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Geodesics in Transitive Graphs

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functions in X and in P respectively. One calls P a *geodesic* if $d(x, y) = d_P(x, y)$, for all vertices x and y in P. We give situations when every edge of a graph belongs to a geodesic or a half-geodesic. Furthermore, we show the existence of geodesics in infinite locally-finite transitive graphs with polynomial growth which are left invariant (set-wise) under "translating" automorphisms. As the main result, we show that an infinite, locally-finite, transitive, 1-ended graph with polynomial growth is planar if and only if the complement of every geodesic has exactly two infinite components.

1. INTRODUCTION

The graphs considered in this paper are simple graphs and may be finite or infinite. The symbols $V(X)$, $E(X)$, and $\mathrm{AUT}(X)$ will denote, respectively, the vertex set, the edge set, and the automorphism group of the graph X. All infinite graphs will be presumed to be *locally finite*; that is, the valence of every vertex is finite. If $x, y \in V(X)$, then the *distance* from x to y, denoted $d_X(x, y)$, is the length of the shortest path (in X) joining x and y. The subscript will be omitted if no ambiguity results.

We say that a group $G \leqslant \mathrm{AUT}(X)$ acts *transitively* on a graph X if for every pair $\{x, y\}$ of vertices in X there exists an automorphism g in G such that $g(x) = y$; or equivalently, G acts on $V(X)$ with just one orbit. If such a group exists we call X *transitive*. If $\mathrm{AUT}(X)$ acts on $E(X)$ with just one orbit then we call X *edge-transitive*. We call X *almost-transitive* if $\mathrm{AUT}(X)$ acts on $V(X)$ with finitely many orbits.

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Let G be a group and H a generating set for G not containing the identity. The *Cayley graph*, denoted $C(G, H)$, of G (with respect to H) is the graph with vertex set G and edge set

$$E(C(G, H)) = \{(g, gh) \mid g \in G, h \in H \cup H^{-1}\}.$$

A *double ray* in a graph is an infinite connected subgraph in which every vertex has valence 2. A *ray* is an infinite connected subgraph in which every vertex has valence 2 except for precisely one vertex, which has degree 1. A ray is said to *emanate from* its vertex of degree 1. Two double rays or rays *intersect* if they share a common vertex.

If Y is a subgraph of a graph X, then $X - Y$ denotes the subgraph of X induced by $V(X) - V(Y)$.

We use the notion of an "end" as formulated by Halin [5]. Two rays P and Q, in a graph X, are *equivalent* if there is a third ray R such that $V(P) \cap V(R)$ and $V(Q) \cap V(R)$ are both infinite. The equivalence classes with respect to this relation are called *ends*. The number of ends of an infinite graph X turns out to be the supremum of the number of infinite components of $X - T$ as T ranges over all finite subsets of $V(X)$. It is well-known that every connected, infinite, almost-transitive graph has either one, two or infinitely many ends. Graphs with zero ends are finite.

The *growth function* of a graph X, with respect to a vertex $x \in V(X)$, is defined by $f_X(x, 0) = 1$ and

$$f_X(x, n) = |\{y \in V(X) \mid d(x, y) \leqslant n\}|, \qquad \text{for all} \quad n \in \mathbf{N}.$$

If X is transitive, then the growth function clearly does not depend on a particular vertex, and therefore we denote it by $f_X(n)$. We say that X has *exponential growth* if there exists a constant $c > 1$ such that $f_X(n) \geqslant c^n$ holds for all $n \in \mathbf{N}$. Otherwise X has *nonexponential growth*. In particular, X has *polynomial growth* if $f_X(n) \leqslant cn^d$ holds for some constants c and d. For a survey of results on graphs with polynomial growth we refer the reader to [9].

A double ray P in a graph X is called a *geodesic* (*in* X) if $d_X(x, y) = d_P(x, y)$ for all vertices x and y of P. An edge in X is said to be *geodetic* if it belongs to some geodesic in X. Similarly, a ray R in X is called a *half-geodesic* (*in* X) if $d_X(x, y) = d_R(x, y)$ for all vertices x and y of R.

Geodesics and half-geodesic were studied by Watkins in [19]. (In this particular work they were called "axes" and "half-axes", respectively.) Watkins proved many results about geodesics in transitive graphs. What we prove in the sequel may be regarded as a continuation of Watkins' paper.

Theorem 4.1 in [19] states that in an infinite transitive graph, every vertex belongs to a geodesic. However this is not usually the case for edges.

EXAMPLE 1.1. We construct an infinite family of infinite Cayley graphs with polynomial growth in which not all edges are geodetic. Let G_n be the group $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}_n$. Clearly

$$H_n = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 0, 1), (0, 1, 1)\}$$

is a generating set for G_n. For every $n \geqslant 2$, the edges corresponding to the generator $(0, 0, 1)$ in $C(G_n, H_n)$ are not geodetic.

In Section 3, we give two situations where we can guarantee that every edge of a graph is geodetic or every edge belongs to a half-geodesic. It is shown that in a 1-ended transitive bipartite graph every edge belongs to a half-geodesic and that with the additional property of planarity, every edge is geodetic.

In [7], Halin discusses the two essentially different types of automorphisms of an infinite graph X. An automorphism g is of *type* 1 if there is a finite non-empty subgraph Y of X such that $g(Y) = Y$; otherwise g is of *type* 2. Automorphisms of type 1 represent, something like "rotations," whereas type 2 automorphisms may be regarded as something like "translations." Halin shows that for every type 2 automorphism g there exists a double ray P and an integer n such that $g^n(P) = P$. In Section 5, we partially extend this result to show that in graphs with polynomial growth, there exists a type 2 automorphism g and a geodesic P such that $g(P) = P$. Such geodesics are said to be of *type* 2.

A double ray P is said to *separate* a connected graph X if $X - P$ has two or more infinite components. In particular, P *bisects* X if $X - P$ has exactly two infinite components. Separating double rays were studied in [2] where a characterization of infinite almost-transitive planar graphs, by double ray separation properties was given. Also stated was the following conjecture which we will confirm, in Sections 6, 7 and 8, for the case where X has polynomial growth.

Conjecture 1.2. A 1-ended transitive graph X is planar if and only if every geodesic bisects X.

2. THE REPLACEMENT PROPERTY FOR GEODESICS

We now introduce some notation for describing various parts of a path, ray or double ray P. If $x, y \in V(P)$, then $P[x, y]$ and $P[y, x]$ will denote the subpath of P joining x and y. The subpath of $P[x, y]$ remaining when x and the edge incident with x have been deleted is denoted by $P(x, y]$. Analogously we define $P[x, y)$ and $P(x, y)$. If P is a double ray and $x \neq y$, then the notation $P_y[x, \infty)$ will denote the unique ray emanating from x

in $P-\{y\}$. A similar definition can be made for $P_y(x, \infty)$. Convenient notations are $P_{[x, y]}$ and $P_{(x, y)}$ for $P-P[x, y]$ and $P-P(x, y)$ respectively. Hence each of $P_{[x, y]}$ and $P_{(x, y)}$ consist of two rays; in the latter case the two rays emanate from x and y respectively.

The following property of geodesics is called the "replacement property."

LEMMA 2.1. *If x and y are distinct vertices of a geodesic P and Q is an xy-path, then $P_{(x, y)} \cup Q$ is a geodesic if and only if Q is a shortest xy-path.*

Proof. The necessity follows immediately from the definition of a geodesic.

