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Bruhat order on classical Weyl groups: minimal chains and covering relation

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Abstract

In this paper we study some aspects of the Bruhat order on classical Weyl groups, obtaining a direct combinatorial description of the minimal chains, that is chains with the lexicographically minimal labelling. Moreover, we find a combinatorial characterization of the covering relation in the hyperoctahedral group and in the even-signed permutation group, providing results analogous to the well-known characterization of the covering relation in the symmetric group.

1. Introduction

Bruhat order on Coxeter groups has been studied extensively (see, e.g., [5–7, 12, 13, 17]). In particular it is known that every Coxeter group, partially ordered by the Bruhat order, is a graded, EL-shellable poset (see [5, 7, 12]).

In this work we study some aspects of the Bruhat order on a particular class of Coxeter groups, namely that of classical Weyl groups, which have nice combinatorial descriptions in terms of permutation groups: the symmetric group S_n, the hyperoctahedral group B_n and the even-signed permutation group D_n.

We obtain, for these groups, a direct combinatorial description of the minimal chains, that is chains with the lexicographically minimal labelling, which play a crucial role in the definition of the EL-shellability.

As a parallel result, we find a combinatorial characterization of the covering relation in the hyperoctahedral group and in the even-signed permutation group, analogous to the well-known characterization in the symmetric group.

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The organization of the paper is as follows. In Section 2 we collect some basic notions and results. In Section 3 we expose some general techniques about posets, which will be used in the rest of this work. Sections 4–6 contain the main results, about, respectively, the symmetric group, the hyperoctahedral group and the even-signed permutation group. Finally, Section 7 is dedicated to the proof of a rather technical result.

2. Notation and preliminaries

We let $\mathbf{N} = \{1, 2, 3, \ldots\}$ and $\mathbf{Z}$ be the set of integers. For $n, m \in \mathbf{Z}$, with $n \leq m$, we let $[n, m] = \{n, n+1, \ldots, m\}$. For $n \in \mathbf{N}$, we let $[n] = [1, n]$ and $[\pm n] = [-n, n] \backslash \{0\}$. We denote by $\equiv$ the congruency modulo 2: $n \equiv m$, with $n, m \in \mathbf{Z}$, means that $n - m$ is even. Finally, we denote simply by $<$ the lexicographic ordering of n-tuples: $(a_1, a_2, \ldots, a_n) < (b_1, b_2, \ldots, b_n)$ means that $a_k < b_k$, where $k = \min\{i \in [n] : a_i \neq b_i\}$.

2.1. Posets

We follow [15, Chapter 3] for poset notation and terminology. In particular, we denote by $\lhd$ the *covering relation*: $x \lhd y$ means that $x < y$ and there is no z such that $x < z < y$. A poset is *bounded* if it has a minimum and a maximum, denoted by $\hat{0}$ and $\hat{1}$ respectively. If $x, y \in P$, with $x \leq y$, we let $[x, y] = \{z \in P : x \leq z \leq y\}$, and we call it an *interval* of P. If $x, y \in P$, with $x < y$, a *chain* from x to y of *length* k is a $(k+1)$-tuple $(x_0, x_1, \ldots, x_k)$ such that $x = x_0 < x_1 < \cdots < x_k = y$. A chain $x_0 < x_1 < \cdots < x_k$ is said to be *saturated* if all the relations in it are covering relations ($x_0 \lhd x_1 \lhd \cdots \lhd x_k$).

A poset is said to be *graded* of *rank* n if it is finite, bounded and if all maximal chains of P have the same length n. If P is a graded poset of rank n, then there is a unique *rank function* $\rho : P \to [0, n]$ such that $\rho(\hat{0}) = 0$, $\rho(\hat{1}) = n$ and $\rho(y) = \rho(x) + 1$ whenever y covers x in P. Conversely, if P is finite and bounded, and if such a function exists, then P is graded of rank n.

Let P be a graded poset and let Q be a totally ordered set. An *EL-labelling* of P is a function $\lambda : \{(x, y) \in P^2 : x \lhd y\} \to Q$ such that for every $x, y \in P$, with $x < y$, two properties hold:

1. There is exactly one saturated chain from x to y with non-decreasing labels:

$$x = x_0 \underset{\lambda_1}{\lhd} x_1 \underset{\lambda_2}{\lhd} \cdots \underset{\lambda_k}{\lhd} x_k = y,$$

   with $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$.
2. This chain has the lexicographically minimal labelling: if

$$x = y_0 \underset{\mu_1}{\lhd} y_1 \underset{\mu_2}{\lhd} \cdots \underset{\mu_k}{\lhd} y_k = y$$

   is a saturated chain from x to y different from the previous one, then

$$(\lambda_1, \lambda_2, \ldots, \lambda_k) < (\mu_1, \mu_2, \ldots, \mu_k).$$

A graded poset P is said to be *EL-shellable* if it has an EL-labelling.

