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Nonadditive Set Functions on a Finite Set and Linear Inequalities

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A set function is a function whose domain is the power set of a set, which is assumed to be finite in this paper. We treat a possibly nonadditive set function, i.e., a set function which does not satisfy necessarily additivity, $\varphi(A) + \varphi(B) = \varphi(A \cup B)$ for $A \cap B = \emptyset$, as an element of the linear space on the power set. Then some of the famous classes of set functions are polyhedral in that linear space, i.e., expressed by a finite number of linear inequalities. We specify the sets of the coefficients of the linear inequalities for some classes of set functions. Then we consider the following three problems: (a) the domain extension problem for nonadditive set functions, (b) the sandwich problem for nonadditive set functions, and (c) the representation problem of a binary relation by a nonadditive set function, i.e., the problem of nonadditive comparative probabilities. © 1997 Academic Press

1. INTRODUCTION

Set functions considered in this paper are normalized real functions defined on the power set of a finite set. The theory of set functions is developed in various fields, for example, fuzzy measure theory, utility theory [7], coorperative game theory [27, 24], and the theory of polymatroids [4]. Various classes of set functions are known, for example, probability measures, belief functions, plausibility functions, supermodular functions, submodular functions, and lower and upper envelopes of probability measures. The relations among these classes were studied by F. Delbaen [2], M. Sugeno and T. Murofushi [29], and J. Kindler [14].

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The above examples of set functions are convex in the space of all set functions; that is, by checking some linear inequalities, we can examine whether a set function belongs to a given class. We pay attention to the coefficients of these inequalities, especially, the set of the coefficients of all valid inequalities to the polytope of a given class of set functions. J. Kindler [14] studied this from a similar view point.

Then, we discuss three problems for nonadditive set functions, say, the domain extension problem, the sandwich problem, and the representation problem of comparative probabilities.

Domain Extension Problem

When can a set function defined on a subset of the power set of a finite set be extended to a set function defined on the whole power set so that it belongs to a certain class of set functions? As for a probability measure, the extension problem has been considered (J. Łós and E. Marczewski [17], K. P. S. Bhjaskara Rao and M. Bhaskara Rao [19]). A necessary and sufficient condition for a real valued set function φ defined on $\mathcal{M} \subset 2^{\Omega}$ to be extended to a probability measure with the domain 2^{Ω} is

$$\sum_{G \in \mathcal{M}} \lambda(G)\varphi(G) \geq 0 \qquad \text{for any } \lambda\colon \mathcal{M} \to \mathbf{Z}$$
$$\text{such that } \sum_{G:\, \omega \in G} \lambda(G) \geq 0 \text{ for all } \omega \in \Omega.$$

(See D. Schmeidler [24] and R. Giles [8]). We generalize this fact for other convex classes of set functions.

Sandwich Problem

When does there exist a set function which belongs to a certain class of set functions and which lies between two given set functions? The case that given two functions are submodular and supermodular is important in combinatorial optimization (J. Edmonds and R. Giles [5]). As for probability measures sandwiched by two set functions, a necessary and sufficient condition is known (J. Kindler [13], P. Plappert [18]). We generalize it for a polyhedral class of nonadditive set functions.

Representation Problem of Binary Relations

A binary relation $\succcurlyeq$ is said to be realizable by a probability measure if there is a probability measure φ such that

$$E \succcurlyeq F \qquad \text{if and only if } \varphi(E) \geq \varphi(F).$$

It is known when a binary relation on the power set of a finite set is realizable by a probability measure (C. H. Kraft, J. W. Pratt, and A. Seidenberg [15], D. Scott [25]). We generalize this for a polyhedral class of nonadditive set functions.

2. COEFFICIENTS OF LINEAR INEQUALITIES AND THEIR DUALS

Let Ω be a finite set throughout this paper. The space of all set functions on a finite set Ω can be regarded as a real vector space with a finite dimension.

Notation 2.1. $\Phi = \{\varphi: 2^{\Omega} \to \mathbf{R} \mid \varphi(\emptyset) = 0,\ \varphi(\Omega) = 1\}$.

By the assumptions $\varphi(\emptyset) = 0$ and $\varphi(\Omega) = 1$, the dimension of the linear space Φ is $2^{|\Omega|} - 2$. Many of the famous classes of set functions on a finite set are convex and closed, moreover polyhedral, i.e., the intersection of closed halfspaces expressed by a finite number of linear inequalities as proved in this section, for example, probability measures, belief functions, upper envelopes of probability measures. For example, the space of probability measures on $\Omega = \{a, b\}$ is

$$\{\varphi \in \Phi \mid \varphi(a) + \varphi(b) - 1 \geq 0, -\varphi(a) - \varphi(b) + 1 \geq 0 \text{ and } \varphi(a), \varphi(b) \geq 0\}.$$

We consider a set of coefficients of such inequalities.

Notation 2.2. $\Lambda = \{\lambda: 2^{\Omega} \to \mathbf{R} \mid \sum_{G \subset \Omega} \lambda(G) = 0\}$.

If we replace the range $\mathbf{R}$ of λ by $\mathbf{Z}$, the discussions made in this paper may be almost the same. Note that for $\lambda \in \Lambda$ such that at least one $\lambda(G)$ in all $G \in 2^{\Omega} - \{\emptyset, \Omega\}$ is nonzero, $\{\varphi \in \Phi \mid \sum_{G \subset \Omega} \lambda(G)\varphi(G) \geq 0\}$ is a closed halfspace in Φ.

DEFINITION 2.3. Given $\mathcal{F} \subset \Lambda$, we denote

$$\mathcal{F}^* = \left\{\varphi \in \Phi \;\middle|\; \sum_{G \subset \Omega} \lambda(G)\varphi(G) \geq 0 \text{ for all } \lambda \in \mathcal{F}\right\}.$$

We call $\mathcal{F}$ a set of determining coefficients of $\mathcal{F}^*$.

That is, $\mathcal{F}^*$ is a convex and closed class of set functions determined by the linear inequalities with the set $\mathcal{F}$ of coefficients. The assumption $\sum_{G \subset \Omega} \lambda(G) = 0$ in Notation 2.2 is not strong because we can set $\lambda(\emptyset)$ arbitrarily without giving any effect to the value of $\sum_{G \subset \Omega} \lambda(G)\varphi(G)$. But it plays an important role when we consider the duality of set functions.

DEFINITION 2.4. Given $\mathsf{G} \subset \Phi$, we denote

$$\mathsf{G}^{\dagger} = \left\{ \lambda \in \Lambda \;\middle|\; \sum_{G \subset \Omega} \lambda(G)\,\varphi(G) \geq 0 \text{ for all } \varphi \in \mathsf{G} \right\}.$$

We call $\mathsf{G}^{\dagger}$ the set of all valid coefficients of G.

Obviously $\mathsf{G}^{\dagger}$ is the maximum set of determining coefficients of G. For convex and closed $\mathsf{G} \subset \Phi$, the set of all valid coefficients $\mathsf{G}^{\dagger}$ determines G [20, Theorem 11.5]. It is clear that the set of all valid coefficients of G consists of all the inequalities whose nonnegative regions contain all the extreme points of G.