To prove sufficiency, set $R = P_{(x, y)} \cup Q$ and let u and v be distinct vertices of R. Suppose $d(u, v) < d_R(u, v)$. We may assume without loss of generality that $u \in P_y(x, \infty)$ and $v \in Q$. Now,

$$\begin{aligned} d(u, y) &\leqslant d(u, v) + d(v, y) \\ &< d_R(u, v) + d(v, y) \\ &= d_R(u, x) + d_R(x, v) + d_R(v, y) \\ &= d_P(u, x) + d_P(x, y) \\ &= d(u, y) \text{ since } P \text{ is a geodesic,} \end{aligned}$$

which is impossible. Hence $d(u, v) = d_R(u, v)$ for all vertices $u, v \in R$. ∎

3. GEODETIC EDGES

LEMMA 3.1. *Let $e = (x, y)$ be an edge in a bipartite graph, and suppose that there exist two intersecting geodesics containing x and y respectively. Then e is geodetic.*

Proof. Let P and Q be two intersecting geodesics containing x and y respectively, and let z be a vertex of $P \cap Q$ such that $V(P[x, z] \cap Q[y, z]) = \{z\}$. Since both P and Q are geodesics, then $d_P(z, x)$ and $d_Q(z, y)$ differ by one; otherwise $P[x, z] \cup Q[y, z] + e$ would be a circuit of odd length, a contradiction. Hence we assume without loss of generality that $d_Q(y, z) = d_P(x, z) - 1$. Thus, by the replacement property for geodesics, $P_{(x, z)} \cup Q[y, z] + e$ is a geodesic containing e, as required. ∎

We now, momentarily, turn our attention to planar graphs. When a planar graph is 3-connected, the cyclic order of the edges incident with each vertex when that graph is embedded in the plane becomes an intrinsic property of the graph and is the same for all planar embeddings. This classic result of H. Whitney [20] for finite graphs has been generalized to infinite graphs in [8] and [17]. Also independent of the embedding is the collection of subgraphs that form the boundaries of the faces. We call each such subgraph a *face boundary*.

It is well known that every 1-ended, locally finite, planar graph has a *proper embedding*, that is, an embedding without accumulation points. (See for example [6].) It will be assumed that only proper embeddings of planar 1-ended graphs are used. Furthermore, it is easy to show (see [2]) that a 1-ended transitive graph is 3-connected and hence there is essentially just one embedding of such a graph.

THEOREM 3.2. *Every edge of a* 1-*ended transitive bipartite planar graph is geodetic.*

Proof. Let X be a 1-ended transitive bipartite planar graph and let $e=(x, y)$ be an arbitrary edge in X. By Theorem 4.1 in [19], there exist geodesics P and Q containing the vertices x and y respectively. If P and Q intersect Lemma 3.1 implies that e is geodetic, and hence we assume otherwise.

Let $E(P)=\{(x_i, x_{i+1}) \mid i \in \mathbf{Z}\}$ where $x=x_0$, and (without loss of generality) $(x_1 \cdots y \cdots x_{-1} \cdots)$ is the clockwise permutation induced on the neighbors of x in the (unique) embedding of X. (See Fig. 1.) Similarly, let $E(Q)=\{(y_i, y_{i+1}) \mid i \in \mathbf{Z}\}$, where $y=y_0$ and (without loss of generality) $(y_1 \cdots y_{-1} \cdots x \cdots)$ is the clockwise permutation induced on the neighbors of y. We denote by P^+ and P^- the rays induced by the vertices of P with nonnegative subscripts and nonpositive subscripts, respectively. Appropriate definitions can be made for Q^+ and Q^-.

Consider the double ray $R=e+P^- \cup Q^+$. If R is not a geodesic then some path S joining a vertex $w \in P^-$ to a vertex $z \in Q^+$ has the property that $d_S(w, z) < d_R(w, z)$. By planarity and 1-endedness, S must meet some vertex t in P^+ or Q^-; otherwise, $S \cup R[w, z]$ contains an end-separating circuit. Without loss of generality assume that $t \in V(Q^-)$. Since Q is a geodesic, we may assume that $S[t, z]=Q[t, z]$. Since P is a geodesic and $d_S(w, y) < d_R(w, y)$ then $d_S(w, y)=d_P(w, x)-1$ or $d_S(w, y)=d_P(w, x)$.

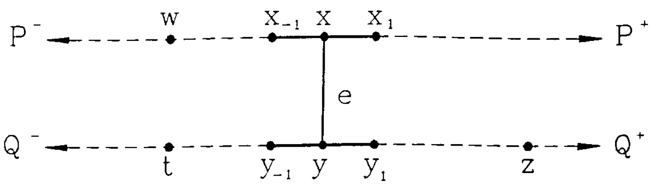

FIGURE 1

However, the latter is ruled out by bipartiteness. The replacement property for geodesics implies $P_{(w,x)} \cup S[w, y] + e$ is a geodesic containing e. ∎

It is not known if one can drop the condition that X is bipartite from Theorem 3.2. Hence we offer the following conjecture.

Conjecture 3.3. Every edge of a 1-ended transitive planar graph is geodetic.

We would like to mention that Thomassen [15] has shown that in an infinite transitive cubic graph every edge is geodetic.

We conclude this section with a result for transitive bipartite graphs in general.

THEOREM 3.4. *Every edge of an infinite connected transitive bipartite graph belongs to a half-geodesic.*

Proof. Let X be an infinite connected transitive bipartite graph and let $e = (x, y)$ be an arbitrary edge in X. By Theorem 4.1 in [19], there exists a geodesic P containing x. If $e \in E(P)$ then we are done and hence we assume otherwise. By Lemma 3.2 in [19], there exists a half-geodesic emanating from y which has the form $R \cup P_v[u, \infty)$, where $v \in V(P)$, $u \in P_v[x, \infty)$ and R is a shortest yu-path. If $e \in E(R)$ then again we are done, and hence we assume otherwise. Since X is bipartite, $d_R(y, u) \neq d_P(x, u)$. Furthermore, $P[x, u]$ is a shortest xu-path, and therefore $d_R(y, u)$ and $d_P(x, u)$ differ by 1. If $d_R(y, u) = d_P(x, u) - 1$ then $P_{(x,u)} \cup R + e$ is a geodesic containing e. On the other hand, if $d_R(y, u) = d_P(x, u) + 1$ then we can assume that $R = P[x, u] + e$, and again we are done. ∎

4. CHARACTERIZING POLYNOMIAL GROWTH

The *growth function* $f_G(n)$ of a group G with respect to a finite generating set H, is defined by $f_G(0) = 1$ and

$$f_g(n) = |\{g \mid g = h_1 \cdots h_n, h_i \in H \cup H^{-1} \cup \{e\}\}|, \qquad \text{for all} \quad n \in \mathbf{N}.$$

We say that G has *exponential growth* if there exists a constant $c > 1$ such that $f_G(n) \geqslant c^n$ holds for all $n \in \mathbf{N}$. Otherwise G has *nonexponential growth.* In particular, G has *polynomial growth* if $f_G(n) \leqslant cn^d$ holds for some constants c and d.

We can identify the growth function of a group G, with respect to a generating set H (not containing the identity), and that of $C(G, H)$. One can consider results about group growth as results about Cayley graphs.

Gromov's [3] characterization of groups with polynomial growth is one of the crucial results which permeates the remainder of this paper.

