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Weak and Strong Extensions of First-Order Differential Operators in R^m

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1. INTRODUCTION

Let L be the differential operator

$$L = \sum_{k=1}^{m} A_k(x)\, D^k + B(x), \tag{1.1}$$

where $x = (x_1, ..., x_m)$, $D^k = \partial/\partial x_k$, $A_1, ..., A_m$ are $\nu \times \nu$ continuously differentiable matrices in R^m, and B is a continuous $\nu \times \nu$ matrix; naturally, L acts on ν-dimensional vector functions. (We assume that all entities involved are real valued, although the results hold equally well for complex-valued functions and matrices.) The *minimal operator* A_0 associated with L is defined by $A_0 u = Lu$ with domain $D(A_0) = \mathscr{D}^\nu$ ($\mathscr{D}$ = the space of Schwartz test functions in R^m), while the *maximal operator* A associated with L (or the *weak extension* of A_0 in $L^p(R^m)^\nu = L^p(R^m, R^\nu)$ $(1 < p < \infty)$) is the adjoint $(A_0')^*$ of At, the formal adjoint of A_0, in $L^{p'}(R^m)^\nu$, $p'^{-1} + p^{-1} = 1$ (A_0' is defined as the minimal operator associated with the formal adjoint $L'u = -\sum D^k(A_k^* u) + B^* u$, where * indicates matrix adjoint). Finally, the *strong extension* of A_0 in $L^p(R^m)^\nu$ is $Cl(A_0)$. A classical result of Friedrichs [1] establishes that the weak and strong extensions of L coincide, i.e.,

$$Cl(A_0) = A \tag{1.2}$$

for $p = 2$ assuming that A_k, $D^j A_k$, and B are all bounded in R^m, and the proof is easily seen to work just as well for $1 < p < \infty$. Although this result has been extended in many directions (see, for example [3]) I have been unable to find in the literature any results on whether (1.2) holds under weaker assumptions on A_k, B at infinity. The object of this note is to show that the boundedness

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conditions on the D^jA_k and B can be entirely discarded; as for the A_k we need only assume that

$$|A_k(x)| \leqslant \rho(|x|) \qquad (x \in R^m) \tag{1.3}$$

for $k = 1, 2,..., m$, where ρ is a positive continuous function growing so slowly at infinity that

$$\int^{\infty} \frac{dr}{\rho(r)} = \infty. \tag{1.4}$$

The proof of this result (Theorem 2.1) is no more than a simple modification of Friedrichs' method of mollifiers via a "reduction to compact support" that has been employed many times (see, for instance, [6]). What makes condition (1.3)–(1.4) interesting, however, is that it is best possible in the following sense: Given a positive nondecreasing continuous ρ which does not satisfy (1.4) there exists an L (with $m = 2$, $\nu = 1$) satisfying (1.3) but not (1.2). It is not by chance, incidentally, that (1.4) is Wintner's condition in [9] for existence in the large of solutions of the ordinary differential system

$$X'(t) = a(X(t)), \tag{1.5}$$

where $a = (a_1 ,..., a_m)$, is a suitably smooth map from R^m into itself satisfying $|a(x)| \leqslant \rho(|x|)$, since there exists a close relation among (1.5) and the hyperbolic equation

$$D_t u = a_1 D^1 u + \cdots + a_m D^m u;$$

namely, solutions of (1.6) are constant along curves $(X(t), t)$ in $(m + 1)$-dimensional space. This relation is in fact used in the construction of the counterexample (Theorem 3.1).

We point in Section 4 an application of Theorem 2.1 to symmetric hyperbolic systems.

2. The Main Result

THEOREM 2.1. *Let L be as in Section* 1, *and assume the coefficients* $A_1 ,..., A_m$ *satisfy* (1.3)–(1.4). *Then* (1.2) *holds, i.e.,* $Cl\, A_0 = A$ *for* $1 < p < \infty$.

Proof. We recall briefly the definition and some properties of Friedrichs' mollifiers [1, 2, 4]. Let ψ be a nonnegative function in $\mathscr{D}$ with integral 1 and vanishing (say) in $|x| \geqslant 1$; the operators J_n $(n \geqslant 1)$ are defined by $J_n = \psi_n *$, where $\psi_n(x) = n^m \psi(nx)$. Each J_n is a bounded operator in $L^p(R^m)^\nu$ $(1 \leqslant p \leqslant \infty)$ with norm $\leqslant 1$; if $1 \leqslant p < \infty$, $J_n u \to u$ in the L^p-norm as $n \to \infty$ (see [4] for proofs). In the following result the only hypotheses on $A_1 ,..., A_m$, B are those at the beginning of Section 1; i.e., no growth conditions at infinity are prescribed.