DEFINITION 2.5. For $\varphi \in \Phi$, $\varphi^{d}(G) = 1 - \varphi(G^{c})$ is said to be the dual function of φ. For $\mathsf{G} \subset \Phi$, $\mathsf{G}^{d} = \{\varphi^{d} \mid \varphi \in \mathsf{G}\}$ is said to be the dual class of G.

DEFINITION 2.6. For $\lambda \in \Lambda$, $\lambda^{d}(G) = -\lambda(G^{c})$ is said to be the dual coefficient of λ. For $\mathsf{F} \subset \Lambda$, $\mathsf{F}^{d} = \{\lambda^{d} \mid \lambda \in \mathsf{F}\}$.

Note that $\lambda^{d} \in \Lambda$ for $\lambda \in \Lambda$.

The next theorem is very useful.

THEOREM 2.7. *Let* $\mathsf{F} \subset \Lambda$. *Then* $(\mathsf{F}^{*})^{d} = (\mathsf{F}^{d})^{*}$.
Let $\mathsf{G} \subset \Phi$. *Then* $(\mathsf{G}^{\dagger})^{d} = (\mathsf{G}^{d})^{\dagger}$.

Proof.

$$\begin{aligned}
&\sum_{G \subset \Omega} \lambda^{d}(G)\,\varphi^{d}(G) \\
&\quad = -\sum_{G \subset \Omega} \lambda(G^{c})(1 - \varphi(G^{c})) = -\sum_{G \subset \Omega} \lambda(G)(1 - \varphi(G)) \\
&\quad = -\sum_{G \subset \Omega} \lambda(G) + \sum_{G \subset \Omega} \lambda(G)\,\varphi(G) = \sum_{G \subset \Omega} \lambda(G)\,\varphi(G).
\end{aligned}$$

Note that $\sum_{G \subset \Omega} \lambda(G) = 0$ by the definition of Λ. ▮

Note that, by this theorem, the dual of a polyhedral class is also polyhedral.

3. VARIOUS CONVEX CLASSES OF SET FUNCTIONS

In this section, we want to find the set of all valid coefficients and the set of determining coefficients of several classes of set functions.

3.1. *Probability Measures*

The class of probability measures may be the most famous class of set functions.

DEFINITION 3.1. A function $\varphi \in \Phi$ is a probability measure if $\varphi(E) + \varphi(F) = \varphi(E \cup F)$ for all $E, F \subset \Omega$ such that $E \cap F = \emptyset$, and $\varphi(G) \geq 0$ for all $G \subset \Omega$.

That is, a probability measure is a normalized additive set function. Obviously, the set of probability measures is convex and closed in Φ and moreover polyhedral because the conditions which define probability measures like $\varphi(E) + \varphi(F) \geq \varphi(E \cup F)$, $\varphi(E) + \varphi(F) \leq \varphi(E \cup F)$, and $\varphi(G) \geq 0$ are linear inequalities in Φ and the number of them is finite.

The next theorem is well known.

THEOREM 3.2. *The set of all valid coefficients of the class of all probability measures is*

$$\left\{ \lambda \colon 2^{\Omega} \to \mathbf{R} \;\middle|\; \sum_{G:\, \omega \in G} \lambda(G) \geq 0 \text{ for all } \omega \in \Omega \right\}.$$

Proof. For $\omega \in \Omega$, define p_ω as

$$p_\omega(G) = \begin{cases} 1 & \text{when } \omega \in G \\ 0 & \text{when } \omega \notin G. \end{cases}$$

Then $\{p_\omega\}_{\omega \in \Omega}$ is the set of extreme points of the set of probability measures. In fact, for a probability measure p, we can write $p = \sum_{\omega \in \Omega} p(\{\omega\}) p_\omega$.

The set of all valid coefficients consists of λ such that

$$\sum_{G \subset \Omega} p_\omega(G) \lambda(G) \geq 0 \qquad \text{for all } \omega \in \Omega.$$

This is equivalent to

$$\sum_{G:\, \omega \in G} \lambda(G) \geq 0 \qquad \text{for all } \omega \in \Omega. \quad \blacksquare$$

Obviously, the dual of the class of probability measures is itself.

3.2. *Belief and Plausibility Functions*

The theory of belief functions is developed as a theory of evidence [26].

DEFINITION 3.3. A function $m \colon 2^{\Omega} \to [0, 1]$ is a basic probability assignment if $\sum_{G \subset \Omega} m(G) = 1$ and $m(\emptyset) = 0$. A function $\varphi \in \Phi$ is a belief function if there exists a basic probability assignment m such that $\varphi(E) = \sum_{G \subset E} m(G)$ for $E \subset \Omega$. A function $\varphi \in \Phi$ is a plausibility function if there exists a basic probability assignment m such that $\varphi(E) = \sum_{G \cap E \neq \varnothing} m(G)$ for $E \subset \Omega$.

Obviously, the set of belief functions is convex and moreover polyhedral in Φ because it is known that φ is a belief function if and only if φ satisfies a finite number of linear inequalities

$$\varphi\left(\bigcup_{i=1}^{n} E_i\right) \geq \sum_{\substack{G \subset \{1,\dots,n\} \\ G \neq \varnothing}} (-1)^{|G|+1} \varphi\left(\bigcap_{i \in G} E_i\right)$$

for all sequences $\{E_i\}$.

The basic probability assignment m is the Möbius inverse of belief function φ, i.e., $m(F) = \sum_{F \subset G \subset \Omega} (-1)^{|G-F|} \varphi(G)$. We can say that a belief function is a set function which has nonnegative Möbius inverse.

THEOREM 3.4. *The set of all valid coefficients set of belief functions is*

$$\left\{\lambda\colon 2^{\Omega} \to \mathbf{R} \;\middle|\; \sum_{G\colon E \subset G} \lambda(G) \geq 0 \textit{ for all } E \subset \Omega\right\}.$$

Proof. Let φ be a belief function. For $G \subset \Omega$, define Bel_G as

$$\mathrm{Bel}_G(F) = \begin{cases} 1 & \text{when } G \subset F \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{\mathrm{Bel}_G\}_{G \subset \Omega}$ is the set of extreme points of the set of belief functions. In fact, we can write $\varphi = \sum_{G \subset \Omega} m(G) \mathrm{Bel}_G$ because $\varphi(F) = \sum_{G \subset F} m(G)$.

The set of all valid coefficients consists of λ such that

$$\sum_{F \subset \Omega} \mathrm{Bel}_G(F) \lambda(F) \geq 0 \qquad \text{for all } G \subset \Omega.$$

This is equivalent to

$$\sum_{F\colon G \subset F} \lambda(F) \geq 0 \qquad \text{for all } G \subset \Omega. \qquad \blacksquare$$

Obviously, the classes of belief functions and plausibility functions are dual for each other.