THEOREM 4.1 (Gromov [3]). *A finitely generated group G has polynomial growth if and only if it contains a nilpotent subgroup of finite index.*

If a group G acts transitively on X, then an *imprimitivity system* of G on X is a partition τ of $V(X)$ into subsets called *blocks*, such that every element of G induces a permutation of the blocks of τ. Among imprimitivity systems we include the partition of $V(X)$ into singletons and into $\{V(X)\}$ itself. If τ is a partition into blocks of a group G which does not act transitively on X then τ is called a *block system* of G on X. The *quotient graph* X_τ is defined as follows: $V(X_\tau)$ is the set of blocks and two vertices $x_\tau, y_\tau \in V(X_\tau)$ are adjacent in X_τ if there exists an edge $(x, y) \in E(X)$ such that $x \in x_\tau$ and $y \in y_\tau$. By G_τ we denote that group acting on X_τ which is induced by G. Clearly, it is a homomorphic image of G and G_τ is a subgroup of $\mathrm{AUT}(X_\tau)$.

A group G *almost* has a property **P** if a normal subgroup of finite index in G has property **P**.

THEOREM 4.2 (Trofimov [18]). *Let X be an infinite connected transitive graph with polynomial growth and suppose that a subgroup G of $\mathrm{AUT}(X)$ acts transitively on X. Then there exists an imprimitivity system τ of G on X with finite blocks such that G_τ is a finitely generated almost nilpotent group and the stabilizer of a vertex of X_τ in G_τ is finite.*

Gromov's [3] deep result about groups with polynomial growth together with [1, Theorem 2] implies that for every group G with polynomial growth we can find constants c_1, c_2 and an integer d such that $c_1 n^d \leqslant f_G(n) \leqslant c_2 n^d$ holds. Trofimov's characterization of automorphism groups of graphs with polynomial growth implies that the same also holds for graphs with polynomial growth; that is, there always exist constants c_1, c_2 such that $c_1 n^d \leqslant f_X(n) \leqslant c_2 n^d$ holds for some integer d. We call the least integer d such that $f_X(n) \leqslant cn^d$ holds for some constant c the *growth degree* d_X of X. Graphs with growth degree 1 are said to have *linear* growth. Similarly, graphs with growth degree 2 are said to have *quadratic* growth. Theorem 4.2, together with a result of Sabidussi [11, Theorem 4], implies that G_τ and X_τ (and hence also X) have the same growth degree.

An automorphism g of a graph X is called *bounded* if there is a constant k, depending upon g, such that $d(x, g(x)) \leqslant k$ for every $x \in V(X)$. We denote by $B(X)$ the group of all bounded automorphisms of X, and by $B_f(X)$ the set of elements of finite order in $B(X)$.

LEMMA 4.3 (Seifter [14, Corollary 2.7]). *Let X be an infinite connected graph with polynomial growth and suppose $G \geqslant \mathrm{AUT}(X)$ acts transitively on X. Then the orbits of $B_f(X) \cap G$ on X give rise to an imprimitivity system τ of G on X such that G_τ is a finitely generated almost nilpotent group and the stabilizer of a vertex of X_τ in G_τ is finite.*

5. TYPE 2 GEODESICS

We now proceed to show the existence of type 2 geodesics in graphs with polynomial growth. Firstly, a preliminary result from [14].

LEMMA 5.1 (Seifter [14, Lemma 4.1]). *Let X be a graph and let O and O' be two orbits of a subgroup of $\mathrm{AUT}(X)$ on X. Suppose O and O' are both finite, have identical size, and there is at least one edge in X joining a vertex in O to a vertex in O'. Then there exists a complete matching of the bipartite subgraph of X with vertex set $O \cup O'$ and edge set $\{(x, y) \mid (x, y) \in E(X), x \in O, y \in O'\}$.*

LEMMA 5.2. *Let X be an infinite connected graph and suppose the group G acts transitively on X. Let τ be an imprimitivity system with finite blocks of G on X such that the blocks of τ coincide with the orbits of a subgroup of G. If there exists a type 2 automorphism in G_τ which leaves invariant a geodesic in X_τ, then G contains a type 2 automorphism which leaves invariant a geodesic in X.*

Proof. Let n denote the cardinality of the blocks of τ. Suppose g is an automorphism in G such that g_τ is a type 2 automorphism which leaves invariant a geodesic P_τ in X_τ. Clearly, g is a type 2 automorphism of X.

Let x_τ be a distinguished vertex of P_τ. Then evidently

$$P_\tau = \bigcup_{j \in \mathbf{Z}} g_\tau^j(P_\tau[x_\tau, g_\tau(x_\tau)]).$$

Let $m = |E(P_\tau[x_\tau, g_\tau(x_\tau)])|$, and label the vertices of P_τ so that $E(P_\tau) = \{(x_\tau^j, x_\tau^{j+1}) \mid j \in \mathbf{Z}\}$ where $x_\tau^0 = x_\tau$ and $g(x_\tau^j) = x_\tau^{j+m}$. Recall that each x_τ^j is a block of τ in X. Evidently any double ray Q with edge set $E(Q) = \{(y_j, y_{j+1}) \mid y_j \in x_\tau^j, j \in \mathbf{Z}\}$ is a geodesic in X; for otherwise any path joining vertices y_i and y_k that has length less than $|i-k|$ would have an $x_\tau^i x_\tau^k$ path of length less than $|i-k|$ as an image in X_τ, contradicting the fact that P_τ is a geodesic. We shall proceed to construct such a double ray in X, and show that it remains invariant under a type 2 automorphism.

Since each vertex x_τ^j is a block of τ in X of cardinality n, we let $x_\tau^j = \{x_1^j, ..., x_n^j\}$. Furthermore, $g(x_\tau^j) = x_\tau^{j+m}$ for all $j \in \mathbf{Z}$, and therefore we may choose this labeling so that $g(x_i^j) = x_i^{j+m}$ for all $1 \leqslant i \leqslant n$ and $j \in \mathbf{Z}$. Hence each $O_i = (\ldots, x_i^{-m}, x_i^0, x_i^m, x_i^{2m}, \ldots)$ is an orbit of g on X.

By a previous observation every preimage of P_τ is a geodesic. Let $P_\tau = (\ldots, x_\tau^{-1}, x_\tau^0, x_\tau^1, \ldots)$. Since each x_τ^j is a block of τ in X and the blocks of τ satisfy the assumptions of Lemma 5.1, there is a complete matching between every pair of blocks x_τ^j, x_τ^{j+1}, $j \in \mathbf{Z}$, in X. Hence P_τ has n pairwise disjoint preimages each of which is a geodesic in X. Let P denote one such preimage. It is not likely that P is invariant under g, even a power of g. However, since P is infinite and meets each (finite) block x_τ^j for $j \in \mathbf{Z}$, it meets an infinite number of blocks of the form x_τ^{im}. Hence, there is an integer $1 \leqslant l \leqslant n$ such that O_l is met infinitely often by P. Let s and t be two integers such that x_l^{sm} and x_l^{tm} both belong to P. Then $Q = \bigcup_{j \in \mathbf{Z}} (g^{|t-s|})^j (P[x_l^{sm}, x_l^{tm}])$ is the required geodesic. ∎

LEMMA 5.3. *Let a finitely generated torsion free nilpotent group N act almost-transitively on a connected graph X. Then there is a type 2 geodesic P in X.*

Proof. We prove this by induction on the growth degree of N. If N has linear growth then $N \cong \mathbf{Z}$ which immediately implies that X is a two-ended graph. Then the conclusion follows easily from Theorem 2.4 in [10].

Let $\{e\} = N_0 \lhd N_1 \lhd N_2 \lhd \cdots \lhd N_p = N$ be the upper central series of N. Since N_1 is the centre of N, then by Theorem 2.3 in [13] it contains a type 2 automorphism g which is central in N. Let ε now denote the block system of N on X which is induced by the orbits of g on X. Then by Theorem 2 in [1] (see also Lemma 6.3 of this paper), N_ϵ has growth degree less than N. Also N_ϵ acts with finitely many orbits on X_ϵ.

