The graded identities of upper triangular matrices of size two

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Received 20 January 2001; received in revised form 29 September 2001
Communicated by M. Barr

Abstract

Let $UT_2$ be the algebra of $2 \times 2$ upper triangular matrices over a field $F$. We first classify all possible gradings on $UT_2$ by a group $G$. It turns out that, up to isomorphism, there is only one non-trivial grading and we study all the graded polynomial identities for such algebra. In case $F$ is of characteristic zero we give a complete description of the space of multilinear graded identities in the language of Young diagrams through the representation theory of the hyperoctahedral group. We finally establish a result concerning the rate of growth of the identities for such algebra by proving that its sequence of graded codimensions has almost polynomial growth. © 2001 Elsevier Science B.V. All rights reserved.

MSC: Primary 16R10, 16W50; secondary 16P90

1. Introduction

The algebra $UT_2$ of $2 \times 2$ upper triangular matrices over a field plays an important role in the combinatorial theory of PI-algebras (or algebras with polynomial identity). In general, if $A$ is a PI-algebra, one can attach to $A$ a numerical sequence $c_n(A)$, $n = 1, 2, \ldots$, called the sequence of codimensions of $A$. In [8] it was proved that such sequence is exponentially bounded and its asymptotic behavior has become an interesting invariant of the algebra. Let $Id(A)$ be the $T$-ideal of the free algebra of all polynomial identities of $A$. As a consequence of a theorem of Kemer [5] it follows that if $A$ is an algebra over a field of characteristic zero, then $c_n(A)$ is polynomially bounded if and only if $Id(A) \nsubseteq Id(UT_2)$ and $Id(A) \nsubseteq Id(G)$, where $G$ is

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the infinite dimensional Grassmann algebra. It follows that $UT_2$ has almost polynomial growth of the codimensions, i.e., $c_n(UT_2)$ grows exponentially but for every $T$-ideal $I \supset Id(UT_2)$, $c_n(I)$ is polynomially bounded.

In this paper, we shall prove that $UT_2$ has a similar property regarded as a graded algebra. From the theory of Kemer concerning the structure of the varieties of associative algebras it turns out that a prominent role is played by the superalgebras (or $\mathbb{Z}_2$-graded algebras) and their identities. Recall that an algebra $A$ is graded by a group $G$ if $A = \bigoplus_{g \in G} A_g$ where for $g \in G$, $A_g$ is a subspace of $A$ and $A_g A_h \subseteq A_{gh}$ for all $g, h \in G$. In this paper, we shall first classify all possible gradings on the algebra $UT_2$ by a group $G$. We shall prove that, up to isomorphism, only two gradings are allowed on $UT_2$: the trivial grading $UT_2 = (UT_2)_1$; and the canonical $\mathbb{Z}_2$-grading given by $UT_2 = (UT_2)_0 \oplus (UT_2)_1$, where $(UT_2)_0 = Fe_{11} + Fe_{22}$ and $(UT_2)_1 = Fe_{12}$ with the $e_{ij}$'s the usual matrix units.

We shall then extensively study the ideal of graded identities of $UT_2$ in characteristic zero. Recall that a polynomial in two distinct sets of non-commutative variables $f(y_1, \ldots, y_m, z_1, \ldots, z_n)$ is a graded identity for the superalgebra $A = A_0 \oplus A_1$ if $f(a_1, \ldots, a_m, b_1, \ldots, b_m) = 0$ for all $a_1, \ldots, a_m \in A_0$, $b_1, \ldots, b_m \in A_1$. By exploiting a well known duality between $\mathbb{Z}_2$-gradings and automorphisms of order two of an algebra $A$, one has naturally the notion of multilinear graded polynomial of degree $n$ in $y_1, z_1, \ldots, y_n, z_n$ (in each monomial either $y_i$ or $z_i$ appears, for all $i = 1, \ldots, n$); we write $P_n^{gr}$ for the space of such polynomials.

In this paper we shall first compute a set of generators for the ideal $Id^{gr}(UT_2)$ of graded identities of $UT_2$, then we shall describe the space of multilinear graded identities of $UT_2$ of any degree through the representation theory of the hyperoctahedral group $H_n$ as follows: the group $H_n$ acts naturally on the space $P_n^{gr} \cap Id^{gr}(UT_2)$ of multilinear graded identities of degree $n$; by complete reducibility, the space $P_n^{gr} \cap Id^{gr}(UT_2)$ splits into the direct sum of irreducibles and we shall compute for each irreducible the corresponding multiplicity.

For a superalgebra $A$, the sequence

$$c_n^{gr}(A) = \dim \frac{P_n^{gr}}{P_n^{gr} \cap Id^{gr}(UT_2)}, \quad n = 1, 2, \ldots$$

is called the sequence of graded codimensions of $A$. As a consequence of our description we shall compute the asymptotic behavior of the sequence $c_n^{gr}(UT_2)$. It turns out that the algebra $UT_2$ with the canonical grading has almost polynomial growth of the graded codimensions; in fact we shall prove that $c_n^{gr}(UT_2)$ has exponential growth equal to two and any ideal of graded identity properly containing $Id^{gr}(UT_2)$ has polynomially bounded growth of the graded codimensions.

