1 \} \) such that

(i) for every \( \alpha < \kappa \), \( |H_\alpha| < \kappa \),
(ii) for every \( \alpha < \kappa \), \( H_\alpha \subseteq H_{\alpha + 1} \),
(iii) for every limit ordinal \( \alpha < \kappa \), \( H_\alpha = \bigcup_{\gamma \in \alpha} H_\gamma \), and
(iv) \( \bigcup_{\alpha < \kappa} H_\alpha = H \).

Note that \( H \) is a disjoint union of sets \( H_{\alpha + 1} \setminus H_\alpha \), where \( \alpha < \kappa \), and \( H_0 \). Define \( \phi : H \to \kappa \) by

\[
\phi(x) = \begin{cases} 
\alpha + 1 & \text{if } x \in H_{\alpha + 1} \setminus H_\alpha, \\
0 & \text{if } x \in H_0.
\end{cases}
\]

Clearly \( \phi \) satisfies (a). To check (c), let \( x \in H \). One may assume that \( x \neq 1 \). Then \( x \in H_{\alpha + 1} \setminus H_\alpha \) for some \( \alpha < \kappa \). Now let \( Y = H \setminus H_\alpha \) and let \( y \in Y \). Then \( y \in H_{\gamma + 1} \setminus H_\gamma \) for some \( \gamma \geq \alpha + 1 \). It follows that both \( xy \) and \( yx \) also belong to \( H_{\gamma + 1} \setminus H_\gamma \). Hence, \( \phi(xy) = \phi(yx) = \phi(y) \). \( \square \)

Let \( \phi : H \to \kappa \) be a function guaranteed by Lemma 5 and let \( \tilde{\phi} : \beta H \to \beta \kappa \) denote the continuous extension of \( \phi \).

**Corollary 6.**

(1) If \( \kappa = \omega \), then for every \( q \in \beta S \) and \( p \in T \), one has \( \tilde{\phi}(qp) = \tilde{\phi}(p) \).

(2) If \( \kappa > \omega \), then for every \( q \in \beta S \) and \( p \in U(S) \), one has \( \tilde{\phi}(qp) = \tilde{\phi}(p) \).

**Proof.** (1) Let \( q \in \beta S \) and \( p \in T \). To see that \( \tilde{\phi}(qp) = \tilde{\phi}(p) \), let \( R \in p \). For every \( x \in S \), there is a neighborhood \( V_x \) of \( 1 \in H \) such that \( \phi(xy) = \phi(y) \) for all \( y \in V \setminus \{ 1 \} \). Since \( p \in T \), \( V_x \setminus \{ 1 \} \in p \). Put \( P_x = R \cap V_x \setminus \{ 1 \} \). Then \( \bigcup_{x \in S} xP_x \in \phi(q) \) and, whenever \( x \in S \) and \( y \in Y_x \), one has \( \phi(xy) = \phi(y) \). \( \phi(R) \).

(2) Let \( q \in \beta S \) and \( p \in U(S) \). To see that \( \tilde{\phi}(qp) = \tilde{\phi}(p) \), let \( R \in p \). For every \( x \in S \), there is \( Y_x \subseteq S \) with \( |S \setminus Y_x| < \kappa \) such that \( \phi(xy) = \phi(y) \) for all \( y \in Y_x \). Since \( p \) is uniform, \( Y_x \subseteq p \). Put \( P_x = R \cap Y_x \). Then \( \bigcup_{x \in S} xY_x \subseteq \phi(q) \) and, whenever \( x \in S \) and \( y \in Y_x \), one has \( \phi(xy) = \phi(y) \). \( \phi(R) \). \( \square \)

**Corollary 7.**

(1) If \( \kappa = \omega \), then for every \( D \subseteq \omega \) such that \( \tilde{\phi}^{-1}(D^*) \cap T \neq \emptyset \), \( \tilde{\phi}^{-1}(D^*) \cap T \) is a left ideal of \( T \).

(2) If \( \kappa > \omega \), then for every \( D \subseteq \phi(S) \) with \( |D| = \kappa \), \( \tilde{\phi}^{-1}(U(D)) \cap \beta S \) is a left ideal of \( \beta S \).