By the induction hypothesis there is a type 2 automorphism h_ε of X_ε which leaves invariant a geodesic P_ε. Let $E(P) = \{(x_\varepsilon^j, x_\varepsilon^{j+1}) \mid j \in \mathbf{Z}\}$. Let n be the positive integer such that $h_\varepsilon(x_\varepsilon^j) = x_\varepsilon^{j+n}$ holds for all $j \in \mathbf{Z}$. Since the x_ε^j are orbits of g, then $P_\varepsilon[x_\varepsilon^0, x_\varepsilon^n]$ lifts to a finite path Q in X with edge set $E(Q) = \{(x^j, x^{j+1}) \mid 0 \leqslant j \leqslant n-1\}$, such that $x^i \in x_\varepsilon^i$.

Let $h = \epsilon^{-1}(h_\epsilon)$. Since h acts on the orbits of g, there exists an m such that $g^m h(x^0) = x^n$. Then $\bigcup_{j \in \mathbf{Z}} (g^m h)^j (Q)$ is the vertex set of a geodesic in X which is left invariant by the type 2 automorphism $g^m h$, as required. ∎

THEOREM 5.4. *Let X be an infinite connected transitive graph with polynomial growth. Then there is a type 2 geodesic P in X.*

Proof. By Lemma 5.2 and Theorem 4.2 we can assume that $\mathrm{AUT}(X)$ is a finitely generated group with polynomial growth. Hence by Theorem 4.1 $\mathrm{AUT}(X)$ contains a finitely generated nilpotent subgroup N of finite index. Furthermore it is known (see, for example, [12, Corollary 10]), that the subgroup N_f of N consisting of the elements of finite order of N is finite.

Hence there are only finitely many values for $d_X(x, y)$ where x and y belong to the same orbit of N_f on X. Therefore N_f cannot contain unbounded automorphisms. Hence by Lemma 5.1 we can assume that N is a torsion free group. Since it has finite index in $\mathrm{AUT}(X)$, it clearly acts with finitely many orbits on X. So Lemma 5.3 completes the proof. ∎

6. The Geodetic Subgraph

The subgraph of an infinite graph X spanned by the geodetic edges is called the *geodetic* subgraph of X. If X is transitive then it is immediate that its geodetic subgraph is transitive and spanning. The section is devoted to proving results on the geodetic subgraph.

THEOREM 6.1. *If X is a* 1*-ended transitive graph such that every geodesic bisects X, then the geodetic subgraph of X is planar.*

Proof. Let Y be the geodetic subgraph of X, and by way of contradiction, suppose that Y is nonplanar. By Kuratowski's theorem, Y contains a subdivision K of $K_{3,3}$ or K_5. We label the edges of K so that $E(K) = \{e_1, ..., e_n\}$, where $n = |E(K)|$. We denote by P_i a geodesic in X containing e_i, for $1 \leqslant i \leqslant n$. Since each P_i bisects X, we let $\{V_i, V'_i\}$ be the partition of $V(X) - V(P_i)$ induced by the two infinite components of $X - P_i$. (There are no finite components in $X - P_i$ by Theorem 4.2 in [19].)

Now suppose that some subgraph with vertex set $V(X)$ and containing P_i has been properly embedded in the plane. Clearly, P_i induces a partition of $V(X) - V(P_i)$ into two classes; the classes are determined by the two sides "sides" of P_i. (See, for example [16, Lemma 8.1].) We say this planar embedding is *faithful to* P_i if this partition is $\{V_i, V'_i\}$.

Evidently it is possible to embed the graph with vertex set $V(X)$ and edge set $E(P_1)$ in the plane so that this embedding is faithful to P_1. Since this graph is not necessarily 3-connected, this embedding is not necessarily unique. Now suppose S is an embedding of the graph with vertex set $V(X)$ and edge set $E(\bigcup_{j=1}^{m} P_i)$ in the plane, that is faithful to $P_1, ..., P_m$, where $1 < m < n$. Moreover, suppose that this is not possible with the graph with vertex set $V(X)$ and edge set $E(\bigcup_{j=1}^{m+1} P_i)$.

Case 1. The addition of the edges of P_{m+1} to S produces a nonplanar embedding. We assume that this is still the case if we consider other planar embeddings that are faithful to $P_1, ..., P_m$. Then there exists an edge $e = (x, y)$ of P_{m+1} that crosses some P_k, where $1 \leqslant k \leqslant m$. Clearly not both x and y can belong to P_k since P_k is a geodesic. If one of x or y belongs to P_k, then the assumption that *all* planar embeddings faithful to $P_1, ..., P_m$ become nonplanar by the addition of P_{m+1} implies there exists a geodesic in $\{P_1, ..., P_m\}$ also crossed by e not containing x or y. Hence we assume that $\{x, y\} \cap P_k = \emptyset$. Then since e crosses P_k and S is faithful to P_k, then x and y belong to different classes of the partition $\{V_k, V'_k\}$. However, this is impossible since e is an edge of $X - P_k$.

Case 2. The addition of the edges of P_{m+1} to S produces an embedding which is not faithful to $P_1, ..., P_{m+1}$. Then it is immediate that some edge of $\bigcup_{j=1}^{m} P_i$ joins vertices in different classes of the partition $\{V_{m+1}, V'_{m+1}\}$, a contradiction.

Hence we conclude that it is possible to find an embedding of the graph with vertex set $V(X)$ and edge set $E(\bigcup_{j=1}^{m+1} P_i)$ in the plane, that is faithful to $P_1, ..., P_{m+1}$. Proceeding inductively, we have a planar embedding of the graph with vertex set $V(X)$ and edge set $E(\bigcup_{j=1}^{n} P_i)$. However, this graph contains K, a contradiction, and hence we deduce that X is planar, as required. ∎

If X is edge-transitive then X is its own geodetic subgraph and hence Conjecture 1.2 is true for edge-transitive graphs.

COROLLARY 6.2. *A* 1-*ended transitive and edge-transitive graph X is planar if and only if every geodesic bisects X.*

We now consider graphs with quadratic growth. Using the following result, due to Bass, one can gain considerable insight into the structure of the automorphism groups of such graphs.

If G is a finitely generated abelian group, then by $\rho(G)$ we denote the torsion-free rank of G.

LEMMA 6.3 (Bass [1, Theorem 2]). *Let N be a finitely generated nilpotent group with lower central series $N = N_1 \geqslant N_2 \geqslant \cdots \geqslant N_n = \{e\}$. Then N has growth degree $\sum_{i \geqslant 1} i\rho(N_i/N_{i+1})$.*

COROLLARY 6.4. *If G is a finitely generated group with quadratic growth, then G contains a free abelian group of rank* 2 *which has finite index in G.*

Proof. By Theorem 4.1, G contains a nilpotent subgroup N of finite index with quadratic growth. Let $N = N_1 \geqslant N_2 \geqslant \cdots \geqslant N_n = \{e\}$ be the lower central series. By Lemma 6.3, $2 = \sum_{i \geqslant 1} i\rho(N_i/N_{i+1})$. This implies that $\rho(N_1/N_2) = 2$ and that $\rho(N_i/N_{i+1}) = 0$ for all $2 \leqslant i \leqslant n-1$. Hence $n = 2$ and N is a free abelian group of rank 2, as required. ∎

LEMMA 6.5. *Let X be a connected transitive graph with quadratic growth such that every geodesic bisects X. Then X contains a connected subgraph* *which is left invariant by a free abelian subgroup of* AUT(X) *of rank two, and in which every edge is geodetic in* X.