Connections between EL-shellable posets and shellable complexes, Cohen–Macaulay complexes and Cohen–Macaulay rings can be found, for example, in [1, 3, 4, 8, 9, 14, 16]. Here we only recall the following important result, due to Björner.

Theorem 2.1. *Let P be a graded poset. If P is EL-shellable then P is shellable and hence Cohen–Macaulay.*

2.2. Coxeter groups and Bruhat order

We refer to [10] for the definition of a Coxeter group. Let W be a Coxeter group, with set of generators S. The *length* of an element $w \in W$, denoted by $l(w)$, is the minimal k such that w can be written as a product of k generators. A *reflection* in a Coxeter group W is a conjugate of some element in S. The set of all reflections is usually denoted by T:

$$T = \{wsw^{-1} : s \in S, w \in W\}.$$

Let W be a Coxeter group with set of generators S. Let us have $u, v \in W$. We set $u \to v$ if and only if $v = ut$, with $t \in T$, and $l(u) < l(v)$. The *Bruhat order* of W is the partial order relation so defined: given $u, v \in W$, then $u \le v$ if and only if there is a chain

$$u = u_0 \to u_1 \to u_2 \to \cdots \to u_k = v.$$

If W is finite it is known that W has a maximum, which is usually denoted by w_0. This element is an involution: $w_0^2 = 1$. Moreover, composition and conjugacy with w_0 induce (anti)automorphisms of the Bruhat order, as we state in the following.

Proposition 2.2. *Let W be a finite Coxeter group, with maximum w_0, and let $u, v \in W$. Then the following are equivalent:*

1. $u \le v$;
2. $w_0 v \le w_0 u$;
3. $v w_0 \le u w_0$;
4. $w_0 u w_0 \le w_0 v w_0$.

Bruhat order on Coxeter groups has been studied extensively (see, e.g., [5–7, 12, 13, 17]). In particular it is known that every Coxeter group, partially ordered by the Bruhat order, is a graded, EL-shellable poset (see [5, 7, 12]).

2.3. Classical Weyl groups

The finite irreducible Coxeter groups have been completely classified (see, e.g., [2, 10]). Among them we find the classical Weyl groups, which have nice combinatorial descriptions in terms of permutation groups: the symmetric group S_n is a representative for type $\mathbf{A}_{n-1}$, the hyperoctahedral group B_n for type $\mathbf{B}_n$ and the even-signed permutation group D_n for type $\mathbf{D}_n$.

2.3.1. The symmetric group

We denote by S_n the *symmetric group*, defined by

$$S_n = \{\sigma : [n] \to [n] : \sigma \text{ is a bijection}\}$$

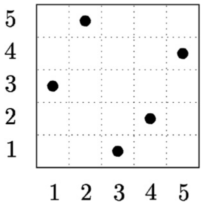

Fig. 1. The diagram of $\sigma = 35124 \in S_5$.

and we call its elements *permutations*. To denote a permutation $\sigma \in S_n$ we often use the *one-line notation*: we write $\sigma = \sigma_1\sigma_2 \ldots \sigma_n$, to mean that $\sigma(i) = \sigma_i$ for every $i \in [n]$. We also write σ in *disjoint cycle form*, omitting to write the 1-cycles of σ: for example, if $\sigma = 364152$, then we also write $\sigma = (1, 3, 4)(2, 6)$. Given $\sigma, \tau \in S_n$, we let $\sigma\tau = \sigma \circ \tau$ (composition of functions) so that, for example, $(1, 2)(2, 3) = (1, 2, 3)$. Given $\sigma \in S_n$, the *diagram* of σ is a square of $n \times n$ cells, with the cell (i, j) (that is, the cell in column i and row j, with the convention that the first column is the leftmost one and the first row is the lowest one) filled with a dot if and only if $\sigma(i) = j$. For example, in Fig. 1 the diagram of $\sigma = 35124 \in S_5$ is represented.

As a set of generators for S_n, we take $S = \{s_1, s_2, \ldots, s_{n-1}\}$, where $s_i = (i, i + 1)$ for every $i \in [n - 1]$. It is known that the symmetric group S_n, with this set of generators, is a Coxeter group of type $\mathbf{A}_{n-1}$ (see, e.g., [2]).

The length of a permutation $\sigma \in S_n$ is given by

$$l(\sigma) = \mathrm{inv}(\sigma),$$

where

$$\mathrm{inv}(\sigma) = |\{(i, j) \in [n]^2 : i < j, \sigma(i) > \sigma(j)\}|$$

is the number of *inversions* of σ.