**Proof.** (1) Let \( D \subseteq \omega \) and suppose that \( L = \tilde{\phi}^{-1}(D^*) \cap T \) is nonempty. To see that \( L \) is a left ideal of \( T \), let \( p \in L \) and \( q \in T \). Then \( qp \in T \) and \( \tilde{\phi}(qp) = \tilde{\phi}(p) \in D^* \). Hence, \( qp \in L \).

(2) Let \( D \subseteq \phi(S) \) and \( |D| = \kappa \). Then \( \phi^{-1}(D) \cap S = \kappa \). It follows that \( L = \tilde{\phi}^{-1}(U(D)) \cap \beta S \) is nonempty. To see that \( L \) is a left ideal of \( \beta S \), let \( p \in L \) and \( q \in S \). Then \( qp \in \beta S \), \( p \in U(S) \) and \( \tilde{\phi}(qp) = \tilde{\phi}(p) \in U(D) \). Hence, \( qp \in L \). \( \square \)

**Lemma 8.** There is \( D \subseteq \kappa \) with \( |D| = \kappa \) such that whenever \( D_0 \subseteq D \) and \( |D_0| = \kappa \), \( \tilde{\phi}^{-1}(U(D_0)) \cap T \) is a left ideal of \( T \).

**Proof.** Consider two cases.

Case 1: \( \kappa = \omega \). Let \( \{ V_n : n < \omega \} \) be a decreasing neighborhood base at \( 1 \in H \). Construct inductively a sequence \( (x_n)_{n < \omega} \) in \( S \) such that for each \( n < \omega \), \( x_n \in V_n \setminus \{ 1 \} \) and \( \phi(x_{n+1}) \notin \{ \phi(x_i) : i \leq n \} \). Then \( (x_n)_{n < \omega} \) converges to \( 1 \) and \( \phi(x_n) \neq \phi(x_{n+1}) \).
if \( n \neq m \). Put \( D = \{ \phi(x_n) : n < \omega \} \). Clearly \( D \) is infinite. Now let \( D_0 \) be an infinite subset of \( D \). Then \( \{ x_n : \phi(x_n) \in D_0 \} \) is a subsequence of \( \{ x_n \}_{n < \omega} \), and so is converging to 1. Consequently, \( \overline{\phi^{-1}(D_0)} \cap T \neq \emptyset \). Hence by Corollary 7(1), \( \overline{\phi^{-1}(D_0)} \cap T \) is a left ideal of \( T \).

Case 2: \( \kappa > \omega \). Put \( D = \phi(S) \). Then \( |D| = \kappa \). Now let \( D_0 \subseteq D \) and \( |D_0| = \kappa \). Then by Corollary 7(2), \( \overline{\phi^{-1}(D_0)} \cap \beta S \) is a left ideal of \( \beta S \). Since \( T \) contains all the idempotents of \( S^* \), \( \overline{\phi^{-1}(D_0)} \cap T \neq \emptyset \). It then follows that \( \overline{\phi^{-1}(U(D_0))} \cap T \) is a left ideal of \( T \). \( \square \)

Let \( D \) be a subset of \( \kappa \) guaranteed by Lemma 8. Pick any sequence \( (D_n)_{n < \omega} \) of pairwise disjoint subsets of \( D \) of cardinality \( \kappa \). Then for each \( n < \omega \), \( \overline{\phi^{-1}(U(D_n))} \cap T \) is a left ideal of \( T \), so there is an idempotent \( p_n \in \overline{\phi^{-1}(U(D_n))} \cap K(T) \). Thus, for every \( n < \omega \), one has \( p_n \in E(K(T)) \) and \( \phi(p_n) \in D_n \). Let \( p \in \phi \{ p_n : n < \omega \} \). Then \( p \in \phi \{ p_n : n < \omega \} \) for each \( n < \omega \). We shall show that \( p \notin (\phi \{ T \}) \cap T \).

Recall that a subset \( C \subseteq S \) is syndetic if there is a finite \( F \subseteq S \) such that \( F^{-1}C = S \). Here, \( F^{-1}C = \bigcup_{x \in F} x^{-1}C \) and \( x^{-1}C = \{ y \in S : xy \in C \} \).