Proof. We first assume that AUT(X) is a finitely generated group. Then it has quadratic growth by Theorem 4.2 and the results in [14, Section 3]. By Corollary 6.4, AUT(X) contains a free abelian subgroup H of rank 2 which has finite index in AUT(X). Let $H = \langle a, b \rangle$.

By Theorem 2.3 in [13], all automorphisms in AUT(X) of infinite order are of type 2. Hence a and b are type 2 automorphisms. We now consider the graph X_σ, where σ is the block system of H on X which is induced by the orbits of b on X. Applying [10, Theorem 2.4], we can assume that a_σ leaves invariant a geodesic P_σ. As in the proof of Lemma 5.3 we can then show that there is a geodesic P in X which is left invariant by ab^m for some $m \in \mathbf{Z}$. But then, for example, the group generated by $\{ab, a^{-1}b^m\}$ is also isomorphic to $\mathbf{Z} \times \mathbf{Z}$ and has finite index in AUT(X). Hence we can without loss of generality assume that a itself leaves invariant a geodesic P.

We claim that there exists a positive number k such that $\{b^{kj}(P)\}_{j \in \mathbf{Z}}$ is a two way infinite sequence of pairwise disjoint geodesics in X. To prove this claim, we first note that a acts with finitely many orbits $O_1, ..., O_m$ on P. Let x_i be a vertex in O_i for $1 \leqslant i \leqslant m$. Suppose infinitely many vertices in $\{b^j(x_i)\}_{j \in \mathbf{Z}}$ belong to P. This implies that there exist integers l and $s \neq t$ such that $b^s(x_i), b^t(x_i) \in O_l$, or equivalently $b^s(x_i) = a^r b^t(x_i)$ for some integer r. Then commutativity implies that $b^{s-t}(x) = a^r(x)$ holds for all $x \in V(X)$ thus contradicting the fact that $\{a, b\}$ generates H. We conclude that for each i such that $1 \leqslant i \leqslant m$, there exists a largest integer m_i such that $b^j(x_i) \cap P \neq \varnothing$ for all $j \leqslant m_i$. Evidently $k = \max\{m_i \mid 1 \leqslant i \leqslant m\} + 1$ is the required positive integer.

Now consider the graph X_ε, where ε is the block system of H on X which is induced by the orbits of a. Evidently the infinite cyclic group $\langle b_\varepsilon \rangle$ acts with finitely many orbits on X_ε. Hence by Lemma 5.3, some b_ε^n leaves invariant a geodesic Q_ε in X_ε. Also Q_ε must meet all of the finite cycles ..., $P_\varepsilon^{-1}, P_\varepsilon^0, P_\varepsilon^1, \ldots$ which are the images of $\ldots, b^{-k}(P), b^0(P), b^k(P), \ldots$ for otherwise one of these geodesics would not bisect X. Also Q_ε immediately leads to a geodesic Q in X which is left invariant by $a^m b^{nk}$. Then

$$\bigcup_{j \in \mathbf{Z}} (a^j(Q) \cup (a^m b^{nk})^j (P))$$

is the required subgraph.

If AUT(X) is not a finitely generated group, then the blocks of the imprimitivity system τ (see Theorem 4.2) are nontrivial. Hence the above and considerations similar to those in the proof of Lemma 5.2 immediately imply that there are non-bisecting geodesics in X. ∎

LEMMA 6.6. *Let X be a connected transitive graph with quadratic growth such that every geodesic bisects X. Then X contains a connected transitive spanning subgraph in which every edge is geodetic in X.*

Proof. By Lemma 6.5, X contains a connected subgraph Y which is left invariant by a free abelian group of rank two, and in which every edge is geodetic. It may be the case that Y does not span X.

Let a and b be two type 2 automorphisms of X, such that $\langle a, b \rangle$ is a free abelian group of rank two which leaves Y invariant. Then there exist type 2 geodesics P and Q containing x such that $a^m(P) = P$ and $b^n(Q) = Q$ for some integers m and n.

Consider the spanning subgraph $Y' = \bigcup_{g \in \mathrm{AUT}(X)} g(Y)$. Evidently Y' is transitive and spans X. Furthermore, every edge of Y' is geodetic in X. By way of contradiction, assume that Y' is disconnected. Then there exist two disjoint copies of Y, say $g_1(Y)$ and $g_2(Y)$. We may assume without loss of generality that g_1 is the identity and that there exists an edge e with one end vertex x in Y and the other end vertex y in $g_2(Y)$.

Clearly there exists a positive integer k and a negative integer l such that $b^{nk}(e)$ and $b^{nl}(e)$ both join a vertex in Y to a vertex in $g_2(Y)$; otherwise X would have growth degree greater than 2. One may choose k and l so that $b^{nk}(x)$ and $b^{nl}(x)$ are vertices in different components of $Y - P$. However, since there exists a path in $g_2(Y)$ joining $b^{nk}(y)$ and $b^{nl}(y)$, $X - P$ has just one infinite component, a contradiction. ∎

Suppose X is a transitive graph with polynomial growth of degree greater than 2. By generalising the arguments in the preceeding lemmas, one can construct a nonbisecting geodesic in X. Hence, we have the following lemma.

LEMMA 6.7. *Let X be a* 1*-ended transitive graph with polynomial growth such that every geodesic bisects X. Then X has quadratic growth.*

7. NORMAL PLANE TILINGS

Suppose X is a transitive graph with quadratic growth such that every geodesic bisects X. In this section we apply a result of Zieschang [21] to show that the geodetic subgraph of X is isomorphic to one of 11 graphs. These 11 graphs correspond to the 3-connected transitive normal plane tilings given by Grünbaum and Shephard [4].

THEOREM 7.1. *Let X be a transitive* 1*-ended graph with polynomial growth such that every geodesic bisects X. Then* $\mathrm{AUT}(X)$ *is isomorphic to a Euclidean motion group.*

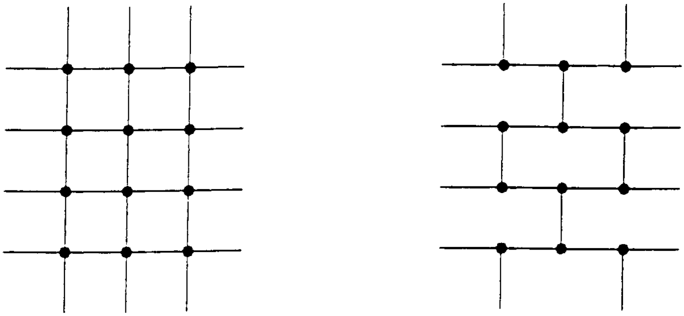

FIGURE 2

Proof. By Lemma 6.7, X has quadratic growth. By Lemma 6.5 there exists a free abelian group $S = \langle a, b \rangle$ of rank 2 which has finite index in AUT(X). Suppose g is a nontrivial automorphism of X of finite order which commutes with all members of S.

Now, consider the subgroup H of AUT(X) generated by a, b and g. Since S acts with finitely many orbits on X then so does H. Furthermore, g commutes with a and b, and therefore the orbits of g give rise to a block system ε of H on X. Clearly S_ε acts with finitely many orbits on X_ε, although X_ε need not be transitive. If we now assume that all geodesics are separating, we can again apply the methods of the proof of Lemma 6.5 to show that X_ε contains a subgraph Y_ε homeomorphic to one of the two graphs in Fig. 2, each of which is spanned by geodesics and left invariant by some subgroup of finite index in S. Then by Lemma 5.1 it follows immediately that Y_ε can be lifted to $|g|$ disjoint copies of Y_ε in X. But, as in Lemma 6.6 this implies that X contains a nonbisecting geodesic. Hence we conclude that the identity is the only automorphism of finite order that commutes with all members of S.

