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Note

On the Falsity of a Conjecture on Orthogonal Steiner Triple Systems

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A pair of orthogonal Steiner triple systems of order v = 27 is constructed, thus showing the conjecture about the non-existence of a pair of orthogonal Steiner triple systems of orders $v \equiv 3 \pmod 6$ to be false.

Two Steiner triple systems (briefly STS) S_1 and S_2 on the same set S are said to be orthogonal if (1) they are disjoint, i.e., have no triple in common, and (2) if two pairs of elements appear with the same third element in triples of S_1 then they appear with distinct third elements in triples of S_2.

Pairs of orthogonal STS are known to exist for infinitely many orders $v \equiv 1 \pmod 6$ [1, 2, 3]. On the other hand, Mullin and Nemeth in [4] attribute the following conjecture to O'Shaughnessy [5]:

CONJECTURE. *There does not exist a pair of orthogonal* STS *of order v for any* $v \equiv 3$ (mod 6).

In the same paper [4] it is proved that, in fact, there is no pair of orthogonal STS of order 9.

The aim of this note is to present a counterexample to the above conjecture by exhibiting a pair of orthogonal STS of order 27.

Let A be the additive group of residues mod 13 and consider the Cartesian product $S' = A \times \{1, 2\}$. For $i \in \{1, 2\}$, we denote the subset $A \times \{i\}$ of S' simply by A_i and any element (a, i) of A_i simply by a_i. Let $S = A_1 \cup A_2 \cup \{\infty\}$, where ∞ is a new symbol. We determine two STS S_1 and S_2 on the set S such that both systems will have

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$\alpha = (\infty)(0_1\ 1, \cdots 12_1)(0_2\ 1_2 \cdots 12,)$ as their automorphism. The systems S_1 and S_2 are then determined by the following base triples:

S_1: $(\infty, 0_1, 0_2)$, $(0_1, 3_1, 4_1)$, $(0_1, 5_1, 7_1)$, $(0_1, 8_2, 9_2)$, $(0_1, 1_2, 3_2)$, $(0_1, 2_2, 5_2)$, $(0_1, 6_2, 10_2)$, $(0_1, 7_2, 12_2)$, $(0_1, 4_2, 11_2)$,

S_2: $(\infty, 0_1, 9_2)$, $(0_1, 4_1, 6_1)$, $(0_2, 2_2, 6_2)$, $(0_1, 1_1, 3_2)$, $(0_1, 3_1, 7_2)$, $(0_1, 5_1, 6_2)$, $(0_2, 1_2, 1_1)$, $(0_2, 3, , 5_1)$, $(0_2, 5_2, 8_1)$.

(The remaining triples of S_1 and S_2, respectively, are obtained by applying α^i, $i = 1, 2, \ldots, 12$, to the base triples.)

Both S_1 and S_2 are easily verified to be STS of order 27. Observe that S_1 contains a subsystem of order 13 on the set A, while the second system S_2 has no non-trivial subsystems.

It remains to be shown that S_1 and S_2 are orthogonal. A convenient way to verify this fact is to use Table I.

In column 1, all pairs of elements of S are listed having a fixed element of A, , say 0, , as the third element of a triple in S_1; in column 2, for the same pairs the third elements of triples in S_2 are listed. If S_1 and S_2 are to

TABLE I

1	2	3	4	5	6
S_1	S_2	S_1	S_2	S_1	S_2
0_1		0_2		∞	
$1_1, 10_1$	3_1	$4, \infty$	9_2	$0_1, 0_2$	12_2
$2_1, 8_1$	6_1	$1_1, 8_2$	4_1		
$3_1, 4_1$	6_2	$2_1, 6_2$	12_1		
$5_1, 7_1$	1_1	$3_1, 9_2$	8_1		
$6_1, 11_1$	12_2	$4_1, 12_2$	2_2		
$9_1, 12_1$	3_2	$5_1, 1_2$	w		
$0_2, \infty$	4_1	$6_1, 5_2$	6_2		
$1_2, 3_2$	7_2	$7_1, 4_2$	12_2		
$2_2, 5_2$	7_1	$8_1, 10_2$	7_1		
$4_2, 11_2$	0_2	$9_1, 7_2$	4_2		
$6_2, 10_2$	4_2	$10_1, 11_2$	5_1		
$7_2, 12_2$	2_1	$11_1, 3_2$	8_2		
$8_2, 9_2$	9_1	$12_1, 2_2$	0_1		

be orthogonal, all the elements in column 2 must be distinct and, of course, different from 0, . In columns 3, 4 and 5, 6, respectively, the same is done for a fixed element of A, and for the element co, respectively. In view of the fact that α is an automorphism of both S_1 and S_2, the table presents a full verification of the orthogonality of S_1 and S_2.

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