COROLLARY 3.5. *The set of all valid coefficients of plausibility functions is*

$$\left\{\lambda \in \Lambda \,\middle|\, \sum_{G:\, G \cap E \neq \varnothing} \lambda(G) \geq 0 \textit{ for all } E \subset \Omega\right\}.$$

Proof. By Theorem 2.7, the set of all valid coefficients of plausibility functions is the set of the duals of the elements of the set of all valid coefficients of belief functions. Let $\lambda \in \Lambda$ be in the set of all valid coefficients of belief functions, that is $\sum_{G:\, E \subset G} \lambda(G) \geq 0$ for all $E \subset \Omega$ by Theorem 3.4.

$$\begin{aligned} \sum_{G:\, E \subset G} \lambda(G) &= -\sum_{G:\, E \subset G} \lambda^d(G^c) = -\sum_{G:\, E \subset G^c} \lambda^d(G) \\ &= -\sum_{G \subset \Omega} \lambda^d(G) + \sum_{G:\, E \cap G \neq \varnothing} \lambda^d(G) = \sum_{G:\, E \cap G \neq \varnothing} \lambda^d(G). \end{aligned}$$

■

3.3. *Submodular and Supermodular Functions*

A set function $\varphi \in \Phi$ is said to be submodular if $\varphi(A) + \varphi(B) \geq \varphi(A \cup B) + \varphi(A \cap B)$ for all $A, B \subset \Omega$. By definition, this class of set functions is expressed by linear inequalities. The class of submodular functions is extensively studied in combinatorial optimization (J. Edmonds [4], S. Fujishige [6]).

$$\left\{\lambda \in \Lambda \,\middle|\, \sum_{G:\, \omega \in G} \lambda(G) \geq 0 \text{ for all } \omega \in \Omega\right.$$

$$\text{and there exist } E, F \subset \Omega \text{ such that } \lambda(E) < 0,\ \lambda(F) < 0,\ E \subset F,$$

$$\left.\text{and } \lambda(D) \geq 0 \text{ for all } D \neq E, F\right\}$$

is a set of determining coefficients of submodular set functions (F. Delbaen [2]).

The dual of a submodular function is a supermodular function. A supermodular function is called a convex game in cooperative game theory (L. S. Shapley [27, 28]). A. Chateauneuf and J. Y. Jaffray [1] consider the Möbius inversion for supermodular functions. The Möbius inversion is an isomorphism in the linear space. So the Möbius inverse of a convex closed class of set functions is also convex and closed.

We have not obtained the explicit form to write the set of all valid coefficients. For that purpose, we have to list the extreme points of submodular functions. The problem is only partially solved (J. Rosenmüller [21–23], H. Q. Nguyen [16], K. Kashiwabara and B. Nakano [12]).

The class of submodular function contains the class of plausibility functions, and the class of supermodular functions contains the class of belief functions.

3.4. *Upper and Lower Envelopes*

The class of upper and lower envelopes is investigated in various fields, for example, game theory [24], robust statistics [10], and fuzzy measure theory [3].

DEFINITION 3.6. Let $\mathsf{G} \subset \Phi$ be a class of set functions. A function $\varphi: 2^{\Omega} \to \mathbf{R}$ is said to be an upper envelope of $\mathsf{G} \subset \Phi$ if there exists a subset $\{\varphi_i\}_{i \in I} \subset \mathsf{G}$ with a finite set I such that

$$\varphi(E) = \max_{i \in I} \varphi_i(E) \qquad \text{for all } E \subset \Omega.$$

D. Schmeidler [24] and R. Giles [8] independently found a necessary and sufficient condition for a set function to be an upper envelope of probability measures.

THEOREM 3.7.

$$\left\{ \lambda \in \Lambda \,\middle|\, \sum_{G:\, \omega \in G} \lambda(G) \geq 0 \textit{ for all } \omega \in \Omega \right.$$
$$\left. \textit{and } \left| \left\{ G \in 2^{\Omega} - \{\Omega, \varnothing\} : \lambda(G) < 0 \right\} \right| \leq 1 \right\}$$

is a set of determining coefficients of upper envelopes of probability measures.

Instead of proving the above theorem, we show the following generalized theorem.

THEOREM 3.8. *Let $\mathsf{G} \subset \Phi$ be a convex closed class of set functions. Then*

$$\mathsf{G}^{\dagger} \cap \left\{ \lambda \in \Lambda : \left| \left\{ G \in 2^{\Omega} - \{\Omega, \varnothing\} : \lambda(G) < 0 \right\} \right| \leq 1 \right\}$$

is a set of determining coefficients of upper envelopes of G.

Proof. Let φ be an upper envelope of G. Then we can write $\varphi(G) = \max_{E \in 2^{\Omega} - \{\Omega, \varnothing\}} \varphi_E(G)$ such that $\varphi_F(F) = \varphi(F)$ for all $F \in 2^{\Omega} - \{\Omega, \varnothing\}$. Let $\lambda \in \mathsf{G}^{\dagger} \cap \{\lambda \in \Lambda: |\{G \in 2^{\Omega} - \{\Omega, \varnothing\}: \lambda(G) < 0\}| \leq 1\}$. When $|\{G \in 2^{\Omega} - \{\Omega, \varnothing\}: \lambda(G) < 0\}| = 1$, take E with $\lambda(E) < 0$; otherwise take E

arbitrarily. Then

$$\sum_{G \subset \Omega} \lambda(G)\varphi(G) \geq \sum_{G \subset \Omega} \lambda(G)\varphi_E(G) \geq 0.$$

This means that φ is contained in all closed halfspaces associated with the above coefficients.

Next, let φ satisfy

$$\sum_{G \subset \Omega} \lambda(G)\varphi(G) \geq 0$$

$$\text{for all } \lambda \in \mathsf{G}^{\dagger} \cap \left\{\lambda \in \Lambda : \left|\{G \in 2^{\Omega} - \{\Omega, \emptyset\} : \lambda(G) < 0\}\right| \leq 1\right\}. \quad (1)$$

If we can show

$$\mathsf{G} \cap \left\{\varphi' \in \Phi \mid \varphi'(G) \leq \varphi(G) \text{ for all } G \subset \Omega \text{ and } \varphi'(E) = \varphi(E)\right\}$$

is not empty for all $E \in 2^{\Omega} - \{\Omega, \emptyset\}$, by letting φ_E be an element of that set, we can write $\varphi(G) = \max \varphi_E(G)$.

Assume that the above set is empty. Then there exists a separating hyperplane with a coefficient vector λ such that

$$\sum_{G \subset \Omega} \lambda(G)\varphi''(G) \geq 0 \qquad \text{for all } \varphi'' \in \mathsf{G}$$

and $\sum_{G \subset \Omega} \lambda(G)\varphi''(G) < 0$ for $\varphi'' \in \{\varphi' \in \Phi \mid \varphi'(G) \leq \varphi(G)$ for all $G \subset \Omega$ and $\varphi'(E) = \varphi(E)\}$.

Then the second inequality implies that $\lambda(G)$ is nonnegative for $G \in 2^{\Omega} - \{\Omega, \emptyset, E\}$. This contradicts assumption (1). ∎

By this theorem we observe that the class of the envelopes of a polyhedral class is also polyhedral.