LEMMA 2.2. *Let $u \in D(A)$ have compact support. Then*

$$\| J_n Au - AJ_n u \| \to 0 \qquad (n \to \infty). \tag{2.1}$$

Proof. Lemma 2.2 is a well-known result of Friedrichs [1] (see also [4]) when A_k, $D^j A_k$, and B are bounded in R^m, with no restrictions on the support of u. We only have to apply this theorem to the operator $\tilde{L} = \sum \tilde{A}_k D^k + \tilde{B}$, where the $\tilde{A}_k$ (resp. $\tilde{B}$) are, say, continuously differentiable (resp. continuous) matrix functions defined in all of R^m, having compact support and coinciding with A_k (resp. B) in the set of all those $x \in R^m$ with $\operatorname{dist}(x, K) \geqslant 1$; in fact, if $\tilde{A}$ is the maximal operator of $\tilde{L}$ we obviously have $AJ_n u = \tilde{A} J_n u$, $J_n Au = J_n \tilde{A} u$.

Proof of Theorem 2.1. We may obviously assume that ρ is infinitely differentiable. Let φ be another infinitely differentiable function of r with $\varphi(r) = 0$ if $r \leqslant 0$, $\varphi(r) = 1$ if $r \geqslant 1$. Given $0 < r < \infty$ define

$$f_r(s) \begin{array}{ll} = 1 & \text{if } \ 0 \leqslant s \leqslant r \\ = 1 - \displaystyle\int_r^s \frac{d\sigma}{\rho(\sigma)} & \text{if } \ r \leqslant s \leqslant s_r \\ = 0 & \text{if } \ s_r \leqslant s, \end{array} \tag{2.2}$$

where s_r is such that $\int_r^{s_r} \rho(s)^{-1}\, ds = 1$. The function

$$\chi_r(x) = \varphi(f_r(| x |)) \tag{2.3}$$

belongs to $\mathscr{D}$, its support being contained in $| x | \leqslant S_r$; its first partials have support in the ring $r \leqslant | x | \leqslant s_r$ and

$$| D^k \chi_r(x) | \leqslant C/\rho(| x |) \qquad (x \in R^m). \tag{2.4}$$

If $u \in L^p(R^m)^\nu$ it is clear that

$$\chi_r u \to u \tag{2.5}$$

as $r \to \infty$. On the other hand, if $u \in D(A)$ a simple computation with adjoints shows that

$$A(\chi_r u) = \left(\sum_{k=1}^m A_k D^k \chi_r \right) u + \chi_r Au. \tag{2.6}$$

In view of (1.3) and (2.4) we obtain, taking into account that the $D^k \chi_r$ vanish for $| x | \leqslant r$,

$$\|(A_k D^k \chi_r) u \| \leqslant C \int_{|x| \geqslant r} | u |^p \, dx,$$

which tends to zero as $r \to \infty$; accordingly,

$$A(\chi_r u) \to Au,$$

which, combined with (2.5), shows that $Cl(A_C) = A$, A_C the restriction of A to elements of $D(A)$ having compact support. Hence we only have to prove that, given $u \in D(A_C)$ there exists a sequence $\{u_n\}$ in $D(A_0)$ with $u_n \to u$, $A_0 u_n \to Au$. In view of (2.2), $\{J_n u\}$ is such a sequence. This ends the proof.

3. A Counterexample

Throughout this section ρ is a positive, nondecreasing differentiable function in $r \geqslant 0$ with $\rho'(0) = 0$, and

$$\int_0^\infty \frac{dr}{\rho(r)} < \infty \tag{3.1}$$

and

$$L = \rho(x_2)D^1 + \rho(x_1)D^2, \tag{3.2}$$

where ρ is extended to $r < 0$ by setting $\rho(r) = \rho(0)$. The operator L is a particular case of that in (1.1) (here $m = 2$, $\nu = 1$).