In the symmetric group the reflections are the transpositions:

$$T = \{(i, j) \in [n]^2 : i < j\}.$$

In order to give a characterization of the covering relation in the Bruhat order of the symmetric group, we introduce the following definition.

Definition 2.3. Let us have $\sigma \in S_n$. A *rise* of σ is a pair $(i, j) \in [n]^2$ such that $i < j$ and $\sigma(i) < \sigma(j)$. A rise (i, j) is said to be *free* if there is no $k \in [n]$ such that $i < k < j$ and $\sigma(i) < \sigma(k) < \sigma(j)$.

For example, the rises of $\sigma = 35124 \in S_5$ are $(1, 2)$, $(1, 5)$, $(3, 4)$, $(3, 5)$ and $(4, 5)$. They are all free except $(3, 5)$. The following is a well-known result.

Proposition 2.4. *Let $\sigma, \tau \in S_n$, with $\sigma < \tau$. Then $\sigma \lhd \tau$ in S_n if and only if*

$$\tau = \sigma(i, j),$$

where (i, j) is a free rise of σ.

In this work we will provide analogous results for the hyperoctahedral group and for the even-signed permutation group.

In order to give a characterization of the Bruhat order relation in S_n, we introduce the following notation: for $\sigma \in S_n$ and for $(h, k) \in [n]^2$, we set

$$\sigma[h, k] = |\{i \in [h] : \sigma(i) \in [k, n]\}|,$$

The characterization is the following (see, e.g., [12]).

Theorem 2.5. *Let $\sigma, \tau \in S_n$. Then $\sigma \leq \tau$ if and only if*

$$\sigma[h, k] \leq \tau[h, k],$$

for every $(h, k) \in [n]^2$.

Finally, the maximum of S_n is

$$w_0 = n(n-1)(n-2)\ldots 3\ 2\ 1.$$

Note that, given $\sigma \in S_n$, the diagrams of the permutations $w_0\sigma$, σw_0 and $w_0\sigma w_0$ are obtained from the diagram of σ by, respectively, reversing the rows, reversing the columns and reversing both rows and columns. So the effects of these operations on the Bruhat order are described in Proposition 2.2.

2.3.2. The hyperoctahedral group

We denote by $S_{\pm n}$ the symmetric group on the set $[\pm n]$:

$$S_{\pm n} = \{\sigma : [\pm n] \to [\pm n] : \sigma \text{ is a bijection}\}$$

(clearly isomorphic to S_{2n}), and by B_n the *hyperoctahedral group*, defined by

$$B_n = \{\sigma \in S_{\pm n} : \sigma(-i) = -\sigma(i) \text{ for every } i \in [n]\}$$

and we call its elements *signed permutations*. To denote a signed permutation $\sigma \in B_n$ we use the *window notation*: we write $\sigma = [\sigma_1, \sigma_2, \ldots, \sigma_n]$, to mean that $\sigma(i) = \sigma_i$ for every $i \in [n]$ (the images of the negative entries are then uniquely determined). We also denote σ by the sequence $|\sigma_1||\sigma_2|\ldots|\sigma_n|$, with the negative entries underlined. For example, $\underline{3}\ \underline{2}\ 1$ denotes the signed permutation $[-3, -2, 1]$. We also write σ in disjoint cycle form. Signed permutations are particular permutations of the set $[\pm n]$, so they inherit the notion of diagram. Note that the diagram of a signed permutation is symmetric with respect to the center. In Fig. 2, the diagram of $\sigma = \underline{3}\ \underline{2}\ 1 \in B_3$ is represented.

As a set of generators for B_n, we take $S = \{s_0, s_1, \ldots, s_{n-1}\}$, where $s_0 = (1, -1)$ and $s_i = (i, i+1)(-i, -i-1)$ for every $i \in [n-1]$. It is known that the hyperoctahedral group B_n, with this set of generators, is a Coxeter group of type $\mathbf{B}_n$ (see, e.g., [2]).

In [11] we introduced a new formula for computing the length in B_n: the length of $\sigma \in B_n$ is given by

$$l_B(\sigma) = \frac{\text{inv}(\sigma) + \text{neg}(\sigma)}{2}, \tag{1}$$

where

$$\text{inv}(\sigma) = |\{(i, j) \in [\pm n]^2 : i < j, \sigma(i) > \sigma(j)\}|$$

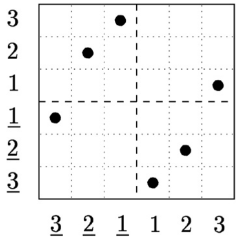

Fig. 2. The diagram of $\sigma = \underline{3}\,\underline{2}\,1 \in B_3$.