2. Gradings on $UT_2$

Let $A$ be an associative algebra over the field $F$ and let $G$ be a group. Recall that a $G$-grading on $A$ is a decomposition of $A$ into the direct sum of subspaces $A = \bigoplus_{g \in G} A_g$
such that \( A_g A_h \subseteq A_{gh} \) for all \( g, h \in G \). We say that an element \( a \in A \) is homogeneous of degree \( g \) if \( a \in A_g \). In case \( A_g = 0 \) for all \( g \neq 1 \) then we say that \( A \) has trivial grading.

We shall next give a complete description of all gradings on \( UT_2 \), the algebra of \( 2 \times 2 \) upper triangular matrices over \( F \). Let \( G \) be an arbitrary group, we say that \( UT_2 \) has the canonical \( G \)-grading if there exists \( g \in G, g \neq 1 \) such that \( UT_2 \cong (UT_2)_1 \oplus (UT_2)_g \), where \( (UT_2)_1 = Fe_{11} + Fe_{22} \) and \( (UT_2)_g = Fe_{12} \). We have the following:

**Theorem 1.** Any \( G \)-grading on \( UT_2 \) is, up to isomorphism, either trivial or canonical.

**Proof.** Write \( A = UT_2 \) and let \( e \in G \) be the unit element of \( G \). If \( \dim A_e = 3 \) then \( A \) has trivial grading. Hence we may assume that \( \dim A_e \leq 2 \).

Suppose first that \( \dim A_e = 2 \). We may clearly assume that \( e_1 + e_2 \) and \( ae_{11} + be_{12} \) form a basis of \( A_e \) over \( F \), for suitable \( a, b \in F \). Since \( \dim F UT_2 = 3 \), there exists \( g \in G \) such that \( \dim A_g = 1 \) and let \( A_g = F(a' e_{11} + b' e_{12} + c' e_{22}) \). In case \( a = 0 \), then the inclusions \( A_g A_e \subseteq A_g \) and \( A_e A_g \subseteq A_g \) lead to \( a' = c' = 0 \). Hence \( A_g = Fe_{12} \subseteq A_e \), a contradiction. Thus \( a \neq 0 \). It follows that the elements \( e_{11} + be_{12} \) and \( e_{22} - be_{12} \) span \( A_e \) over \( F \). Suppose first that \( b \neq 0 \). Since \( (a' e_{11} + b' e_{12} + c' e_{22}) (e_{11} + be_{12}) = d' (e_{11} + be_{12}) \in A_g \cap A_e = 0 \), we obtain that \( a' = 0 \). Similarly, by multiplying \( b' e_{12} + c' e_{22} \) on the left by \( e_{22} - be_{12} \), we obtain \( c' = 0 \). Hence \( A_g = Fe_{12} A_e = F(e_{11} + e_{22}) + F(e_{11} + be_{12}) \) and \( A_g \oplus A_g \) is isomorphic to \( UT_2 \) with the canonical \( G \)-grading. In case \( b = 0 \), \( A_e = Fe_{11} + Fe_{22} \) and it easily follows that \( A_g = Fe_{12} \). Thus we are done in this case too.

Suppose now that \( \dim A_e = 1 \) that is \( A_e = F(e_{11} + be_{12}) \). So either \( UT_2 = A_e \oplus A_g \oplus A_h \) where \( \dim A_g = \dim A_h = 1 \) or \( UT_2 = A_e \oplus A_g \) with \( \dim A_g = 2 \). Let \( UT_2 = A_e \oplus A_g \oplus A_h \) and suppose first that \( gh \neq e \). Then \( A_g A_h = 0 \) and, in case \( g^2 \neq e \) and \( h^2 \neq e \), we get that \( A_g \oplus A_h \) is a two-dimensional nilpotent ideal of \( UT_2 \), contradicting the fact that \( \dim J = 1 \) where \( J \) is the Jacobson radical of \( UT_2 \). Hence \( (g^2 \neq e, h^2 \neq e) \) or \( g^2 = h^2 = e \). In the first case one easily gets that \( A_g = J \) and let \( A_h = F(ae_{11} + be_{12} + ce_{22}) \). From \( A_g A_h = A_h A_g = 0 \) we easily obtain \( a = c = 0 \). Hence \( A_h = A_g \), a contradiction. In case \( g^2 = h^2 = e \), since \( A_g = Fu, A_h = Fv \), where \( u^2 = v^2 = 1 \), we get that \( 0 \neq uv \in A_g A_h \), a contradiction.

Suppose now that \( gh = e \). If \( g \neq e \) then \( g^2 \neq g^{-1} \) and \( g \neq e \) and \( g^{-1} \neq g \). Hence \( (A_g)^2 = (A_h)^2 = 0 \). a contradiction. In case \( g^2 = e \) we obtain that \( A_g = Fa \) where \( a = x e_{11} + ye_{22} \) with \( x \) a third root of the unity. Then we would get that \( \dim (Fa + Fa^2 + Fe) = 2 \), a contradiction. We are left with \( UT_2 = A_e \oplus A_g \) and \( \dim A_g = 2 \). If \( g^2 \neq e \) it follows that \( A_g \) is a nilpotent ideal, hence \( A_g \subseteq J \) and this is a contradiction. Thus \( g^2 = e \). Since \( A_g A_g \subseteq A_e \) we easily obtain a contradiction also in this case.