THEOREM 3.1. *Let A_0 (resp. A) be the minimal* (resp. *maximal*) *operator associated with L. Then, for $p = 2$*

$$Cl(A_0) \neq A. \tag{3.3}$$

Proof. Since $D^1\rho(x_2) + D^2\rho(x_1) = 0$, the operator $A_0' = -A_0$ is dissipative (in fact, conservative, since $(A_0'w, w) = 0$ for $w \in D(A_0'))$ thus it follows from standard duality arguments ([4]) that

$$(I \pm A)\, D(A) = L^2(R^2).$$

If $Cl(A_0) = A$ then A itself is conservative; hence it generates a group $S(t)$, $-\infty < t < \infty$ of isometric operators in $L^2(R^2)$. Consider now the ordinary differential system

$$X_1'(t) = \rho(X_2(t)), \qquad X_2'(t) = \rho(X_1(t)). \tag{3.4}$$

Its trajectories can be described as follows. Solutions starting (say, at $t = 0$) in the negative quadrant $\pi_{--}(x_1\,, x_2 < 0)$ are straight lines

$$X_1(t) = X_1(0) + t\rho(0), \qquad X_2(t) = X_2(0) + t\rho(0)$$

until they leave π_{--}. If $(X_1(0), X_2(0)) \in \pi_{+-}$ $(x_1 \geqslant 0, x_2 < 0)$, the corresponding trajectory is

$$X_1(t) = X_1(0) + t\rho(0),$$

$$X_2(t) = X_2(0) + \int_0^t \rho(X_1(0) + s\rho(0))\, ds$$

until it leaves π_{+-} (which it will eventually do, since its slope is > 1). An entirely symmetric expression holds in π_{-+}. Finally, if $(X_1(0), X_2(0))$ belongs to the positive quadrant π_{++}, $X_1(t)$ is obtained inverting the function

$$t(X_1) = \int_0^{X_1} \frac{dr}{\rho(R^{-1}(R(r) + C_1)} + C_2 \tag{3.5}$$

and setting

$$X_2(t) = R^{-1}(R(X_1(t)) + C_1),$$

where

$$R(r) = \int_0^r \rho(s)\, ds \qquad (s \geqslant 0),$$

R^{-1} is the functional inverse of R, and C_1, C_2 are constants determined by the initial conditions

$$C_1 = R(X_2(0)) - R(X_1(0)),$$

$$C_2 = -\int_0^{X_1(0)} \frac{dr}{\rho(R^{-1}(R(r) + C_1))}.$$

If $C_1 \geqslant 0$, $R^{-1}(R(r) + C_1) \geqslant r$, and thus $\rho(R^{-1}(R(r) + C_1)) \geqslant \rho(r)$. Hence, t tends to a finite limit as X_1 (thus X_2) tends to infinity; consequently, solutions with $X_1(0) \geqslant X_2(0)$ cease to exist at a time $t = t(X_1(0), X_2(0))$ given by

$$t = \int_{X_1(0)}^{\infty} \frac{dr}{\rho(R^{-1}(R(r) + C_1))} \leqslant \int_{X_1(0)}^{\infty} \frac{dr}{\rho(r)}. \tag{3.6}$$

(Since we can argue in the same way interchanging X_1 and X_2, *all* solutions starting in π_{++} have a finite life expectancy.) Putting together all these observations we conclude that an arbitrary trajectory of (2.4) will eventually enter π_{++} and it will blow itself up in finite time; on the hand, if time is run backward, the trajectory will enter π_{--} and will then exist forever (see Fig. 1).

Let $T > 0$, u_0 be a vector function in $\mathscr{D}^\nu$. Given $(x_1, x_2) \in R^2$ denote by $X_1(x_1, x_2, t)$, $X_2(x_1, x_2, t)$ the solution of (3.5) with initial conditions

$$X_1(x_1, x_2, 0) = x_1, \qquad X_2(x_1, x_2, t) = x_2, \tag{3.7}$$

and let $u(x_1, x_2, t)$ be defined by

$$u(x_1, x_2, t) = u_0(X_1(x_1, X_2, T - t), X_2(x_1, x_2, T - t))$$

for $(x_1, x_2) \in R^2$, $0 \leqslant t \leqslant T$ ($u(x_1, x_2, t) = 0$ if $(X_1(x_1, x_2, s), X\,(x_1, x_2, s))$ blows up before $T - t$). It follows from standard theorems on dependence on initial data (see [5]) and from our previous observations on the trajectories of