(the length of σ in the symmetric group $S_{\pm n}$), and

$$\mathrm{neg}(\sigma) = |\{i \in [n] : \sigma(i) < 0\}|.$$

For example, for $\sigma = \underline{3}\ \underline{2}\ 1 \in B_3$, we have $\mathrm{inv}(\sigma) = 8$, $\mathrm{neg}(\sigma) = 2$, so $l_B(\sigma) = 5$.

It is known (see, e.g., [2]) that the set of reflections of B_n is

$$T = \{(i, -i) : i \in [n]\} \cup \{(i, j)(-i, -j) : 1 \le i < |j| \le n\}.$$

It is useful to extend a notation introduced for the symmetric group: for $\sigma \in B_n$ and for $(h, k) \in [\pm n]^2$ we set

$$\sigma[h, k] = |\{i \in [-n, h] : \sigma(i) \in [k, n]\}|.$$

Definition 2.6. Let $\sigma, \tau \in B_n$. We say that the pair (σ, τ) satisfies the B-condition if

$$\sigma[h, k] \le \tau[h, k]$$

for every $h, k \in [\pm n]^2$.

The following result gives a combinatorial characterization of the Bruhat order relation in B_n (see, e.g., [2, Theorem 8.1.8]).

Theorem 2.7. *Let $\sigma, \tau \in B_n$. Then $\sigma \le \tau$ if and only if the pair (σ, τ) satisfies the B-condition.*

Comparing Theorems 2.5 and 2.7, we can conclude the following.

Proposition 2.8. *Let $\sigma, \tau \in B_n$. Then $\sigma \le \tau$ in the Bruhat order of B_n if and only if $\sigma \le \tau$ in the Bruhat order of the symmetric group $S_{\pm n}$.*

The maximum of B_n is

$$w_0 = \underline{1}\,\underline{2} \cdots \underline{n}$$

and the effects on the diagram of a signed permutation of composing and conjugating with w_0 are the same as described for the symmetric group.

2.3.3. The even-signed permutation group

We denote by D_n the *even-signed permutation group*, defined by

$$D_n = \{\sigma \in B_n : \mathrm{neg}(\sigma) \text{ is even}\}.$$

The notation and terminology are inherited from the hyperoctahedral group. For example the signed permutation $\sigma = \underline{3}\,\underline{2}\,1$, whose diagram is represented in Fig. 2, is also in D_3.

As a set of generators for D_n, we take $S = \{s_0, s_1, \ldots, s_{n-1}\}$, where $s_0 = (1, -2)(-1, 2)$ and $s_i = (i, i + 1)(-i, -i - 1)$ for every $i \in [n - 1]$. It is known that the even-signed permutation group D_n, with this set of generators, is a Coxeter group of type $\mathbf{D}_n$ (see, e.g., [2]).

As regards the length function in D_n, it is known (see, e.g., [2]) that

$$l_D(\sigma) = l_B(\sigma) - \mathrm{neg}(\sigma).$$

Thus, by (1), the length of $\sigma \in D_n$ is given by

$$l_D(\sigma) = \frac{\mathrm{inv}(\sigma) - \mathrm{neg}(\sigma)}{2}.$$

For example, for $\sigma = \underline{3}\,\underline{2}\,1 \in D_3$, we have $l_D(\sigma) = 3$.

Finally, it is known (see, e.g., [2]) that the set of reflections of D_n is

$$T = \{(i, j)(-i, -j) : 1 \le i < |j| \le n\}.$$

In order to give a combinatorial characterization of the Bruhat order relation in D_n, we introduce the following notation: for $\sigma \in D_n$ and $(h, k) \in [-n] \times [n]$, we set

$$\begin{aligned}
\sigma_{\text{center}}[h, k] &= \sigma_{[\pm|h|]\times[\pm k]},\\
\sigma_{NW}[h, k] &= \sigma_{[-n,h-1]\times[k+1,n]},\\
\sigma_{N\text{left}}[h, k] &= \sigma_{[h]\times[k+1,n]},\\
\sigma_{W\text{up}}[h, k] &= \sigma_{[-n,h-1]\times[k]}.
\end{aligned}$$

We say that $(h, k) \in [-n] \times [n]$ is *free* for σ if

$$\sigma_{\text{center}}[h, k] = 0.$$

Definition 2.9. Let $\sigma, \tau \in D_n$. We say that $(h, k) \in [-n] \times [n]$ is a *D-cell* of the pair (σ, τ) if it is free for both σ and τ and

$$\sigma_{NW}[h, k] = \tau_{NW}[h, k].$$

If (h, k) is a D-cell of (σ, τ), then we say that it is valid if

$$\sigma_{N\text{left}}[h, k] \equiv \tau_{N\text{left}}[h, k],$$

or, equivalently, if

$$\sigma_{W\text{up}}[h, k] \equiv \tau_{W\text{up}}[h, k].$$

Finally, we say that the pair (σ, τ) satisfies the *D-condition* if every D-cell of (σ, τ) is valid.