Then a result of Zieschang (see [21, Theorem 24.1]) implies that AUT(X) is isomorphic to a Euclidean motion group, as required. ∎

COROLLARY 7.2. *Let X be a* 1*-ended transitive graph with polynomial growth such that every geodesic bisects X. Then the geodetic subgraph of X is isomorphic to one of the* 11 3*-connected transitive normal plane tilings.*

8. A CHARACTERIZATION OF PLANAR GRAPHS WITH POLYNOMIAL GROWTH

Throughout this section X is a 1-ended transitive graph of polynomial growth in which every geodesic bisects. By Corollary 7.2 the geodetic subgraph of X is one of the eleven normal plane tilings. This geodetic subgraph will be denoted by Y. For the proof of our main theorem we need several preliminary results:

LEMMA 8.1. *If $a \in \mathrm{AUT}(Y)$, then $a^k \in \mathrm{AUT}(X)$ for some $k \geqslant 1$.*

Proof. If a has finite order, then our assertion is trivial.

Let a be of infinite order. Then it is an automorphism of type 2. It is easy to see that there exists a double ray (not necessarily a geodesic) left invariant by a power of a. Let P be such a double ray with $a^n(P) = P$ for some integer n. If $n \neq 1$ we replace a^n by a. Furthermore we choose an $x_0 \in V(P)$ and set $x_i = a^i(x_0)$, $i \in \mathbf{Z}$.

Since X is transitive, there is a $g_i \in \mathrm{AUT}(X)$ for each x_i such that $x_i = g_i(x_0)$. From the construction of Y it is obvious that $\mathrm{AUT}(X) \subseteq \mathrm{AUT}(Y)$ and therefore also $g_i^{-1} a^i \in \mathrm{AUT}(Y)$ for every $i \in \mathbf{Z}$. Each of these automorphisms fixes x_0 and since the stabilizer of x_0 in $\mathrm{AUT}(Y)$ is finite, there are integers $j \neq l$ such that $g_j^{-1} a^j = g_l^{-1} a^l$. This implies that $a^k = g_l^{-1} g_j \in \mathrm{AUT}(X)$ for $k = l - j$. ▮

Let φ be a homomorphism of a graph X_1 onto a graph X_2 and let $S(v)$, $v \in V(X_1)$, denote the star consisting of v and all edges incident with v. If $\varphi(S(v))$ is isomorphic to $S(v)$ for every $v \in V(X_1)$ we call φ a *covering map*.

LEMMA 8.2. *Let $\{\mathscr{L}_i : i \in \mathbf{Z}\}$ be a set of lines in the plane and let P denote a double ray in Y with the following properties*:

(a) *There is a translation b of the plane that leaves Y and every line $\mathscr{L}_i$ invariant and therefore is a type 2 automorphism of Y.*

(b) *If a line $\mathscr{L}_i$ meets an edge e of Y it either meets an endpoint of e or contains e.*

(c) *There is a translation a of the plane which leaves Y and P invariant and maps every $\mathscr{L}_i$ onto $\mathscr{L}_{i+1}$.*

(d) *P meets every $\mathscr{L}_i$ in exactly one vertex, say x_i.*

(e) *The distance between every pair x_i, x_{i+1} on P is k, where k is the length of a shortest path between $\mathscr{L}_i$ and $\mathscr{L}_{i+1}$ in Y.*

Then P is a type 2 geodesic in X.

Proof. Figure 3 shows the eleven normal plane tilings. We drew them such that they facilitate following the arguments of this and the next proof. We first note that P is a geodesic in Y, since every double ray in Y which satisfies (d) and (e) is a geodesic in Y.

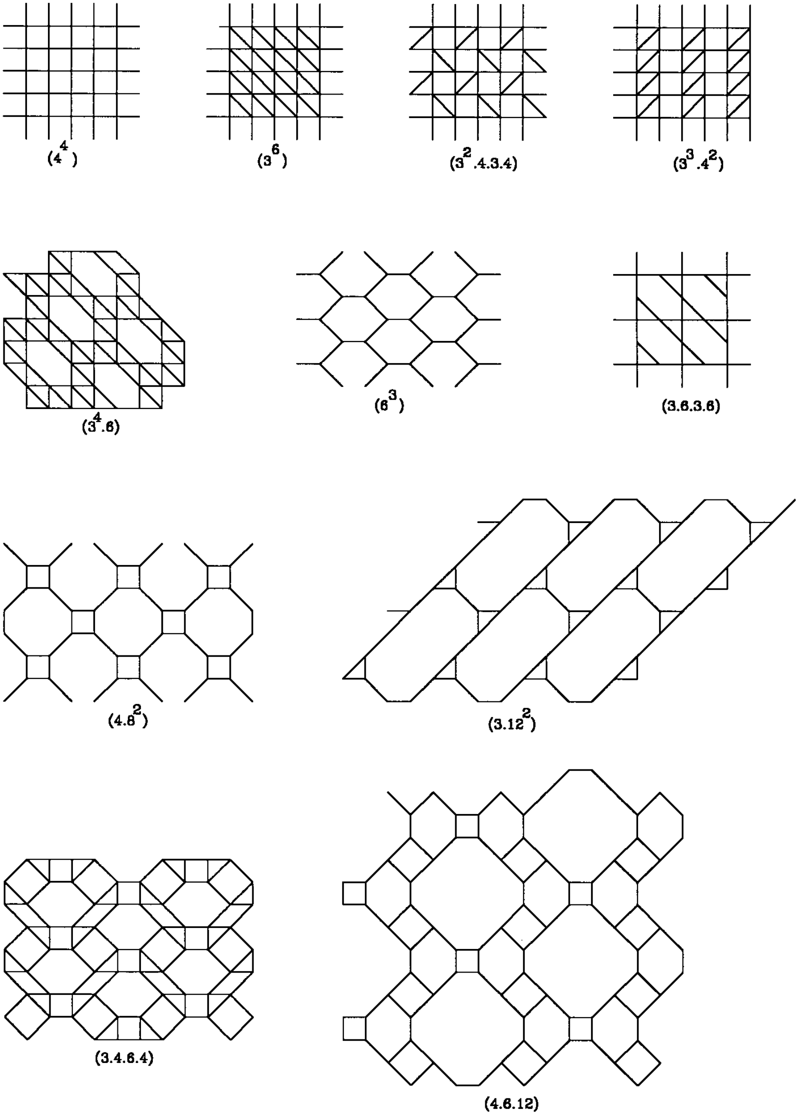

FIGURE 3

Suppose P is not a geodesic in X. Then $d_X(x_0, x_i) < k \cdot i$ for some $i \geqslant 1$. Then, by (c), we have

$$d_X(x_0, x_{i \cdot j}) \leqslant k \cdot i \cdot j - j = d_Y(x_0, x_{i \cdot j}) - j$$

for every $j \geqslant 1$.

Thus, for every $m \geqslant 1$ there is an index s such that $d_X(x_0, x_s) < k \cdot s - m$. From the structure of $\mathrm{AUT}(Y)$ it follows immediately that the group generated by a and b is isomorphic to $\mathbf{Z}^2$. Then, by 8.1, a group isomorphic to $\mathbf{Z}^2$ also acts on X. Without loss of generality we can assume that this group is again generated by a and b. We also assume that no orbit of b on X contains adjacent vertices of X and that no vertex of X is adjacent to more than one vertex of an orbit of b on X. (If the orbits of b do not have this property, we consider some b^p, $p \geqslant 1$. Since X is locally finite we can always find some p such that the orbits of b^p on X have the required properties.) By ϵ we denote the block system of $\langle a, b \rangle$ on X which is induced by the orbits of b. Again X_ϵ denotes the quotient of X with respect to the blocks of ϵ and because of the assumed properties of the orbits of b, the quotient map $\varphi\colon X \to X_\epsilon$, induced by the construction of X_ϵ is a covering map. Clearly X_ϵ is a graph with linear growth and a_ϵ acts almost transitively on it. Thus, by Lemma 5.3, X_ϵ contains a geodesic R_ϵ.