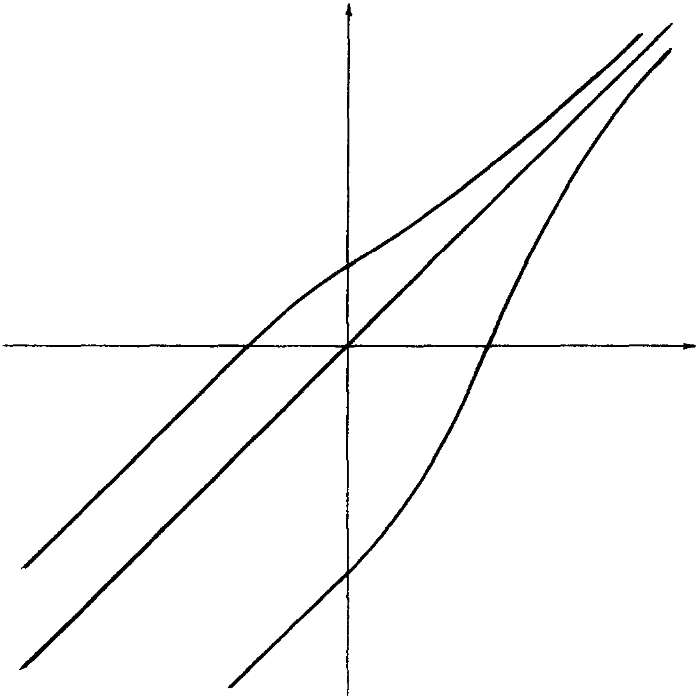

FIGURE 1

(3.4) that u is continuously differentiable in $R^2 \times [0, T]$ and has compact support there. It is obvious that u is constant on trajectories of (3.4), thus $D_t u = \rho(x_2) D^1 u + \rho(x_1) D^2 u$, and it is easy to see that the $L^2(R^2)$-valued function $u(t) = u(\cdot, \cdot, t)$ is a solution of $u'(t) = Au(t)$ in $0 \leqslant t \leqslant T$ with $u(T) = u_0$; thus

$$u(0) = S(-T)u_0 . \qquad (3.8)$$

It follows from (3.6) and the comments preceding and following this inequality that

$$u(x_1, x_2, 0) = 0 \qquad ((x_1, x_2) \in \Omega), \qquad (3.9)$$

where Ω is the quadrant $x_1, x_2 \geqslant \omega$ with ω so large that

$$\int_\omega^\infty \frac{dr}{\rho(r)} \leqslant T.$$

We end the proof as follows. Let u be an arbitrary element of $L^2(R^2)$; choose a sequence $\{u_n\}$ in $\mathscr{D}^v$ with $u_n \to u$, write (3.8) for u_n, and take limits. We obtain using (3.9) that $S(-T)u = 0$ a.e. in Ω, which contradicts the fact that $S(\cdot)$ is a group. We must then conclude that (3.3) holds.

4. An Application

Consider the symmetric hyperbolic system

$$D_t u = \sum_{k=1}^{m} A_k D^k u + Bu, \tag{4.1}$$

where $A_1, \ldots, A_m$, B satisfy the smoothness assumptions at the beginning of Section 2; moreover, each A_k is symmetric. Under the customary assumption that A_k, $D^k A_k$, and B are bounded in R^m it is known that the Cauchy problem for (4.1) is properly posed in $L^2(R^m)^\nu$; precisely, $A - \omega I$ is maximal dissipative there for some ω. This is proved by showing that $A_0 - \omega I$ is dissipative (by integration by parts) and extending this property to $A - \omega I$ using (1.2). We can then obtain the same conclusion under weaker hypotheses, namely,

$$B(x) - \tfrac{1}{2} \sum_{k=1}^{m} D^k A_k(x) \leqslant \omega I \qquad (x \in R^m), \tag{4.2}$$

which suffices to ensure dissipativity of $A_0 - \omega I$ and (1.3)–(1.4), which yields (1.2) via Theorem 2.1.

The example in Theorem 3.1 has some intriguing features. It is not difficult to see that the operator A there, although not a group generator, is maximal dissipative, and thus generates a contraction semigroup $S(t)$ $(t \geqslant 0)$ in $L^2(R^2)$ which is explicitly given by the formula

$$S(t)u(x_1, x_2) = u(X_1(x_1, x_2, -t), X_2(x_1, x_2, -t)), \qquad (x_1, x_2) \in R^2.$$

However, the equation $u'(t) = Au(t)$ does not possess the "finite-domain-of-dependence" properties usually expected of hyperbolic equations; in other words, perturbations arbitrarily far away in space at time $t = 0$ may reach a given point all at (or near) the same time $T > 0$. To see this, let $(X_1(t), X_2(t))$ be an arbitrary trajectory of (3.4), T its escape time (we may assume, translating time if necessary that $T > 0$), and φ a function in $\mathscr{D}$ with $\varphi(0) \neq 0$. If

$$u_n(x_1, x_2) = \varphi(x_1 - X_1(T - 1/n), x_2 - X_2(T - 1/n))$$

the supports of the u_n wander off to infinity; however, $S(t)u_n(x_1, x_2) = \varphi(x_1 - X_1(-1/n), x_2 - X_2(-1/n))$, whose support lies arbitrarily near that of φ.

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