COROLLARY 3.9. *Let* G *be a convex closed class of set functions.*

$$(\mathsf{G}^{\dagger})^{d} \cap \left\{\lambda \in \Lambda : \left|\{G \in 2^{\Omega} - \{\Omega, \emptyset\} : \lambda(G) > 0\}\right| \leq 1\right\}$$

is a coefficient set of lower envelopes of G.

Proof. This follows from Theorem 2.7, Theorem 3.8, and the fact that lower envelopes are G are the dual class of upper envelopes of G. ∎

3.5. *Other Convex Classes*

There are many convex classes which are not already mentioned, for example, subadditive set functions, superadditive set functions, balanced set functions [28], totally balanced set functions [11, 2], and so on.

4. DOMAIN EXTENSION PROBLEM

We consider a condition for the domain of a function which is partially defined on 2^{Ω} to be extended to 2^{Ω} so that the function belongs to a given class of set functions.

Notation 4.1. Let M satisfy $\{\Omega, \emptyset\} \subset M \subset 2^{\Omega}$. For $G \subset \Phi$, we denote

$$G|_M = \{\varphi|_M : \varphi \in G\},$$

where $\varphi|_M$ is the restriction of φ to M.

We can consider that $G|_M$ is the projection of G to M in the linear space Φ. Note that when G is convex and closed, $G|_M$ is also convex and closed in the projected linear space.

Obviously, φ with the domain M can be extended to $\varphi' \in G \subset \Phi$ if and only if $\varphi \in G|_M$.

Notation 4.2. Let M satisfy $\{\Omega, \emptyset\} \subset M \subset 2^{\Omega}$. For $F \subset \Lambda$, we denote

$$F|_M = \left\{\lambda|_M : \lambda \in F;\ \lambda(E) = 0 \text{ for all } E \in 2^{\Omega} - M\right\}.$$

The next theorem is the main theorem in this section.

THEOREM 4.3. *Let M satisfy $\{\Omega, \emptyset\} \subset M \subset 2^{\Omega}$. For convex and closed $G \subset \Phi$.*

This theorem gives us a method to examine whether a partial function can be extended to one which belongs to a given class of set functions. It tells us that to be able to extend a function, it suffices to consider the restriction of the set of all valid coefficients derived from a given convex and closed class of set functions.

Proof. For the part $\{G^{\dagger}|_M\}^* \supset G|_M$, assume that $\varphi \in G|_M$. Then we can extend it to $\varphi' \in G$. By the definition of $G^{\dagger}$, $\sum_{G \subset \Omega} \lambda(G)\varphi'(G) \geq 0$ for all $\lambda \in G^{\dagger}$. Therefore $\sum_{G \subset \Omega} \lambda(G)\varphi'(G) \geq 0$ for any $\lambda \in G^{\dagger}$ such that $\lambda(G) = 0$ for all $G \neq M$. Hence $\sum_{G \in M} \lambda(G)\varphi(G) \geq 0$ for $\lambda \in G^{\dagger}|_M$. Therefore $\varphi \in \{G^{\dagger}|_M\}^*$.

For the part $\{G^{\dagger}|_M\}^* \subset G|_M$, assume that $\varphi \notin G|_M$. Since $G|_M$ is a convex and closed, there exists a hyperplane which separates φ and $G|_M$.

Therefore there exists a λ such that $\sum_{G \in M} \lambda(G)\varphi'(G) \geq 0$ for all $\varphi'(G) \in G|_M$ and $\sum_{G \in M} \lambda(G)\varphi(G) < 0$. So $\lambda \in G^{\dagger}|_M$ but $\sum_{G \in M} \lambda(G)\varphi(G) < 0$. Therefore $\varphi \notin \{G^{\dagger}|_M\}^*$ ∎

By this theorem, we can discuss extension problems for a probability measure and a belief function.

COROLLARY 4.4. *Let* $\mathcal{M}$ *satisfy* $\{\Omega, \emptyset\} \subset \mathcal{M} \subset 2^{\Omega}$. *A function* $\varphi\colon \mathcal{M} \to \mathbf{R}$ *with* $\varphi(\Omega) = 1$ *and* $\varphi(\emptyset) = 0$ *can be extended to a probability measure on* Ω *if and only if* $\sum_{G \in \mathcal{M}} \lambda(G)\varphi(G) \geq 0$ *for any* $\lambda\colon \mathcal{M} \to \mathbf{R}$ *such that* $\sum_{G:\, \omega \in G \in \mathcal{M}} \lambda(G) \geq 0$ *for all* $\omega \in \Omega$.

Proof. This is by Theorem 3.2 and Theorem 4.3. ∎

COROLLARY 4.5. *Let* $\mathcal{M}$ *satisfy* $\{\Omega, \emptyset\} \subset \mathcal{M} \subset 2^{\Omega}$. *A function* $\varphi\colon \mathcal{M} \to \mathbf{R}$ *with* $\varphi(\Omega) = 1$ *and* $\varphi(\emptyset) = 0$ *can be extended to a belief function on* Ω *if and only if* $\sum_{G \in \mathcal{M}} \lambda(G)\varphi(G) \geq 0$ *for any* $\lambda\colon \mathcal{M} \to \mathbf{R}$ *such that* $\sum_{G:\, E \subset G \in \mathcal{M}} \lambda(G) \geq 0$ *for all* $E \subset \Omega$.

Proof. This is by Theorem 3.4 and Theorem 4.3. ∎

Next, we consider the domain extension problem for an upper envelope. Since the coefficient set of upper envelopes specified above is not maximum, we have to consider the domain extension problem in another way.

THEOREM 4.6. *Let* $\mathcal{M}$ *satisfy* $\{\Omega, \emptyset\} \subset \mathcal{M} \subset 2^{\Omega}$. *Let* $\varphi\colon \mathcal{M} \to \mathbf{R}$ *with* $\varphi(\Omega) = 1$ *and* $\varphi(\emptyset) = 0$. *We assume that if* $\lambda\colon \mathcal{M} \to \mathbf{R}$ *satisfies* $\sum_{G:\, \omega \in G \in \mathcal{M}} \lambda(G) \geq 0$ *for all* $\omega \in \Omega$, *and* $|\{G \in \mathcal{M} - \{\Omega, \emptyset\} | \lambda(G) < 0\}| \leq 1$, *then* $\sum_{G \in \mathcal{M}} \lambda(G)\varphi(G) \geq 0$. *Then there exists a set of probability measures* $\{p_i\}_{i \in I}$ *on* Ω *such that* $\varphi(G) = \max_{i \in I} p_i(G)$ *for all* $G \in \mathcal{M}$.