In case of a finite abelian group \( G \) all possible \( G \)-gradings of \( UT_n(F) \) the algebra of \( n \times n \) upper triangular matrices are described in [9] provided that \( F \) is an algebraically closed field of characteristic zero.

### 3. Graded cocharacters and codimensions

Throughout \( F \) is a field of characteristic zero and \( A = A_0 \oplus A_1 \) a \( \mathbb{Z}_2 \)-graded algebra. Let \( F(X) \) be the free associative algebra over the countable set \( X = \{ x_1, x_2, \ldots \} \). If we
write $X = Y \cup Z$ where $Y = \{y_1, y_2, \ldots\}, Z = \{z_1, z_2, \ldots\}$ and $Y \cap Z = \emptyset$, then $F(X)$ has a natural structure of free superalgebra on $X$ by assuming that the variables $y_i \in Y$ and $z_i \in Z$ are homogeneous of degree zero and one, respectively. More precisely, if $\mathcal{F}_0$ is the subspace of $F(X)$ generated by all monomials in the variables of $X$ having even degree in the variables of $Z$ and $\mathcal{F}_1$ the subspace of $F(X)$ generated by all monomials of odd degree in $Z$, then $F(X) = \mathcal{F}_0 \oplus \mathcal{F}_1$ is the induced grading; moreover for any superalgebra $A = A_0 \oplus A_1$, any map $Y \cup Z \to A$ preserving the grading can be extended in a unique way to a homomorphism of superalgebras $F(X) \to A$.

It is well known, that, for any $F$-algebra $A$, there is a duality between $\mathbb{Z}_2$-gradings of $A$ and $\phi$-actions where $\phi \in \text{Aut}(A)$ is an automorphism of order two: in fact if $A = A_0 \oplus A_1$ is a $\mathbb{Z}_2$-grading on $A$, then $\phi : A \to A$ such that $\phi(a_0 + a_1) = a_0 - a_1$ is the defined automorphism. Viceversa, if $\phi$ is an automorphism of order two then $A = A_0 \oplus A_1$ where $A_0 = \{a \in A | \phi(a) = a\}$ and $A_1 = \{a \in A | \phi(a) = -a\}$. Set $\tilde{x}_i = y_i + z_i$ and $\tilde{x}_i^\phi = y_i - z_i$, $i = 1, 2, \ldots$ and require that $\phi$ acts as an automorphism of order two on $F(X)$. Then $F(X)$ becomes the free algebra on $X$ with $\phi$-action. This means that if $A$ is any algebra with an automorphism $\phi, \phi^2 = 1$, then any map $f : \{\tilde{x}_1, \tilde{x}_2, \ldots\} \to A$ extends uniquely to a homomorphism $\tilde{f} : F(X) \to A$ such that $\tilde{f}(\tilde{x}_i^\phi) = f(x_i^\phi) = f(x_i)^\phi$.

The notion of $\phi$-identity or graded identity for a superalgebra $A = A_0 \oplus A_1$, are the obvious ones: $f(y_1, \ldots, y_n, z_1, \ldots, z_n) = f(x_1^\phi, \ldots, x_n^\phi)$ is a graded identity (or a $\phi$-identity) for $A$ if $f(a_1, \ldots, a_n, b_1, \ldots, b_n) = 0$ for all $a_1, \ldots, a_n \in A_0, b_1, \ldots, b_n \in A_1$ ($f(c_1, c_1^\phi, \ldots, c_n, c_n^\phi) = 0$ for all $c_1, \ldots, c_n \in A$, respectively). Let $\text{Id}^g(A) = \text{Id}^\phi(A)$ be the ideal of $F(X)$ of graded identities (or $\phi$-identities) of $A$. It is clear that $\text{Id}^g(A)$ is a $T_2$-ideal of $F(X)$ i.e., an ideal invariant under all endomorphisms of $F(X)$ preserving the grading. Also, since char $F = 0$, by standard arguments it is well known that $\text{Id}^g(A)$ is completely determined by its multilinear polynomials.

Let $P_n^\phi$ be the vector space of multilinear $\phi$-polynomials of degree $n$ in $\tilde{x}_1, \tilde{x}_1^\phi, \ldots, \tilde{x}_n, \tilde{x}_n^\phi$. It follows that

$$P_n^\phi = P_n^g = \text{Span}_F \{w_{\sigma(1)} \cdots w_{\sigma(n)} | \sigma \in S_n, w_i = y_i \text{ or } w_i = z_i, i = 1, \ldots, n\}.$$ 

Let $H_n$ be the hyperoctahedral group of degree $n$. Recall that $H_n = \mathbb{Z}_2 \sim S_n$ is the wreath product of $\mathbb{Z}_2 = \{1, \phi\}$ and $S_n$, the symmetric group of degree $n$. The space $P_n^\phi$ has a natural structure of left $H_n$-module induced by defining for $h = (a_1, \ldots, a_n; \sigma) \in H_n$, $h y_i = y_{\sigma(i)}$, $h z_i = z_{\sigma(i)}$, $h \tilde{x}_i = \pm z_{\sigma(i)}$ (see [2]).