The following result gives a combinatorial characterization of the Bruhat order relation in D_n (see [2, Theorem 8.2.8]).

Theorem 2.10. *Let $\sigma, \tau \in D_n$. Then $\sigma \le \tau$ if and only if the pair (σ, τ) satisfies both the B-condition and the D-condition.*

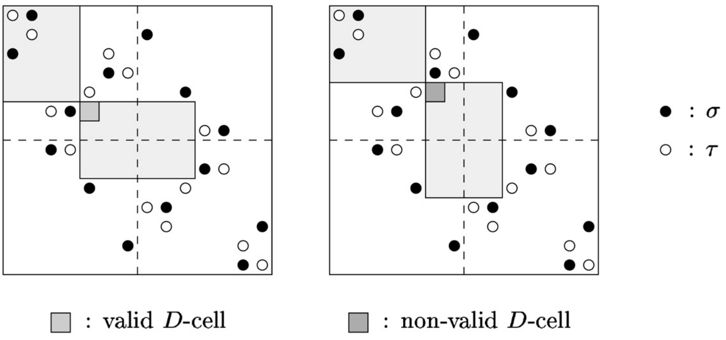

Fig. 3. D-cells.

Note that $\sigma \leq_D \tau$ implies $\sigma \leq_B \tau$, while the converse is not true.

For example, consider the two even-signed permutations $\sigma = 6\,\underline{4}\,3\,\underline{2}\,1\,\underline{7}\,\underline{5}$ and $\tau = \underline{4}\,\underline{5}\,\underline{3}\,1\,\underline{2}\,\underline{6}\,\underline{7}$ in D_7, whose diagrams are shown in Fig. 3.

It is easy to check that the pair (σ, τ) satisfies the B-condition, so $\sigma \leq_B \tau$.

The D-cells of the pair (σ, τ) are $(-3, 1)$, $(-3, 2)$, $(-2, 3)$ and $(-1, 3)$. Among these, $(-3, 1)$ and $(-3, 2)$ are valid, while $(-2, 3)$ and $(-1, 3)$ are not valid. Thus the pair (σ, τ) does not satisfy the D-condition, so $\sigma \not\leq_D \tau$.

The maximum of D_n is

$$w_0 = \begin{cases} \underline{1}\,\underline{2}\ldots\underline{n}, & \text{if } n \text{ is even,} \\ 1\,\underline{2}\ldots\underline{n}, & \text{if } n \text{ is odd.} \end{cases}$$

3. General techniques

In this section we expose some general techniques about posets, concerning gradedness, covering relation and EL-shellability.

Let P be a finite bounded poset.

Definition 3.1. A *successor system* of P is a subset

$$H \subseteq \{(x, y) \in P^2 : x < y\}.$$

An *insertion system* of P is a successor system H of P such that

(insertion property) for every $x, y \in P$, with $x < y$, there exists $z \in P$ such that

$$(x, z) \in H \quad \text{and} \quad z \leq y.$$

A covering system of P is a pair (H, ρ), where H is an insertion system of P and $\rho : P \to \mathbf{N} \cup \{0\}$ is a statistic on P such that

(ρ-base property) $\rho(\hat{0}) = 0$;

(ρ-increasing property) for every $(x, y) \in H$, we have

$$\rho(y) = \rho(x) + 1.$$

Next proposition gives a general method to prove that a poset is graded with a given rank function: it suffices to find a covering system of P.

Proposition 3.2. *If there exists a covering system* (H, ρ) *of* P*, then* P *is graded with rank function* ρ*.*

Proof. By the ρ-base property, $\rho(\hat{0}) = 0$. Now let $x, y \in P$, with $x \lhd y$. By the insertion property, there is $z \in P$, such that $(x, z) \in H$ and $z \leq y$. Since H is a successor system, we have $x < z$, and since $x \lhd y$, necessarily $z = y$. By the ρ-increasing property, we have $\rho(y) = \rho(z) = \rho(x) + 1$. $\square$

A covering system (H, ρ) also gives a complete description of the covering relation in P: the pairs of elements which are in covering relation are exactly the pairs in H, as we state in the following.

Proposition 3.3. *Let* (H, ρ) *be a covering system of* P*. Let* $x, y \in P$*. Then*

$$x \lhd y \;\Leftrightarrow\; (x, y) \in H.$$

Proof. If $x \lhd y$, then we have already observed, in the proof of Proposition 3.2, that $(x, y) \in H$. On the other hand, for every $(x, y) \in H$ we have $x \lhd y$. In fact, from the insertion property and the ρ-increasing property, it follows that ρ is order-preserving, that is, for every $s, t \in P$, $s < t$ implies $\rho(s) < \rho(t)$. If we suppose, by contradiction, that there is $z \in P$ such that $x < z < y$, then we have $\rho(y) \geq \rho(x) + 2$, which is in contradiction with $\rho(y) = \rho(x) + 1$. $\square$

Now let Q be a totally ordered set, the set of *labels*.