Proof. Necessity is obvious by Theorem 3.7. Sufficiency is proved by the next lemma and Theorem 3.7 and mathematical induction. ∎

LEMMA 4.7. *Let* $\mathcal{M}$ *satisfy* $\{\Omega, \emptyset\} \subset \mathcal{M} \subset 2^{\Omega}$. *Let* $\varphi\colon \mathcal{M} \to \mathbf{R}$ *satisfy the hypothesis of Theorem* 4.6. *Let* $M \in 2^{\Omega}$ *satisfy* $M \notin \mathcal{M}$. *Then we can extend the domain of* φ *to* $\mathcal{M} \cup \{M\}$ *so that it satisfies the hypothesis of Theorem* 4.6 *again, that is,* $\sum_{G \in \mathcal{M} \cup \{M\}} \lambda(G)\varphi(G) \geq 0$ *for any* $\lambda\colon \mathcal{M} \cup \{M\} \to \mathbf{R}$ *such that* $\sum_{G:\, \omega \in G \in \mathcal{M} \cup \{M\}} \lambda(G) \geq 0$ *for all* $\omega \in \Omega$, *and* $|\{G \in \mathcal{M} \cup \{M\} - \{\Omega, \emptyset\} \mid \lambda(G) < 0\}| \leq 1$.

Proof. It suffices to show that we can define the value of $\varphi(M)$ such that if λ satisfies that $\sum_{G:\, \omega \in G \in \mathcal{M} \cup \{M\}} \lambda(G) \geq 0$ for all $\omega \in \Omega$, and $|\{G \in \mathcal{M} \cup \{M\} - \{\Omega, \emptyset\}\colon \lambda(G) < 0\}| \leq 1$, then it satisfies the inequality

$$\sum_{G \in \mathcal{M} \cup \{M\}} \lambda(G)\,\varphi(G) \geq 0.$$

This condition is equivalent to the condition that if λ satisfies that

$$\lambda(M) \geq -\sum_{G:\, \omega \in G \in \mathcal{M}} \lambda(G) \text{ for any } \omega \in M, \quad \text{and} \quad \sum_{G:\, \omega \in G \in \mathcal{M}} \lambda(G) \geq 0$$

$$\text{for any } \omega \notin M, \text{ and } \left|\{G \in \mathcal{M} \cup \{M\} - \{\Omega, \emptyset\}\colon \lambda(G) < 0\}\right| \leq 1, \quad (2)$$

then

$$\lambda(M)\,\varphi(N) \geq -\sum_{G \in \mathcal{M}} \lambda(G)\,\varphi(G). \tag{3}$$

Note that condition (2) implies

$$\lambda(M) \geq \max_{\omega \in M} \left\{ -\sum_{G:\, \omega \in G \in \mathcal{M}} \lambda(G) \right\}. \tag{4}$$

Assuming that λ satisfies the above hypothesis, we separate the four cases according to the sign of $\lambda(M)$ and the sign of

$$\max_{\omega \in M} \left\{ -\sum_{G:\omega \in G \in \mathcal{M}} \lambda(G) \right\}.$$

After that, we define $\varphi(M)$ so that it satisfies the conditions in all cases.

The Case $\lambda(M) = 0$. In this case, $\sum_{G:\, \omega \in G \in \mathcal{M}} \lambda(G) \geq 0$ for any $\omega \in \Omega$ and $|\{G \in \mathcal{M} - \{\Omega, \varnothing\}: \lambda(G) < 0\}| \leq 1$. Therefore, by the assumption on φ, we have $\lambda(M)\varphi(M) \geq -\sum_{G \in \mathcal{M}} \lambda(G)\varphi(G)$ regardless of the value of $\varphi(M)$.

The Case $\lambda(M) > 0$ *and* $\max_{\omega \in M}\{-\sum_{G:\, \omega \in G \in \mathcal{M}} \lambda(G)\} > 0$. We have to define $\varphi(M) \geq 0$ such that

$$\varphi(M) \geq -\frac{\sum_{G \in \mathcal{M}} \lambda(G)\,\varphi(G)}{\lambda(M)}.$$

By assumption (4), in order that this inequality holds, it is sufficient that

$$\varphi(M) \geq -\frac{\sum_{G \in \mathcal{M}} \lambda(G)\,\varphi(G)}{\max_{\omega \in M}\left\{-\sum_{G:\, \omega \in G \in \mathcal{M}} \lambda(G)\right\}} \qquad \text{and} \qquad \varphi(M) \geq 0.$$

Note that λ satisfies $|\{G \in \mathcal{M} - \{\Omega, \varnothing\}: \lambda(G) < 0\}| \leq 1$ in this case.

The Case $\lambda(M) > 0$ *and* $\max_{\omega \in M}\{-\sum_{G:\, \omega \in G \in \mathcal{M}} \lambda(G)\} \leq 0$. By the assumption that $\max_{\omega \in \mathcal{M}}\{-\sum_{G:\, \omega \in G \in \mathcal{M}} \lambda(G)\} \leq 0$, $\sum_{G:\, \omega \in G \in \mathcal{M}} \lambda(G) \geq 0$ for all $\omega \in \Omega$. Therefore by the assumption on φ,

$$\sum_{G \in \mathcal{M}} \lambda(G)\,\varphi(G) \geq 0.$$

If we define $\varphi(M)$ such that $\varphi(M) \geq 0$, inequality (3) holds by the above inequality.

The Case $\lambda(\mathcal{M}) < 0$. In this case, by inequality (4),

$$\max_{\omega \in M} \left\{ -\sum_{G:\, \omega \in G \in \mathsf{M}} \lambda(G) \right\} < 0.$$

We have to define $\varphi(M) \geq 0$ such that

$$\varphi(M) \leq -\frac{\sum_{G \in \mathsf{M}} \lambda(G)\varphi(G)}{\lambda(M)}$$

The assumption

$$\sum_{G:\, \omega \in G \in \mathsf{M} \cup \{M\}} \lambda(G) \geq 0 \qquad \text{for all } \omega \in \Omega$$

implies

$$\sum_{G:\, \omega \in G \in \mathsf{M}} \lambda(G) \geq 0 \qquad \text{for all } \omega \in \Omega.$$

By the hypothesis of Theorem 4.5, we obtain $\sum_{G \in \mathsf{M}} \lambda(G)\varphi(G) \geq 0$. So if we define $\varphi(M)$ such that

$$\varphi(M) \leq -\frac{\sum_{G \in \mathsf{M}} \lambda(G)\varphi(G)}{\max_{\omega \in M}\{-\sum_{G:\, \omega \in G \in \mathsf{M}} \lambda(G)\}},$$

inequality (3) holds by the above inequality.

To sum up, it is sufficient to define $\varphi(M)$ satisfying the following conditions.

$$0 \leq \varphi(M)$$

$$\frac{-\sum_{G \in \mathsf{M}} \lambda(G)\varphi(G)}{\max_{\omega \in M}\{-\sum_{G:\, \omega \in G \in \mathsf{M}} \lambda(G)\}} \leq \varphi(M) \qquad \text{for any } \lambda \text{ such that}$$

$$\max_{\omega \in M} \left\{ -\sum_{G:\, \omega \in G \in \mathsf{M}} \lambda(G) \right\} > 0 \text{ and } \left| \{G \in \mathsf{M} - \{\Omega, \emptyset\} : \lambda(G) < 0\} \right| \leq 1;$$

$$\varphi(M) \leq \frac{-\sum_{G \in \mathsf{M}} \lambda(G)\varphi(G)}{\max_{\omega \in M}\{-\sum_{G:\, \omega \in G \in \mathsf{M}} \lambda(G)\}} \qquad \text{for any } \lambda \text{ such that}$$

$$\max_{\omega \in M} \left\{ -\sum_{G:\, \omega \in G \in \mathsf{M}} \lambda(G) \right\} < 0 \text{ and } \lambda(G) \geq 0 \text{ for all } G \in \mathsf{M} - \{\Omega, \emptyset\}.$$

It remains to show that this range is not empty.