Let $P_n^g(A) = (P_n^g / P_n^g \cap \text{Id}^g(A))$ be the space of multilinear elements of degree $n$ in the relatively free algebra $F(X) / \text{Id}^g(A)$. Since $P_n^g \cap \text{Id}^g(A)$ is a $T_2$-invariant under the above action we can view $P_n^g(A)$ as an $H_n$-module; let $\chi^g(A)$ be its character. The sequence

$$c_n^g(A) = \chi^g(A)(1) = \dim_F P_n^g(A), \quad n = 1, 2, \ldots$$

is called the sequence of graded-codimensions of $A$.

Recall that there is a one-to-one correspondence between irreducible $H_n$-characters and pairs of partitions $(\lambda, \mu)$, where $\lambda \vdash r, \mu \vdash n - r$, for all $r = 0, 1, \ldots, n$. If $\chi_{\lambda, \mu}$
denotes the irreducible $H_n$-character corresponding to $(\lambda, \mu)$ then we can write

$$
\chi_n^{gr}(A) = \sum_{r=0}^{n} \sum_{\mu \vdash n-r} \sum_{\lambda \vdash r} m_{\lambda, \mu} \zeta_{\lambda, \mu},
$$

where $m_{\lambda, \mu} \geq 0$ are the corresponding multiplicities.

For fixed $r \in \{0, \ldots, n\}$, let

$$
P_{r,n-r} = \text{Span}_F \{ w_{\sigma(1)} \cdots w_{\sigma(n)} \mid \sigma \in S_n, w_i = y_i \text{ for } i = 1, \ldots, r, \text{ and } w_i = z_i \text{ for } i = r + 1, \ldots, n \}
$$

be the space of multilinear polynomials in the variables $y_1, \ldots, y_r, z_{r+1}, \ldots, z_n$. By the multihomogeneity of $T_2$-ideals, in order to study $P_{n}^{gr}(A)$ it is enough to study

$$
P_{r,n-r}^{gr}(A) = \frac{P_{r,n-r}}{P_{r,n-r} \cap \text{Id}^{gr}(A)}
$$

for all $r = 0, \ldots, n$.

If we let $S_r$ act on the variables $y_1, \ldots, y_r$ and $S_{n-r}$ act on the variables $z_{r+1}, \ldots, z_n$, we obtain an action of $S_r \times S_{n-r}$ on $P_{r,n-r}$ and $P_{r,n-r}^{gr}(A)$ becomes a left $S_r \times S_{n-r}$-module.

Let $\chi_{r,n-r}(A)$ be its character. It is well known that the irreducible $S_r \times S_{n-r}$-characters are obtained by taking the outer tensor product of $S_r$ and $S_{n-r}$ irreducible characters, respectively. Then, we can write

$$
\chi_{r,n-r}(A) = \sum_{\lambda \vdash r} \sum_{\mu \vdash n-r} m_{\lambda, \mu} (\chi_\lambda \otimes \chi_\mu),
$$

where $\chi_\lambda$ (respectively, $\chi_\mu$) denotes the irreducible $S_r$-character (respectively $S_{n-r}$-character) and $m_{\lambda, \mu} \geq 0$ are the corresponding multiplicities.

The relation between $H_n$-characters and the $S_r \times S_{n-r}$-characters is given for instance in [1, Theorem 1.3] as follows: for any superalgebra $A$ we have

$$
\chi_n^{gr}(A) = \sum_{r=0}^{n} \sum_{\lambda \vdash r} \sum_{\mu \vdash n-r} m_{\lambda, \mu} \zeta_{\lambda, \mu} \quad \text{and} \quad \chi_{r,n-r}(A) = \sum_{\lambda \vdash r} \sum_{\mu \vdash n-r} m_{\lambda, \mu} (\chi_\lambda \otimes \chi_\mu)
$$

for all $r \leq n$. Moreover

$$
c_n^{gr}(A) = \sum_{r=0}^{n} \binom{n}{r} \dim_F P_{r,n-r}^{gr}(A).
$$

4. Graded identities of $UT_2$

Throughout this section $A = UT_2$ is the algebra of $2 \times 2$ upper triangular matrices with canonical $\mathbb{Z}_2$-grading. It is easy to verify that $z_1 z_2 \equiv 0$ and $[y_1, y_2] \equiv 0$ are graded identities of $A$. We wish to show that these two identities generate $\text{Id}^{gr}(A)$ as a $T_2$-ideal.
Now, if \( S \subseteq F(X) \), is any set of polynomials, we denote by \( \langle S \rangle_{T_2} \) the \( T_2 \)-ideal of \( F(X) \) generated by \( S \). We start with the following:

**Remark 1.** For any variable \( x = y + z \), \( z_1x_z \in \langle z_1z_2, [y_1, y_2] \rangle_{T_2} \).

**Proof.** Write \( F(X) = \mathbb{F}_0 \oplus \mathbb{F}_1 \) as in the previous section. Since \( z_1y \in \mathbb{F}_1 \) it follows that \( z_1y_z \in \langle z_1z_2 \rangle_{T_2} \). Hence \( z_1(y + z)z_2 \in \langle z_1z_2 \rangle_{T_2} \) and, so, \( z_1x_z \in \langle z_1z_2 \rangle_{T_2} \). □

In Theorem 2 we exhibit a set of generators for \( \text{Id}^{\psi}(A) \).