Definition 3.4. Let H be a successor system of P. A *good labelling* of H is a function $\lambda : H \to Q$ such that

(injectivity property) for every $(x, y), (x, z) \in H$, we have

$$\lambda(x, y) = \lambda(x, z) \;\Rightarrow\; y = z.$$

Let H be a successor system of P and let λ be a good labelling of H. Let $x \in P$. An element $i \in Q$ is a *suitable label* of x if there is $y \in P$ such that $(x, y) \in H$ and $\lambda(x, y) = i$. By the injectivity property, such a y is unique, and we call it the *transformation* of x with respect to the label i, and denote it by

$$t_i^P(x).$$

The set of all suitable labels of x is denoted by $\Lambda(x)$.

The following is an equivalent version of the insertion property, once a good labelling of H is given:

(insertion property) for every $x, y \in P$, with $x < y$, there exists a label $i \in \Lambda(x)$ such that

$$t_i^P(x) \leq y.$$

If (H, ρ) is a covering system of P, then by Proposition 3.3 we have $x \lhd y$ if and only if $(x, y) \in H$. In this case a good labelling λ of H is an edge-labelling of P. It is useful to introduce the following terminology: if $x \in P$ and $i \in \Lambda(x)$ then we call $t_i^P(x)$ the *covering transformation* of x with respect to the label i, and denote it by

$$ct_i^P(x).$$

Thus, for every $x \in P, i \in \Lambda(x)$ we have $x \lhd ct_i^P(x)$. On the other hand, if $x \lhd y$, then $y = ct_i^P(x)$ for a unique $i \in \Lambda(x)$, and we write also

$$x \underset{i}{\lhd} y.$$

We are now able to define the *minimal chains* in P. Note that, if (H, ρ) is a covering system of P, then by the insertion property, for every $x, y \in P$, with $x < y$, the set

$$\{i \in \Lambda(x) : ct_i^P(x) \leq y\}$$

is not empty. This allows us to give the following definition.

Definition 3.5. Let (H, ρ) be a covering system of P. Let $x, y \in P$, with $x < y$. The *minimal label* of x with respect to y, denoted by $mi_y(x)$ (or simply mi), is

$$mi_y(x) = \min\{i \in \Lambda(x) : ct_i(x) \leq y\}.$$

The *minimal covering transformation* of x with respect to y, denoted by $mct_y^P(x)$, is the covering transformation of x with respect to the minimal label:

$$mct_y^P(x) = ct_{mi}^P(x).$$

It is useful to state the following, which is a consequence of the definitions.

Proposition 3.6. *Let $x, y \in P$, with $x < y$. Then*

$$x \lhd mct_y^P(x) \leq y.$$

By Proposition 3.6, the following definition is well-posed.

Definition 3.7. Let $x, y \in P$, with $x < y$. The *minimal chain* from x to y is the saturated chain

$$x = x_0 \lhd x_1 \lhd \cdots \lhd x_k = y,$$

defined by

$$x_i = mct_y^P(x_{i-1}),$$

for every $i \in [k]$.

By the definition of a minimal covering transformation, this chain has, among all the saturated chains from x to y, the lexicographically minimal labelling. The minimal chains are crucial in the definition of the EL-shellability: a poset is EL-shellable if its minimal chains have increasing labels and if any other saturated chain in it has at least one decrease in the labels.

According to Definition 3.7, the minimal chains are completely described if we give a combinatorial description of the minimal covering transformation. This is what we do for classical Weyl groups in next three sections.

4. The symmetric group

By Proposition 2.4, we can define a natural edge-labelling of S_n (the same as introduced by Edelman in [7] to prove the EL-shellability of S_n).

Definition 4.1. The *standard labelling* of S_n is the edge-labelling

$$\lambda : \{(x, y) \in S_n^2 : x \lhd y\} \to \{(i, j) \in [n]^2 : i < j\}$$

defined in the following way: for every $\sigma, \tau \in S_n$, with $\sigma \lhd \tau$, we set

$$\lambda(\sigma, \tau) = (i, j),$$

where (i, j) is the free rise of σ such that $\tau = \sigma(i, j)$.

With the terminology introduced in Section 3, we can say that the *suitable labels* of σ are its free rises and that, if (i, j) is a free rise of σ, then the *covering transformation* of σ with respect to (i, j) is

$$ct_{(i,j)}^{S_n}(\sigma) = \sigma(i, j).$$

In order to describe the minimal covering transformation, we give the following definitions.