All vertices contained in a line $\mathscr{L}_i$ give rise to end separating sets $\mathscr{L}_i^\epsilon$ in X_ϵ. By m we denote twice the diameter of each $\mathscr{L}_i^\epsilon$ in X_ϵ. This number m clearly is the same for all $\mathscr{L}_i^\epsilon$, $i \in \mathbf{Z}$. Furthermore, let s be chosen as above and let $y_0^\epsilon = R_\epsilon \cap \mathscr{L}_0^\epsilon$, $y_s^\epsilon = R_\epsilon \cap \mathscr{L}_s^\epsilon$. Then

$$\begin{aligned} k \cdot s &= \frac{m}{2} + (k \cdot s - m) + \frac{m}{2} > d_{X_\epsilon}(y_0^\epsilon, x_0^\epsilon) + d_{X_\epsilon}(x_0^\epsilon, x_s^\epsilon) + d_{X_\epsilon}(x_s^\epsilon, y_s^\epsilon) \\ &\geqslant d_{R_\epsilon}(y_0^\epsilon, y_s^\epsilon). \end{aligned}$$

Since φ is a covering map, every double ray $R \in \varphi^{-1}(R_\epsilon)$ is a geodesic in X and thus also in Y. Of course the distances in R are the same as in R_ϵ. But then R connects $\mathscr{L}_0$ and $\mathscr{L}_s$ with fewer than $k \cdot s$ edges, in contradiction to (e). ∎

LEMMA 8.3. *Let* $\{\mathscr{L}_i\colon i \in \mathbf{Z}\}$ *be a set of lines in the plane and let* P *denote a double ray in* X *such that the* $\mathscr{L}_i$ *and* P *satisfy properties* (a)–(d) *of Lemma* 8.2 *and in addition*:

(e′) *The distance between every pair* x_i, x_{i+1} *on* P *is* $l \leqslant k$ *where* k *is the length of a shortest path between* $\mathscr{L}_i$ *and* $\mathscr{L}_{i+1}$ *in* Y.

Then $l = k$ *and* P *is in* Y.

Proof. If the assertions of the lemma do not hold, we procede analogously to the proof of Lemma 8.2 until we obtain a geodesic in X—and therefore in Y—that connects $\mathscr{L}_0$ and $\mathscr{L}_s$ with fewer than $l \cdot s \leqslant k \cdot s$ edges, in contradiction to (e′). ▮

THEOREM 8.4. *Let X be a* 1-*ended transitive graph with polynomial growth. Then X is planar if and only if every geodesic bisects X.*

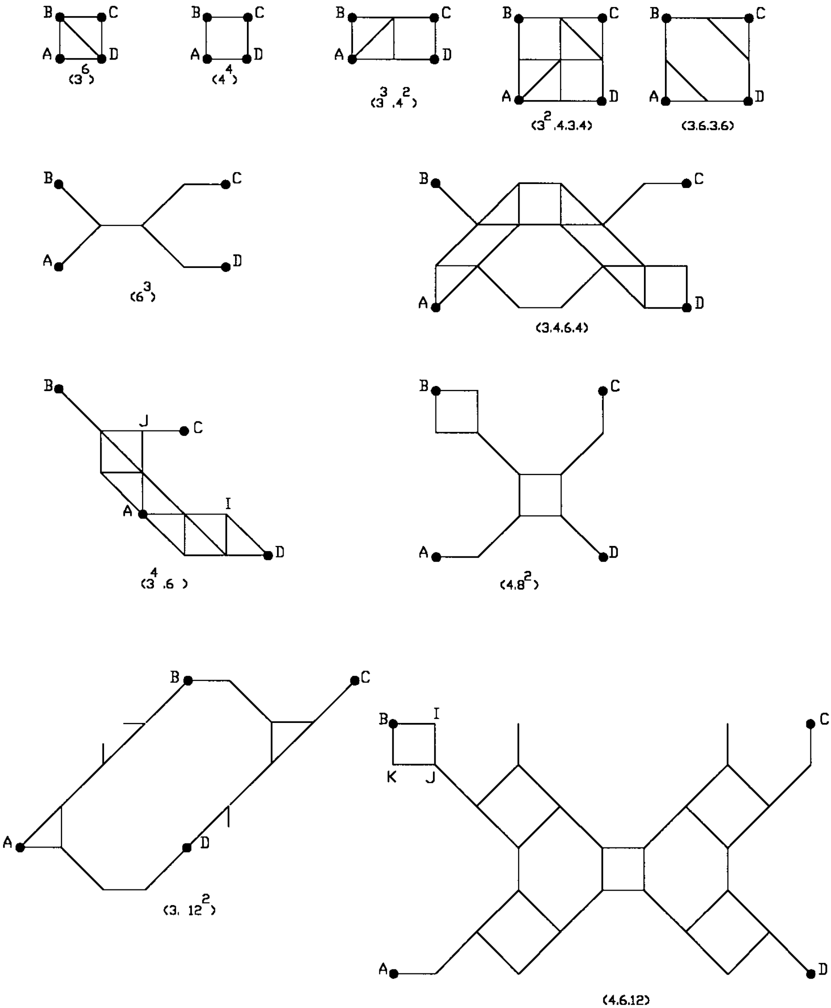

FIGURE 4

Proof. If X is planar, then by Theorem 5.3 in [2] every geodesic bisects X. Therefore it remains to prove the converse.

Let X be a 1-ended transitive graph with polynomial growth such that every geodesic bisects X. By Corollary 7.2, the geodetic subgraph Y of X is isomorphic to one of the 3-connected transitive normal plane tilings (see Figure 3). The vertices are the points with integer coordinates and the edges are lines which connect points at Euclidean distance either 1 or $\sqrt{2}$ from each other.

Furthermore, for each of the eleven graphs Y in consideration $\mathrm{AUT}(Y)$ contains subgroups isomorphic to $\mathbf{Z}^2$. If the graphs are embedded as in Figure 3 it is easy to find such subgroups and fundamental domains with respect to these subgroups. Figure 4 shows such fundamental domains for each of our eleven graphs. The dark vertices are called corner vertices of the fundamental domains.

We now prove that every edge of X is geodetic, which implies that X itself is isomorphic to one of the graphs of Figure 3.

We assume that $E(X)-E(Y)$ contains an edge e, otherwise there is nothing to show. Furthermore $S=\langle g, h\rangle$ always denotes the subgroup isomorphic to $\mathbf{Z}^2$ which has the fundamental domains of Figure 4. In addition we assume that $h(A)=D$, $h(B)=C$ and $g(A)=B$, $g(D)=C$ hold and call a shortest path $Q(L, M)$, $L, M \in \{A, B, C, D\}$, $L \neq M$, *extendable* to a geodesic P in X if $P=\bigcup_{i \in \mathbf{Z}} a^i(Q)$ is a type 2 geodesic in X, where $a(L)=M$, $a \in S$.

Since X is transitive, we can assume that $e \in E(X)-E(Y)$ is incident with a corner vertex of a fundamental domain.