**Theorem 2.** The identities \( z_1z_2 \equiv 0 \) and \( [y_1, y_2] \equiv 0 \) generate \( \text{Id}^{\psi}(UT_2) \) as a \( T_2 \)-ideal.

**Proof.** Write \( I = \langle z_1z_2, [y_1, y_2] \rangle_{T_2} \) and let \( f(y_1, \ldots, y_1, z_1, \ldots, z_i) \) be a multilinear polynomial in \( \text{Id}^{\psi}(A) \). We wish to show that, modulo \( I \), \( f \) is the zero polynomial. From the previous remark it is clear that we can write \( f(y_1, \ldots, y_1, z_1, \ldots, z_i) = f_1(y_1, \ldots, y_1) + f_2(z, \ldots, z_i) \) (mod \( I \)) and, by the multihomogeneity of \( T_2 \)-ideals, it follows that \( f_1 \) and \( f_2 \) are both identities of \( A \). Since \( [y_1, y_2] \subset I \), we obtain that \( f_1 \equiv z_1y \cdot y \cdot y \). But then, by substituting \( y_1 = \cdots = y_n = e_{11} \) we obtain \( x = 0 \) and, so, \( f_1 \equiv 0 \) (mod \( I \)). Write \( f_2 = \sum \alpha y_0 \cdots y_n \cdot y \cdot y_{j_{k+1}} \cdots y_{j_n} \), where \( i_1 < \cdots < i_2 \) and \( j_1 < \cdots < j_n \). Fix one non-zero monomial of \( f_2 \), let it be \( \alpha y_1 \cdots y_n z_{y_{j+1}} \cdots y_n \). By substituting \( y_1 = \cdots = y_n = e_{11}, y_1+1 = \cdots = y_n = e_{22}, z = e_{12} \) we get \( f = \alpha e_{12} \). Hence \( x = 0 \), a contradiction. It follows that \( f_2 \equiv 0 \) (mod \( I \)) and we are done. □

For any partition \( \lambda \vdash n \) let \( T_\lambda \) be a Young tableau of shape \( \lambda \) and \( e_{T_\lambda} \) the corresponding minimal essential idempotent of the group algebra \( FS_n \). Recall that \( e_{T_\lambda} = \sum_{\sigma \in R_{T_\lambda}} (\text{sgn} \sigma) \sigma \tau \) where \( R_{T_\lambda} \) and \( C_{T_\lambda} \) are the subgroups of row and column permutations of \( T_\lambda \), respectively.

Let \( \lambda \vdash r, \mu \vdash n - r \) and let \( W_{\lambda, \mu} \) be a left irreducible \( S_r \times S_{n-r} \)-module. It is well known that if \( T_\lambda \) is \( r \)-tableau of shape \( \lambda \) and \( T_\mu \) a \( n-r \)-tableau of shape \( \mu \), then \( W_{\lambda, \mu} \cong F(S_r \times S_{n-r}) e_{T_\lambda} e_{T_\mu} \) where \( S_r \) and \( S_{n-r} \) act on disjoint sets of integers.

We can now write the explicit decomposition of the \( n \)th graded cocharacter of \( A \) into irreducibles. For a partition \( \lambda \vdash n \) we denote by \( h(\lambda) \) the height of the diagram associated to \( \lambda \).

**Theorem 3.** Let \( h^n(UT_2) = \sum_{r=0}^{n} \sum_{\lambda \vdash r} \sum_{\mu \vdash n-r} m_{\lambda, \mu} T_{\lambda, \mu} \) be the \( n \)th graded cocharacter of \( UT_2 \) with canonical \( \mathbb{Z}_2 \)-grading. Then

1. \( m_{\lambda, \mu} = q + 1 \) if \( \lambda = (p + q, p) \), \( \mu = (1) \);
2. \( m_{(n), 0} = 1 \);
3. \( m_{(n), \lambda} = 0 \) in all other cases.

**Proof.** The proof here is similar to that of Lemma 1 in [7]. We reproduce it for the convenience of the reader. Since \( \dim A_0 = 2 \) and \( \dim A_1 = 1 \), any polynomial alternating on three even variables or in two odd variables vanishes in \( A \). From the general form of the elements \( e_{T_\lambda} e_{T_\mu} \) it follows that \( m_{n, \lambda} = 0 \) if either \( h(\lambda) > 2 \) or \( h(\mu) > 1 \). Moreover by
Remark 1, for every variable \( x, z_1 x z_2 \in Id^{\mathfrak{g}}(A) \) and this implies that \( m_{\lambda, \mu} = 0 \) whenever \( |\mu| \geq 2 \). So let us assume that \( |\mu| \leq 1 \) and suppose first that \( \mu = 0 \). Then, \([y_1, y_2] = 0\) on \( A \) implies that \( y_1 y_2 \cdots y_n \) is a basis of \( P_{n, 0} \) (mod \( Id^{\mathfrak{g}}(A) \)). Hence \( m_{(n, 0)} = 1 \) and \( m_{\lambda, 0} = 0 \) if \( \lambda \neq (n) \).