The discussion of the case $\lambda(M) < 0$ leads you to know that the right side in the last inequality is more than or equal to 0. So it suffices to show that

$$\frac{-\sum_{G \in \mathsf{M}} \lambda(G)\varphi(G)}{\max_{\omega \in M}\{-\sum_{G:\, \omega \in G \in \mathsf{M}} \lambda(G)\}} - \frac{-\sum_{G \in \mathsf{M}} \lambda(G)\varphi(G)}{\max_{\omega \in M}\{-\sum_{G:\, \omega \in G \in \mathsf{M}} \lambda'(G)\}} \geq 0 \tag{5}$$

for λ and λ' such that $\max_{\omega \in M}\{-\sum_{G:\, \omega \in G \in \mathsf{M}} \lambda(G)\} < 0$ and $\lambda(G) \geq 0$ for all $G \in \mathsf{M} - \{\Omega, \emptyset\}$ and $\max_{\omega \in M}\{-\sum_{G:\, \omega \in G \in \mathsf{M}} \lambda'(G)\} > 0$ and $|\{G \in \mathsf{M} - \{\Omega, \emptyset\}\colon \lambda'(G) < 0\}| \leq 1$ and λ, λ' satisfy assumption (2).

The left side of inequality (5) is equal to, by letting

$$\alpha = \max_{\omega \in M}\left\{-\sum_{G:\, \omega \in G \in \mathsf{M}} \lambda'(G)\right\} > 0$$

and

$$\beta = \max_{\omega \in M}\left\{-\sum_{G:\, \omega \in G \in \mathsf{M}} \lambda(G)\right\} < 0,$$

$$\frac{1}{\alpha\beta}\sum_{G \in \mathsf{M}}(\lambda'(G)\beta - \lambda(G)\alpha)\varphi(G).$$

Let $\lambda''(E) = -\lambda'(E)\beta + \lambda(E)\alpha$. Then the above formula is equal to $(-1/\alpha\beta)\sum_{G \in M} \lambda''(G)\varphi(G)$.

By the assumption on φ, in order to prove that this is not negative, it suffices to show that $\sum_{G:\, \omega \in G \in \mathsf{M}} \lambda''(G) \geq 0$ for any $\omega \in \Omega$ because of $\alpha\beta < 0$. Note that it is easy to show $|\{G \in \mathsf{M} - \{\Omega, \emptyset\}\colon \lambda''(G) < 0\}| \leq 1$ since $\lambda(G) \geq 0$ for all $G \in \mathsf{M} - \{\omega, \emptyset\}$ and $|\{G \in \mathsf{M} - \{\Omega, \emptyset\}\colon \lambda'(G) < 0\}| \leq 1$.

So it suffices to show

$$-\frac{1}{\alpha\beta}\sum_{G:\, \omega \in G \in \mathsf{M}} \lambda''(G) = \frac{\sum_{G:\, \omega \in G \in \mathsf{M}} \lambda'(G)}{\alpha} - \frac{\sum_{G:\, \omega \in G \in \mathsf{M}} \lambda(G)}{\beta} \geq 0.$$

In case of $\omega \in M$,

$$-\sum_{G:\, \omega \in G \in \mathsf{M}} \lambda'(G) \leq \alpha > 0 \qquad \text{and} \qquad -\sum_{G:\, \omega \in G \in \mathsf{M}} \lambda(G) \leq \beta < 0,$$

$$\frac{\sum_{G:\, \omega \in G \in \mathsf{M}} \lambda'(G)}{\alpha} \geq -1 \qquad \text{and} \qquad \frac{\sum_{G:\, \omega \in G \in \mathsf{M}} \lambda(G)}{\beta} \leq -1.$$

In the case of $\omega \notin M$, use assumption (2). ∎

5. SANDWICH THEOREM

In this section, we consider so-called sandwich problems. When does there exist a set function which belongs to a certain class of set functions and which lies between two given set functions? J. Kindler [13] gave a solution when a sandwiched set function is a probability measure. We generalize his result for a polyhedral class of nonadditive set functions.

THEOREM 5.1. *Let $\mathsf{G} \subset \Phi$ be polyhedral. For $\mu, \nu \in \Phi$, there exists a set function $\varphi \in \mathsf{G}$ such that $\nu(G) \leq \varphi(G) \leq \mu(G)$ for all $G \subset \Omega$ if and only if*

$$\sum_{G \subset \Omega} \lambda(G)\,\mu(G) - \sum_{G \subset \Omega} \lambda'(G)\,\nu(G) \geq 0$$

for all $\lambda, \lambda' \in \Lambda$ such that $\lambda - \lambda' \in \mathsf{G}^{\dagger}$ and $\lambda(G), \lambda'(G) \geq 0$ for all $G \subset \Omega$.

Proof. To prove necessity, let φ satisfy the above conditions. Then

$$\begin{aligned} &\sum_{G \subset \Omega} \lambda(G)\,\mu(G) - \sum_{G \subset \Omega} \lambda'(G)\,\nu(G) \\ &\qquad \geq \sum_{G \subset \Omega} \lambda(G)\,\varphi(G) - \sum_{G \subset \Omega} \lambda'(G)\,\varphi(G) \\ &\qquad = \sum_{G \subset \Omega} \left(\lambda(G) - \lambda'(G)\right)\varphi(G). \end{aligned}$$

Since $\lambda - \lambda' \in \mathsf{G}^{\dagger}$, this is nonnegative.

To prove sufficiency, we prepare the notation that for $E \neq \Omega$,

$$\lambda_{E,\mu}(G) = \begin{cases} -1, & G = E \\ \mu(E), & G = \Omega \\ 0, & \text{otherwise}. \end{cases}$$

Then it suffices to show that there exists a set function φ satisfying the linear inequalities

$$\sum_{G \subset \Omega} \lambda(G)\,\varphi(G) \geq 0 \qquad \text{for all } \lambda \in \mathsf{G}^{\dagger},$$

$$\sum_{G \subset \Omega} \lambda_{E,\mu}(G)\,\varphi(G) \geq 0 \qquad \text{for all } E \in 2^{\Omega} - \{\Omega, \varnothing\},$$

and

$$\sum_{G \subset \Omega} \left(-\lambda_{E,\nu}(G)\right)\varphi(G) \geq 0 \qquad \text{for all } E \in 2^{\Omega} - \{\Omega, \varnothing\}.$$

By the well-known theorem of the alternatives (for example, see [20, Theorem 22.1]), these inequalities are consistent if the following condition holds.