Case 1. $e \in E(X)-E(Y)$ connects two vertices of one fundamental domain.

Case 1.1. All tessellations except $(3^4 \cdot 6)$ and $(4 \cdot 6 \cdot 12)$.

Case 1.1.1. Let $\mathscr{L}_0$ be the line through A and B left invariant by the translation g and let $\mathscr{L}_1$ be the line through C and D. Note that the distance k between $\mathscr{L}_0$ and $\mathscr{L}_1$ is given as follows:

$$
\begin{array}{lll}
k=1 & \text{for} & (3^6) \text{ and } (4^4) \\
k=2 & \text{for} & (3^3 \cdot 4^2), (3^2 \cdot 4 \cdot 3 \cdot 4), (3 \cdot 6 \cdot 3 \cdot 6) \\
k=4 & \text{for} & (6^3), (4 \cdot 8^2), (3 \cdot 12^2) \\
k=6 & \text{for} & (3 \cdot 4 \cdot 6 \cdot 4).
\end{array}
$$

Suppose the edge e is on a path Q from $\{A, B\}$ to $\{C, D\}$ of length $\leqslant k$. Then an application of Lemma 8.3 shows that Q is extendable, where $a \in \{h, g^{\pm 1} h\}$. But then e is geodetic, a contradiction.

This situation occurs for all edges e in (3^6), (4^4), $(3^3 \cdot 4^2)$ and for all e, except $e=(A, B)$ or $e=(C, D)$, in the tessellations $(3^2 \cdot 4 \cdot 3 \cdot 4)$, $(3 \cdot 6 \cdot 3 \cdot 6)$ and (6^3).

Case 1.1.2. If such an e does not exist, let $\mathscr{L}_0$ be the line through A and D left invariant by the translation h and let $\mathscr{L}_1$ be the line through B and C. Then k has the values

$$k=2 \qquad \text{for} \quad (6^3) \quad \text{and}$$
$$k=3 \qquad \text{for} \quad (3 \cdot 4 \cdot 6 \cdot 4),$$

the other values of k remaining unchanged. We leave it to the reader to verify that e again is on an extendable path Q from $\{A, D\}$ to $\{B, C\}$ of length $\leqslant k$ in all those cases.

Case 1.2. For the tessellation (4.6.12) we observe that the above procedure works, unless $e=(B, J)$. Let the lines $\mathscr{L}_0$, $\mathscr{L}_1$ be defined as in Case 1.1.1. Note that $k=10$ in this case. Since $\mathrm{AUT}(X)$ acts transitively on X and leaves Y invariant, we thus also have the edge (K, I) in X, since the square (B, I, J, K) is only square in Y containing B. Similarly we conclude that the edges $h(B, J)$, $h(K, I)$ are also in X. Then there is a path Q of length 20 between B and $h^2(B)$ which contains e. But Q is an extendable path by Lemma 8.3, a contradiction.

Case 1.3. For the tessellation $(3^4 \cdot 6)$ we first note that every vertex in the fundamental domain either is one of the corner vertices or has distance 1 from a corner vertex. Thus every edge e either connects two corner vertices or is on a path Q of length 2 between two corner vertices.

If Q connects A to B or C, respectively, we let $\mathscr{L}_0$ be the line through A, I and let $\mathscr{L}_1$ be the parallel through B or C, respectively.

If Q connects A to D we let $\mathscr{L}_0$ be the line through A, J and $\mathscr{L}_1$ be the parallel through D.

If Q is a path from B to a vertex in $\{C, D\}$ or from C to D, then we similarly choose $\mathscr{L}_0$ and $\mathscr{L}_1$ as the parallels through the origin and endpoint of Q.

Case 2. The edge e connects vertices of distinct fundamental domains.

Case 2.1. e connects corner vertices of distinct fundamental domains.

Let $e=(x, y)$. Then we can always find numbers p and q such that x and y are corner vertices of fundamental domains with respect to $S=\langle g^p, h^q \rangle$. But in this case it is immediately clear that e is in an extendable path, a contradiction.

Case 2.2. e connects a corner vertex to an internal vertex of a distinct domain.

Let $e=(x, y)$ where x is a corner vertex. Then it is as simple as above to show that e is in a path Q in X from x to a corner vertex w of the domain which contains y, which is shorter than any path between x and w in Y. As in Case 2.1 we can find $S=\langle g^p, h^q\rangle$ and fundamental domains of S, such that x and w are corner vertices of one fundamental domain of S. Thus we again have that Q is an extendable path, a contradiction. Since a detailed proof of this fact is quite similar to the above discussion, we leave it to the reader to deal with the details. ▮

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References

1. H. Bass, The degree of polynomial growth of finitely generated nilpotent groups, *Proc. London Math. Soc. (3)* **25** (1972), 603-614.
2. C. P. Bonnington, W. Imrich, and M. Watkins, Separating double rays in locally finite planar graphs, *Discrete Math.* **145** (1995), 61–72.
3. M. Gromov, Groups of polynomial growth and expanding maps, *Inst. Hautes Études Sci. Publ. Math.* **53** (1981), 53–78.
4. B. Grűnbaum and G. C. Shephard, The geometry of planar graphs, *in* “Combinatorics, Proceedings of the 8th British Combinatorial Conference (Cambridge),” pp. 124–150, Cambridge Univ. Press, Cambridge, 1981.
5. R. Halin, Über unendliche Wege in Graphen, *Math. Ann.* **157** (1964), 125–137.
6. R. Halin, Zur häufungspunktfreien Darstellung abzählbarer Graphen in der Ebene, *Arch. Math. (Basel)* **17** (1966), 239–243.
7. R. Halin, Automorphisms and endomorphisms of infinite locally finite graphs, *Abh. Math. Sem. Univ. Hamburg* **39** (1973), 251–283.
8. W. Imrich, On Whitney's theorem on the unique imbeddability of 3-connected planar graphs, *in* “Recent Advances in Graph Theory” (M. Fiedler, Ed.), pp. 303–306, Academia Praha, Prague, 1975.
9. W. Imrich and N. Seifter, A survey on graphs with polynomial growth, *Discrete Math.* **95** (1991), 101–117.
10. N. Polat and M. Watkins, On translations of double rays in graphs, submitted for publication.
11. G. Sabidussi, Vertex transitive graphs, *Monatsh. Math.* **68** (1964), 385–401.
12. D. Segal, “Polycyclic Groups,” Cambridge Tracts in Mathematics, Vol. 82, Cambridge Univ. Press, Cambridge, 1983.
13. N. Seifter, Groups acting on graphs with polynomial growth, *Discrete Math.* **89** (1991), 269–280.
14. N. Seifter, Properties of graphs with polynomial growth, *J. Combin. Theory Ser. B* **52** (1991), 222–235.
15. C. Thomassen, private communication.

16. C. Thomassen, Planarity and duality of finite and infinite graphs, *J. Combin. Theory Ser. B* **29** (1980), 244–271.
17. C. Thomassen, Duality of infinite graphs, *J. Combin. Theory Ser. B* **33** (1982), 137–166.
18. V. I. Trofimov, Graphs with polynomial growth, *Math. USSR-Sb.* **51** (1985), 405–417.
19. M. E. Watkins, Infinite paths that contain only shortest paths, *J. Combin. Theory Ser. B* **41** (1986), 341–355.
20. A. Whitney, Non-separable and planar graphs, *Trans. Amer. Math. Soc.* **34** (1932), 339–362.
21. H. Zieschang, "Finite Groups of Mapping Classes of Surfaces," Lecture Notes in Mathematics, Vol. 875, Springer-Verlag, Berlin/Heidelberg, 1981.