Suppose now that \( h(\lambda) \leq 2 \) and \( \mu = (1) \). Let \( \lambda = (p + q, p) \), \( p \geq 0, q \geq 0 \) and \( \mu = (1) \). We want to prove that \( m_{\lambda, \mu} = q + 1 \).

For every \( i = 0, \ldots, q \) define the following two tableaux:

\[
T^{(i)}_{\lambda} = \begin{array}{ccccccc}
i + 1 & i + 2 & \cdots & i + p & 1 & 2 & \cdots & i + 2p + 2 & \cdots & n \\
i + p + 2 & i + p + 3 & \cdots & i + 2p + 1 & & & & & & \\
\end{array}\\
T^{(i)}_{\mu} = \begin{array}{c}
i + p + 1 \\
\end{array}
\]

We associate to \( T^{(i)}_{\lambda} \) and \( T^{(i)}_{\mu} \) the polynomial

\[
a^{(i)}_{p, q}(y_1, y_2, z) = y_1^{i} \overline{y}_1 \overline{\cdots} \overline{y}_1 \overline{\cdots} \overline{y}_2 y_2^{q - i},
\]

where \(-, \overline{\cdot}\) mean alternation on the corresponding elements.

Notice that the polynomial \( a^{(i)}_{p, q} \) is obtained from the essential idempotent corresponding to the pair of tableaux \((T^{(i)}_{\lambda}, T^{(i)}_{\mu})\) by identifying all the elements in each row of \( \lambda \)

We shall prove that (mod \( Id^{\mathfrak{g}}(A) \)) the \( q + 1 \) polynomials \( a^{(i)}_{p, q}(y_1, y_2, z), i = 0, \ldots, q \), are linearly independent over \( F \).

Suppose not. Let \( \sum_{i=0}^{q} x_i a^{(i)}_{p, q}(y_1, y_2, z) = 0 \) (mod \( Id^{\mathfrak{g}}(A) \)) and let \( t = \max\{i : x_i \neq 0\} \).

Then \( x_t a^{(t)}_{p, q} + \sum_{i < t} x_i a^{(i)}_{p, q} = 0 \) (mod \( Id^{\mathfrak{g}}(A) \)).

If we substitute \( y_1 \) with \( y_1 + y_3 \), we obtain

\[
x_t (y_1 + y_3)(y_1 + y_3) \cdots (y_1 + y_3) \overline{z} \overline{y}_2 \cdots \overline{y}_2 (y_1 + y_3)^{q - t}
\]

\[
+ \sum_{i < t} x_i (y_1 + y_3)(y_1 + y_3) \cdots (y_1 + y_3) \overline{z} \overline{y}_2 \cdots \overline{y}_2 (y_1 + y_3)^{q - i}
\]

\[
= 0 \text{ (mod } Id^{\mathfrak{g}}(A) \).
\]

Let us consider the homogeneous component of degree \( t + p \) in \( y_1 \) and of degree \( q - t \) in \( y_3 \). If we make the substitution \( y_1 = e_{11}, y_2 = y_3 = e_{22} \) and \( z = e_{12} \), we obtain \( x_t e_{12} = x_t = 0 \), a contradiction. Hence the polynomials \( a^{(i)}_{p, q}, i = 0, \ldots, q \) are linearly independent (mod \( Id^{\mathfrak{g}}(A) \)).

Notice that, for all \( i, e^{(i)}_{12} e^{(i)}_{p, q}(y_1, \ldots, y_{n-1}, z) \) is the complete linearization of \( a^{(i)}_{p, q}(y_1, y_2, z) \). It follows that the polynomials \( e^{(i)}_{12} e^{(i)}_{p, q}, i = 0, \ldots, q \), are linearly independent (mod \( Id^{\mathfrak{g}}(A) \)) and this implies that \( m_{\lambda, \mu} \geq q + 1 \).
Let now, $T_\lambda$ and $T_\mu$ be any two tableaux and $f = e_{T_\lambda}e_{T_\mu}(y_1, \ldots, y_{n-1}, z)$ the corresponding polynomial.

If $f \not\in \langle z_1z_2, [y_1, y_2] \rangle_{T_2}$, then any two alternating variables in $f$ must lie on different sides of $z$. Since $f$ is a linear combination (mod $Id_{gr}(A)$) of polynomials each alternating on $p$ pairs of $y_i$'s, we obtain that $f$ is a linear combination of the polynomials $e_{\tau_i}e_{\tau_j}$, $i = 0, \ldots, q$. Hence $m_{\lambda, \mu} = q + 1$. □

5. Some numerical invariants

For an algebra $A$ we shall denote by $Id(A)$ the $T$-ideal of the free algebra $F\langle X \rangle$ of (ordinary) polynomial identities of $A$. If we denote with $P_n$ the space of multilinear polynomials in $x_1, \ldots, x_n$, then $c_n(A) = \dim F P_n/P_n \cap Id(A)$, $n = 1, 2, \ldots$, is the sequence of codimensions of $A$.