Definition 4.2. Let $\sigma, \tau \in S_n$, with $\sigma < \tau$. The *difference index* of σ with respect to τ, denoted by $di_\tau(\sigma)$ (or simply di), is the minimal index on which σ and τ differ:

$$di_\tau(\sigma) = \min\{i \in [n] : \sigma(i) \neq \tau(i)\}.$$

We write di, instead of $di_\tau(\sigma)$, when there is no ambiguity about the permutations σ and τ which we are referring to.

Lemma 4.3. *Let $\sigma, \tau \in S_n$, with $\sigma < \tau$. Then*

$$\sigma(di) < \tau(di).$$

Proof. First note that, by definition, we have

$$\sigma(di) \neq \tau(di).$$

Now suppose, by contradiction, that $\sigma(di) > \tau(di)$. In this case we would have $\sigma[di, \sigma(di)] = \tau[di, \sigma(di)] + 1$. But $\sigma < \tau$ and, by Theorem 2.5, this implies $\sigma[di, \sigma(di)] \leq \tau[di, \sigma(di)]$, which is a contradiction. $\square$

Lemma 4.4. *Let $\sigma, \tau \in S_n$, with $\sigma < \tau$. Then the set*

$$\{j \in [di + 1, n] : \sigma(j) \in [\sigma(di) + 1, \tau(di)]\} \tag{2}$$

is not empty.

Proof. Set $k = \sigma^{-1}(\tau(di))$. If $k \in [di - 1]$, then $\sigma(k) = \tau(k)$, that is $k = di$, which is a contradiction. If $k = di$, then σ and τ agree at the index di, which is also a contradiction. Thus $k \in [di + 1, n]$. Also, $\sigma(k) = \tau(di)$, so k belongs to the set (2). $\square$

Previous lemmas ensure that next definition is well-posed.

Definition 4.5. Let $\sigma, \tau \in S_n$, with $\sigma < \tau$. The *covering index* of σ with respect to τ, denoted by $ci_\tau(\sigma)$ (or simply ci), is

$$ci_\tau(\sigma) = \min\{j \in [di+1, n] : \sigma(j) \in [\sigma(di)+1, \tau(di)]\}.$$

By definition (di, ci) is a free rise of σ, so it is one of its suitable labels. In the next two propositions we prove that it is the *minimal label* of σ with respect to τ, in the sense of Definition 3.5.

Proposition 4.6. *Let $\sigma, \tau \in S_n$, with $\sigma < \tau$. Then*

$$ct^{S_n}_{(di,ci)}(\sigma) = \sigma(di, ci) \leq \tau.$$

Proof. Let $\chi = \sigma(di, ci)$. We may assume, without loss of generality, that $di = 1$. Set $R = [1, ci-1] \times [\sigma(1)+1, \sigma(ci)]$. For every $(h, k) \in [n]^2$, we have

$$\chi[h, k] = \begin{cases} \sigma[h, k] + 1, & \text{if } (h, k) \in R, \\ \sigma[h, k], & \text{if } (h, k) \notin R. \end{cases}$$

Thus, by Theorem 2.5, to prove that $\chi \leq \tau$, we only have to show that $\tau[h, k] \geq \sigma[h, k]+1$ for every $(h, k) \in R$. But if $(h, k) \in R$, then we have

$$\sigma[h, k] = \sigma[h, \tau(1)+1] \leq \tau[h, \tau(1)+1] \leq \tau[h, k] - 1,$$

so $\chi \leq \tau$. $\square$

Proposition 4.7. *Let $\sigma, \tau \in S_n$, with $\sigma < \tau$. Then*

$$mi_\tau(\sigma) = (di, ci).$$

Proof. Let (i, j) be a free rise of σ such that

$$ct^{S_n}_{(i,j)}(\sigma) = \sigma(i, j) \leq \tau.$$

We want to prove that $(di, ci) \leq (i, j)$. Suppose, by contradiction, that $(i, j) < (di, ci)$, so either $i < di$, or $i = di$ and $j < ci$. If $i < di$, since σ and τ must differ at the index i, the minimality of di is contradicted. If $i = di$ and $j < ci$, set $\xi = \sigma(i, j)$. We have $\xi(di) = \sigma(j)$ and, since $\xi \leq \tau$, by Lemma 4.3, $\xi(di) \leq \tau(di)$. So $\sigma(j) \leq \tau(di)$ and this contradicts the minimality of ci. $\square$

Thus in the symmetric group the *minimal covering transformation* of σ with respect to τ is

$$mct^{S_n}_\tau(\sigma) = \sigma(di, ci),$$

and, as discussed in Section 3, this gives a description of the minimal chains in the symmetric group.