For all nonnegative sequences $\{k_E\}, \{k'_E\}, \{k_i\}$, if

$$\sum_{E \in 2^{\Omega} - \{\Omega, \varnothing\}} k_E \lambda_{E,\mu}(G) - \sum_{E \in 2^{\Omega} - \{\Omega, \varnothing\}} k'_E \lambda_{E,\nu}(G) + \sum_{i<n} k_i \lambda_i(G) = 0$$

for all $G \in 2^{\Omega} - \{\Omega, \varnothing\}$, then

$$\sum_{E \in 2^{\Omega} - \{\Omega, \varnothing\}} k_E \lambda_{E,\mu}(\Omega) - \sum_{E \in 2^{\Omega} - \{\Omega, \varnothing\}} k'_E \lambda_{E,\nu}(\Omega) + \sum_{i<n} k_i \lambda_i(\Omega) \geq 0.$$

By the definition of $\lambda_{E,\mu}$, this condition is equivalent to

$$-k_G + k'_G + \sum_{i<n} k_i \lambda_i(G) = 0 \qquad \text{for all } G \in 2^{\Omega} - \{\Omega, \varnothing\}$$

which implies

$$\sum_{E \in 2^{\Omega} - \{\Omega, \varnothing\}} k_E \mu(E) - \sum_{E \in 2^{\Omega} - \{\Omega, \varnothing\}} k'_E \nu(E) + \sum_{i<n} k_i \lambda_i(\Omega) \geq 0.$$

Let $k_{\Omega} = \max\{\sum_{i<n} k_i \lambda_i(\Omega), 0\}$, $k'_{\Omega} = \max\{-\sum_{i<n} k_i \lambda_i(\Omega), 0\}$, $k_{\varnothing} = \max\{\sum_{i<n} k_i \lambda_i(\varnothing), 0\}$, and $k'_{\varnothing} = \max\{-\sum_{i<n} k_i \lambda_i(\varnothing), 0\}$.

Then it suffices to prove that $\sum_{E \subset \Omega} k_E \mu(E) - \sum_{E \subset \Omega} k'_E \nu(E) \geq 0$ when $k_G - k'_G = \sum_{i<n} k_i \lambda_i(G)$ for all $G \subset \Omega$. Since $\sum_{i<n} k_i \lambda_i(\cdot) \in \mathsf{G}^{\dagger}$, we have completed the proof. ∎

6. COMPARATIVE PROBABILITIES

In this section, we consider nonadditive comparative (subjective) probabilities. It is known when a binary relation on the power set of a finite set can be represented by a probability measure (C. H. Kraft, J. W. Pratt, and A. Seidenberg [15], D. Scott [25]). We generalize this fact for polyhedral classes of set functions, which are represented by a finite number of hyperplanes.

Let $\succcurlyeq$ be a binary relation on 2^{Ω}, where $E \succ F$ means that the event E is more probable than the event F in some subjective sense. We shall write $E \succ F$ for not $F \succcurlyeq E$.

Definition 6.1. Let $\succcurlyeq$ be a binary relation on 2^{Ω}. A binary relation $\succcurlyeq$ is said to be realizable by a class of set functions $\mathsf{G} \subset \Phi$ if there is a set function $\varphi \in \mathsf{G}$ such that for all $E, F \subset \Omega$,

$$E \succcurlyeq F \qquad \textit{if and only if } \varphi(E) \geq \varphi(F).$$

Before we give nonadditive versions of conditions for the realizability of a binary relation, we prove a lemma similar to Theorem 1.1 in D. Scott [25].

DEFINITION 6.2. Let L be a finite-dimensional real linear vector space. A subset $X \subset L$ is symmetric if $X = -X = \{-x \mid x \in X\}$. A subset $N \subset X$ is called realizable in X with respect to $Y \subset L$ if there is a linear functional φ on L satisfying the following two conditions.

(a) $\varphi(x) \geq 0$ for all $x \in Y$,

(b) $x \in N$ if and only if $\varphi(x) \geq 0$.

We shall employ a more suggestive notation writing $x \succcurlyeq 0$ to mean $x \in N$. Further we write $x \preccurlyeq 0$ to mean $-x \in N$, and $x \succ 0$ for not $x \preccurlyeq 0$.

LEMMA 6.3. *Let X be a finite symmetric subset of L, and let Y be a finite subset of L. For a subset $\{x \in X \mid x \succcurlyeq 0\}$ to be realiable in X with respect to Y it is necessary and sufficient that the conditions*

(*a*) $x \succcurlyeq 0$ *or* $x \preccurlyeq 0$,

(*b*) $\sum_{i<n} k_i x_i + \sum_{i<m} k'_i y_i = 0$ *implies* $x_0 \preccurlyeq 0$

hold for all sequences $x_0, \ldots, x_{n-1} \in X$ and $y_0, \ldots, y_{m-1} \in Y$, and for all positive scalars $k_0, \ldots, k_{n-1}$ and $k'_0, \ldots, k'_{m-1}$, where $x_i \succcurlyeq 0$ for all $i < n$; $n > 0$, and $m \geq 0$.

Proof. To prove the necessity, let φ be the realizing functional. Then the necessity of (a) is clear since $\varphi(x) \geq 0$ or $\varphi(x) \leq 0$ must hold. The necessity of (b) becomes at once clear when it is considered

$$0 = \varphi(0) = \varphi\left(\sum_{i<n} k_i x_i + \sum_{i<m} k'_i y_i\right) = \sum_{i<n} k_i \varphi(x_i) + \sum_{i<m} k'_i \varphi(y_i)$$

because k_i and k'_i are positive and $\varphi(x_i)$ and $\varphi(y_i)$ are nonnegative, and since the vector sum is actually 0, $\varphi(x_i)$ cannot be strictly positive.

To prove the sufficiency, assume that the two conditions hold. Let Q be the convex polyhedral cone generated by the set $\{x \in X \mid x \preccurlyeq 0\} \cup (-Y)$. Let P be the convex polyhedron generated by (the convex closure of) the set $\{x \in X \mid x \succ 0\}$. We can assume P is nonempty, since otherwise $x \preccurlyeq 0$ holds for all $x \in X$ by condition (a); and therefore $x \succcurlyeq 0$ would hold for all $x \in X$, because X is symmetric. If we can show that P and Q are disjoint, it follows at once (see Theorem 2 in the book by Kuhn and Tucker [9, p. 50]) that there is a linear functional φ on L such that for all $x \in L$,

$$x \in P \quad \text{implies} \quad \varphi(x) > 0,$$
$$x \in Q \quad \text{implies} \quad \varphi(x) \leq 0.$$

Thus if $x \in X$ and $x \succcurlyeq 0$, then $-x \preccurlyeq 0$, $-x \in Q$, $\varphi(-x) \leq 0$, and $\varphi(x) \geq 0$. If $\varphi(x) \geq 0$, then $\varphi(-x) \not\succ 0$, so $-x \notin P$, and $-x \preccurlyeq 0$ and $x \succcurlyeq 0$. Moreover if $x \in Y$, then $-x \in Q$, $\varphi(-x) \leq 0$, and $\varphi(x) \geq 0$. Hence, φ is the required functional.