Suppose now that $A$ is a superalgebra satisfying a non-trivial polynomial identity. As we mentioned before, the graded identities coincide with the $\phi$-identities where $\phi$ is an automorphism of order two. But then by [3] it follows that

$$c_n(A) \leq c_n^{gr}(A) \leq 2^n c_n(A).$$

In order to capture the exponential behavior of the sequence of graded codimensions we then define $\text{exp}_{gr}(A) = \lim_{n \to \infty} \sup \sqrt[n]{c_n^{gr}(A)}$.

Another numerical sequence that can be attached to a superalgebra is given by the sequence of colengths: if $\chi_{gr,n}(A) = \sum_{r=0}^{n} \sum_{\mu \vdash n-r} m_{\lambda, \mu} \chi_{\lambda, \mu}$ is the decomposition of the $n$th graded cocharacter of $A$, then one defines the $n$th graded-colength of $A$ as

$$\ell_{gr,n}(A) = \sum_{r=0}^{n} \sum_{\mu \vdash n-r} m_{\lambda, \mu}.$$

In the next corollary we shall compute $\text{Exp}_{gr}(UT_2)$ and $\ell_{gr,n}(UT_2)$ for all $n \geq 1$, in case $UT_2$ is endowed with the canonical $\mathbb{Z}_2$-grading. We need to introduce some notation. For a partition $\lambda \vdash n$, let $d_{\lambda} = \chi_{\lambda}(1)$ be the degree of the irreducible $S_n$-character $\chi_{\lambda}$. If $d_{\lambda, \mu} = \chi_{\lambda, \mu}(1)$ is the degree of the irreducible $H_n$-character corresponding to the pair $(\lambda, \mu)$, where $\lambda \vdash r$, $\mu \vdash n-r$, then it is well known that $d_{\lambda, \mu} = \binom{n}{r} d_{\lambda} d_{\mu}$.

Corollary 1.

1. $\text{Exp}_{gr}(UT_2) = 2$.

2. For all $n \geq 1$, $\ell_{n}(UT_2) = \begin{cases} \frac{n^2 - 2n + 9}{4} & \text{if } n \text{ is odd,} \\ \frac{n^2 - 2n + 8}{4} & \text{if } n \text{ is even.} \end{cases}$

Proof. The codimensions of $UT_2$ are known and can be deduced for instance from [6]. It follows that $c_n(UT_2) = an^2 2^n$, for some constants $a, t$. On the other hand, by recalling
that the multiplicities $m_{i,\mu}$ are polynomially bounded, we get
\[ c_n(UT_2) \leq c_n^{\text{gr}}(UT_2) = \sum_{\lambda, \mu \vdash n} m_{\lambda,\mu} \sum_{h(\lambda) \leq 2, \mu_1 \leq 1} d_{\lambda,\mu} \leq \alpha n^e \sum_{h(\lambda) \leq 2, \mu_1 \leq 1} d_{\lambda,\mu} \]

\[ = \alpha n^e \sum_{\lambda, \mu \vdash n} \binom{n}{|\lambda|} d_{\lambda,\mu} \leq \alpha n^{e+1} \sum_{h(\lambda) \leq 2} d_{\lambda} \leq \alpha' n^{e'} 2^n, \]

where the last inequality is deduced from the hook formula for the degrees of the irreducible representations of the symmetric group (see [4]). This proves (1).

The second part of the corollary is obtained by a direct calculation by making use of Theorem 3. \[\square\]

6. Growth of the graded codimensions

In this section, we shall be involved with superalgebras and the growth of their codimensions. If $A$ is a superalgebra the growth of the sequence of graded codimensions of $A$ is called the PI-growth of $A$. We say that $A$ has almost polynomial PI-growth if $A$ has exponential growth but for every $T_2$-ideal $I \supset \text{Id}^{\text{gr}}(UT_2)$ the sequence of graded codimensions of $I$ is polynomially bounded.

In Theorem 4 we shall prove that $UT_2$ with the canonical $\mathbb{Z}_2$-grading, is a superalgebra with almost polynomial PI-growth.

**Theorem 4.** Let $A$ be a $\mathbb{Z}_2$-graded algebra. If $\text{Id}^{\text{gr}}(A) \supset \text{Id}^{\text{gr}}(UT_2)$ then there exists a constant $N$ such that for all $n$ and $|\lambda| + |\mu| = n$ we have that $m_{\lambda,\mu}(A) \leq N$. Moreover $c_n^{\text{gr}}(A)$ is polynomially bounded.