**Lemma 9.** Let \( B \subseteq S \). If \( \phi \{ T \} \) is not syndetic, then \( B \cap K(T) = \emptyset \).

**Proof.** Assume the contrary that there is \( q \in B \cap K(T) \) and let

\[ C = \{ x \in S : x^{-1}B = q \} \]

Then by [2, Theorem 4.39], \( C \) is syndetic. But \( C \subseteq \phi \{ T \} \). Hence, \( \phi \{ T \} \) is syndetic as well — a contradiction. \( \square \)

Construct inductively a sequence \( (W_n)_{n < \omega} \) of neighborhoods of 1 in \( H \) and a sequence \( (x_n)_{n < \omega} \) in \( S \) such that \( W_0 = H \) and for every \( n < \omega \), the following conditions are satisfied:

(i) \( W_{n+1} \subseteq W_n \),
(ii) \( x_n \in (W_n \setminus W_{n+1}) \cap S \), and
(iii) \( x_n W_{n+1} \subseteq W_n \).

These conditions imply that

(iv) \( \phi \{ T \} W_{n+1} \subseteq W_n \), and
(v) whenever \( F \subseteq H \) and \( |F| < n \), one has \( F^{-1}W_{n+1} \setminus S \neq \emptyset \).

Indeed, to see (iv), let \( x \in \phi \{ T \} W_{n+1} \). Then \( W_{n+1}x \cap W_{n+1} \neq \emptyset \). Consequently, \( x \in W_{n+1}^{-1}W_{n+1} \subseteq W_n \).

To see (v), assume the contrary. Then there is \( x \in F \) and \( i < j < n \) such that \( x_i, x_j \in x^{-1}W_{n+1} \). Hence,

\[ x_j^{-1}x_i \in W_{n+1}^{-1}xW_{n+1} = W_{n+1}^{-1}W_{n+1} \subseteq W_n \]

It follows that

\[ x_i x_j^{-1} \subseteq W_{n+1} \]

which is a contradiction.

We have that \( S \) is a disjoint union of sets \( (W_n \setminus W_{n+1}) \cap S \), where \( n < \omega \), and the set \( B = \bigcap_{n < \omega} W_n \cap S \). It follows from (iv) that \( B \) is closed in \( T \), and from (v) that \( B \) is not syndetic. Hence by Lemma 9, \( B \cap K(T) = \emptyset \).

For every \( n < \omega \), let \( A_n = \phi^{-1}(D_n) \cap W_n \cap S \) and \( A = \bigcup_{n < \omega} A_n \). Note that \( A \in p \).

Now to show that \( p \notin (\phi \{ T \})T \), assume on the contrary that \( p = qr \), for some \( q \in \phi \{ T \} \) and \( r \in T \). Choose \( Q \in q \) such that \( Qr \subseteq A \). Since \( q \in \phi \{ T \} \), one has \( B \notin q \), so \( Q \subseteq B \). Pick \( x \in Q \setminus B \). Then \( x \in A \) and \( x \in W_n \setminus W_{n+1} \) for some \( n < \omega \). Since \( r \in T \) and \( \phi \{ T \} W_{n+2} \subseteq W_{n+1} \), we obtain that \( x \in S \setminus W_{n+2} \). It then follows that \( x \in \bigcup_{i < n+1} \overline{A_i} \). Hence, \( \overline{\phi(r)} = \overline{\phi(xr)} \subseteq \bigcup_{i < n+1} \overline{D_i} \). But \( \overline{\phi(r)} = \overline{\phi(qr)} = \overline{\phi(p)} \neq \overline{D_i} \) for each \( i < n \) — a contradiction.

The proof of Theorem 4 is complete.

**Remark.** For subsemigroups of compact groups of countable weight Theorem 4 can be strengthened as follows: There are elements in \( (\phi \{ T \}) \) which are not in \( T^2 \). Indeed, in this case the sequence \( (W_n)_{n < \omega} \) of neighborhoods of 1 in \( H \) may be chosen in addition so that \( \bigcap_{n < \omega} W_n \cap S = \{ 1 \} \). Then for every \( q \in T \), one has \( \bigcup_{n < \omega} (W_n \setminus W_{n+1}) \cap S \subseteq q \).

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