Let us then suppose that there is a vector $z \in P \cap Q$. Now we write

$$z = \sum_{i<n} k_i x_i = \sum_{i<m} k'_i x'_i - \sum_{i<l} k''_i y_i,$$

where $k_i \geq 0$ and $x_i \in X$ with $x_i \succ 0$ for $i < n$, and $\sum_{i<n} k_i = 1$, where $k'_i \geq 0$, $x'_i \in X$ with $x'_i \preccurlyeq 0$, for $i < m$, and where $y_i \in Y$, and $k''_i \geq 0$, for $i < l$. By condition (a), $x_i \succcurlyeq 0$ holds for $i < n$. Thus

$$\sum_{i<n} k_i x_i + \sum_{i<m} k'_i(-x'_i) + \sum_{i<l} k''_i y_i = 0.$$

We can assume that all the scalars are strictly positive, and because $\sum_{i<n} k_i = 1$, we know $n > 0$. Since $-x'_i \succ 0$ for $i < m$, we can apply (b) to conclude that $x_0 \preccurlyeq 0$. But this contradicts the assumption that $x_0 \succ 0$. ∎

Notation 6.4. For distinct $E, F \subset \Omega$ denote

$$\lambda_{E,F}(G) = \begin{cases} 1, & G = E \\ -1, & G = F \\ 0, & \text{otherwise.} \end{cases}$$

Denote $\lambda_{E,E}(G) = 0$ for $E \subset \Omega$.

THEOREM 6.5. *Let* $\mathsf{G} \subset \{\varphi \in \Phi \mid \varphi(G) \geq 0$ *for all* $G \subset \Omega\}$ *be a polyhedral class of set functions and let* $\succcurlyeq$ *be a binary relation on* 2^Ω. *For* $\succcurlyeq$ *to be realizable by* G *it is necessary and sufficient that the conditions*

(a) $\Omega \succ \varnothing$,

(b) $G \succcurlyeq \varnothing$ *for all* $G \subset \Omega$,

(c) $E \succcurlyeq F$ *or* $F \succcurlyeq E$ *for all* $E, F \subset \Omega$,

(d) $\sum_{i<n} k_i \lambda_{E_i, F_i}(G) + k\lambda(G) = 0$ *for all* $G \in 2^\Omega - \{\varnothing\}$ *implies* $E_0 \preccurlyeq F_0$

hold for all sequences $k_0, \ldots, k_{n-1}$ *of positive real numbers and for* $E_i, F_i \in 2^\Omega - \{\varnothing\}$ *such that* $E_i \succcurlyeq F_i$, *for* $i < n$ *where* $n > 0$, *and for all* $\lambda \in \mathsf{G}^\dagger$ *and* $k \geq 0$.

Proof. The necessity of (a) follows from the assumptions that $\varphi(\Omega) = 1$ and $\varphi(\varnothing) = 0$. The necessity of (b) follows from the hypothesis that

$\varphi(G) \geq 0$ for all $G \subset \Omega$.

$$\begin{aligned} 0 &= \sum_{G \subset \Omega} \varphi(G)\Big(\sum_{i<n} k_i \lambda_{E_i, F_i}(G) + k\lambda(G)\Big) \\ &= \sum_{i<n} k_i \Big(\sum_{G \subset \Omega} \varphi(G)\lambda_{E_i, F_i}(G)\Big) + k\Big(\sum_{G \subset \Omega} \varphi(G)\lambda(G)\Big) \\ &= \sum_{i<n} k_i(\varphi(E_i) - \varphi(F_i)) + k\Big(\sum_{G \subset \Omega} \varphi(G)\lambda(G)\Big). \end{aligned}$$

Since $k(\sum_{G \subset \Omega} \varphi(G)\lambda(G)) \geq 0$, $\varphi(E_i) = \varphi(F_i)$ for all $i < n$. So we complete the proof of the necessity of (d).

Now we prove the sufficiency. To apply Lemma 6.3, we take $L = 2^\Omega - \{\emptyset\}$, $X = \{\lambda_{E,F} \mid E, F \in L\}$, and $N = \{\lambda_{E,F} \in X \mid E \succcurlyeq F\}$. Then X is symmetric. Let Y be generators of the polyhedral cone $\mathsf{G}^\dagger|_L$. Then Y is finite since G is polyhedral class. Thus, by Lemma 6.3, there will be a linear functional φ on L such that (e) $\sum_{G \in L} \lambda_{E,F}(G)\varphi(G) \geq 0$ if and only if $E \succcurlyeq F$ for $E, F \in L$; and (f) $\sum_{G \in L} \varphi(G)\lambda(G) \geq 0$ for all $\lambda \in Y$. Letting $\varphi(\emptyset) = 0$, φ is a linear functional on 2^Ω such that (e′) $\sum_{G \subset \Omega} \lambda_{E,F}(G)\varphi(G) \geq 0$ if and only if $E \succcurlyeq F$; and (f′) $\sum_{G \subset \Omega} \varphi(G)\lambda(G) \geq 0$ for $\lambda \in Y$ by the assumptions (a), (b), and $\mathsf{G} \subset \{\varphi \in \Phi \mid \varphi(G) \geq 0$ for all $G \subset \Omega\}$. Let $\varphi'(G) = \varphi(G)/\varphi(\Omega)$. Then $\varphi' \in \Phi$. Note that $\varphi(\Omega) > \varphi(\emptyset) = 0$. Since $\sum_{G \subset \Omega} \lambda(G)\varphi'(G) \geq 0$ for all $\lambda \in \mathsf{G}^\dagger$ if and only if $\varphi' \in \mathsf{G}$, and since $\varphi'(E) \geq \varphi'(F)$ if and only if $\sum_{G \subset \Omega} \lambda_{E,F}(G)\varphi'(G) \geq 0$, φ' is a desired set function. ∎

You can show the theorem for probability measures a little weaker than one in [25, Theorem 1.1] as a corollary of the above theorem and Theorem 3.2 since the set of probability measures is polyhedral.

COROLLARY 6.6. *Let $\succcurlyeq$ be a binary relation on 2^Ω. For $\succcurlyeq$ to be realizable by probability measures it is necessary and sufficient that the conditions*

(a) $\Omega \succ \emptyset$,

(b) $G \succcurlyeq \emptyset$ *for all* $G \subset \Omega$,

(c) $E \succcurlyeq F$ *or* $F \succcurlyeq E$ *for all* $E, F \subset \Omega$,

(d) $\sum_{G:\, \omega \in G} \sum_{i<n} k_i \lambda_{E_i F_i}(G) \leq 0$ *for all* $\omega \in \Omega$ *implies* $E_0 \preccurlyeq F_0$

hold for all sequences $k_0, \ldots, k_{n-1}$ of positive real numbers and for $E_i, F_i \in 2^\Omega - \{\emptyset\}$ such that $E_i \succcurlyeq F_i$ for $i < n$ where $n > 0$.

Proof. The necessity of this corollary is easy to show directly. To prove sufficiency, it suffices to show that condition (d) of this corollary implies condition (d) of Theorem 6.5. To prove this, it suffices to show that the premise part of condition (d) of Theorem 6.5 implies the premise part of condition (d) of this corollary. It follows from Theorem 3.2. ∎

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