**Proof.** Since $\text{Id}^{\text{gr}}(A) \supset \text{Id}^{\text{gr}}(UT_2)$, there exist $\lambda, \mu$ with $|\lambda| + |\mu| = n$ such that $m_{\lambda,\mu}(A) = m_{\lambda,\mu}(UT_2)$. It follows that the polynomials $a_{p,q}^{(i)}$ introduced in the proof of Theorem 3 are linearly dependent modulo $\text{Id}^{\text{gr}}(UT_2)$. Hence

\[ \sum_i \beta_i y_1^i \tilde{y}_1 \cdots \tilde{y}_1 z \tilde{y}_2 \cdots \tilde{y}_2 y_1^{q-i} = 0 \quad (\text{mod } \text{Id}^{\text{gr}}(UT_2)). \]

Let $t = \max\{i : \beta_i \neq 0\}$. By substituting $y_2$ with $y_1^t$, we obtain

\[ f(y_1, z) = \beta_1 y_1^t \tilde{y}_1 \cdots \tilde{y}_1 z (y_1)^2 \cdots (y_1)^2 y_1^{q-t} \]

\[ + \sum_{i < t} \beta_i y_1^i \tilde{y}_1 \cdots \tilde{y}_1 z (y_1)^2 \cdots (y_1)^2 y_1^{q-i} = 0 \quad (\text{mod } \text{Id}^{\text{gr}}(UT_2)). \]

Let $N = 3p + q = \deg(f(y_1, z)) - 1$ and $M = t + 2p$. Then, from the above, it follows that

\[ y_1^M x y_1^{N-M} = \sum_{i < M} \delta_i y_1^i z y_1^{N-i} \quad (\text{mod } \text{Id}^{\text{gr}}(UT_2)) \]  

(1)

for some $\delta_i \in F$. 

\[\square\]
We shall prove that \( m_{\hat{\lambda}, \mu}(A) \leq N \), for any partitions \( \hat{\lambda}, \mu \). Clearly, by Theorem 3, it is enough to consider the case when \( \hat{\lambda} = (p + q, p) \), \( \mu = (1) \). Notice that
\[
\tilde{y}_1 \cdots \tilde{y}_1 \tilde{z} \tilde{y}_2 \cdots \tilde{y}_2 \in \mathcal{F}_1.
\]

Hence, if \( q \geq N \) we can apply (1) to any polynomial \( a^{(i)}_{p,q}(y_1, y_2, z) \) as soon as \( i \geq M \).

We obtain
\[
a^{(i)}_{p,q} = \sum_{j<M} \delta_j a^{(j)}_{p,q} \pmod{\text{Id}^B(UT_2)}
\]
and \( m_{\hat{\lambda}, \mu}(A) \leq M \leq N \) follows.

We next show that the sequence \( c_n^g(A) \), \( n = 1, 2, \ldots \), is polynomially bounded. By multilinearizing (1) we obtain
\[
\sum_{\sigma \in S_N} y_{\sigma(1)} \cdots y_{\sigma(M)} \tilde{z} y_{\sigma(M+1)} \cdots y_{\sigma(N)}
\]
\[
= \sum_{i<M} \sum_{\sigma \in S_N} \delta_i y_{\sigma(1)} \cdots y_{\sigma(i)} \tilde{z} y_{\sigma(i+1)} \cdots y_{\sigma(N)} \pmod{\text{Id}^B(UT_2)}.
\]

We multiply the above expression on the right by \( y_{21} \cdots y_{2M} \) and we then alternate \( y_{1i} \) with \( y_{2j} \) for \( i = 1, \ldots, M \). As a result we obtain
\[
\tilde{y}_{11} \tilde{y}_{12} \cdots \tilde{y}_{1M} \tilde{z} \tilde{y}_{21} \tilde{y}_{22} \cdots \tilde{y}_{2M} y_{11} \cdots y_{1N} = 0 \pmod{\text{Id}^B(UT_2)}.
\]

If we now multiply on the left by \( y_{2M+1} \cdots y_{2N} \) and then we alternate \( y_{1j} \) with \( y_{2j} \) for all \( j = M + 1, \ldots, N \) we get
\[
\tilde{y}_{11} \tilde{y}_{12} \cdots \tilde{y}_{1N} \tilde{z} \tilde{y}_{21} \tilde{y}_{22} \cdots \tilde{y}_{2N} = 0 \pmod{\text{Id}^B(UT_2)}.
\]

This relation shows that the irreducible \( S_{N^2} \times S_1 \)-character corresponding to the pair of partitions \( \hat{\lambda} = (N^2), \mu = (1) \) participates into the \( (N^2 + 1) \)th graded cocharacter of \( A \) with zero multiplicity, i.e., \( m_{((N^2), (1))}(A) = 0 \).

It follows that if \( \hat{\lambda} \) is a partition of \( n \) such that \( \hat{\lambda}_2 \geq N \) then \( m_{\hat{\lambda}, \mu}(A) = 0 \). Recalling Theorem 2, it follows that if \( y_{\lambda, \mu} \) participates in the graded cocharacter with non-zero multiplicity then \( \hat{\lambda} \) must contain at most \( N - 1 \) boxes below the first row and \( |\mu| \leq 1 \). Therefore
\[
m_{\mu}^g(A) = \sum_{|\lambda| + |\mu| = n \atop |\lambda| - \hat{\lambda} \leq N - 1 \atop |\mu| \leq 1} m_{\lambda, \mu} y_{\lambda, \mu}
\]
and, by [2], \( A \) has polynomial growth. \( \Box \)

Acknowledgements

The author was partially supported by MURST of Italy.
References