5. The hyperoctahedral group

Definition 5.1. Let $\sigma \in B_n$. A rise (i, j) of σ is *central* if

$$(0, 0) \in [i, j] \times [\sigma(i), \sigma(j)].$$

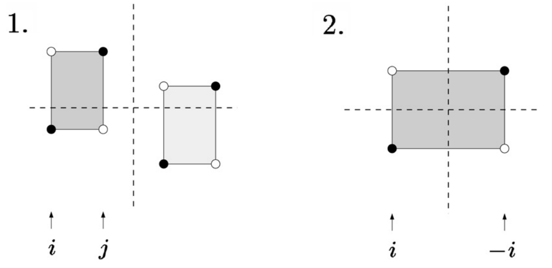

Fig. 4. The covering relation in B_n.

A central rise (i, j) of σ is *symmetric* if $j = -i$.

In order to find a characterization of the covering relation in B_n, we start defining a successor system.

Definition 5.2. Let $\sigma, \tau \in B_n$. We say that (σ, τ) is *a good pair* in B_n if either

1. $\tau = \sigma(i, j)(-i, -j)$, where (i, j) is a non-central free rise of σ, or
2. $\tau = \sigma(i, j)$, where (i, j) is a central symmetric free rise of σ.

Definition 5.2 is illustrated in Fig. 4, where black and white circles denote respectively σ and τ, inside the gray areas there are no other dots of σ and τ, and the diagrams of the two permutations are supposed to be the same anywhere else.

We set

$$H_{B_n} = \{(\sigma, \tau) \in B_n^2 : (\sigma, \tau) \text{ is a good pair in } B_n\},$$

and define the *standard labelling* λ of B_n by associating with every good pair $(\sigma, \tau) \in H_{B_n}$ the pair $(i, j) \in [\pm n]^2$ mentioned in Definition 5.2, which is obviously unique.

By Propositions 2.4 and 2.8, it follows that H_{B_n} is a successor system of B_n and, since τ is uniquely determined by σ and by the label (i, j), λ is a good labelling.

Given $\sigma \in B_n$, the *suitable labels* of σ are then the non-central free rises of σ and the central symmetric free rises of σ. If (i, j) is a suitable label of σ then the *transformation* of σ with respect to (i, j) is

$$t^{B_n}_{(i,j)}(\sigma) = \begin{cases} \sigma(i, j)(-i, -j), & \text{if } (i, j) \text{ is non-central,} \\ \sigma(i, j), & \text{if } (i, j) \text{ is central symmetric.} \end{cases}$$

Now let $\sigma, \tau \in B_n$, with $\sigma < \tau$. In order to prove that the insertion property holds, we define the label

$$i_{B_n}(\sigma, \tau) = \begin{cases} (di, ci), & \text{if } (di, ci) \text{ is non-central,} \\ (di, -di), & \text{if } (di, ci) \text{ is central.} \end{cases}$$

Note that $i_{B_n}(\sigma, \tau)$ is always a suitable label of σ. So we can define the signed permutation

$$\chi_{B_n}(\sigma, \tau) = t^{B_n}_{i_{B_n}(\sigma,\tau)}(\sigma) = \begin{cases} \sigma(di, ci)(-di, -ci), & \text{if } (di, ci) \text{ is non-central,} \\ \sigma(di, -di), & \text{if } (di, ci) \text{ is central.} \end{cases}$$

Proposition 5.3. *Let $\sigma, \tau \in B_n$, with $\sigma < \tau$. Then*

$$\chi_{B_n}(\sigma, \tau) \leq \tau.$$

Proof. Let $\chi = \chi_{B_n}(\sigma, \tau)$. If (di, ci) is non-central then

$$\chi = \sigma(di, ci)(-di, -ci) = mct_\tau^{S_{\pm n}}(w_0(mct_\tau^{S_{\pm n}}(\sigma))w_0).$$

Thus, by Propositions 2.2, 2.8 and 3.6 (applied twice), we have

$$\chi \leq \tau.$$

If (di, ci) is central, then $\chi = \sigma(di, -di)$. We may assume, without loss of generality, that $di = -n$. So necessarily $\sigma(di) = -1$. Set $R = [\pm n] \times \{1\}$. For every $(h, k) \in [\pm n]^2$ we have

$$\chi[h, k] = \begin{cases} \sigma[h, k] + 1, & \text{if } (h, k) \in R, \\ \sigma[h, k], & \text{if } (h, k) \notin R. \end{cases}$$

Thus to prove that $\chi \leq \tau$ it suffices to show that $\tau[h, k] \geq \sigma[h, k] + 1$ for every $(h, k) \in R$. By the symmetry of the diagram, it is enough to show that $\tau[h, 1] \geq \sigma[h, 1]+1$ for every $h \in [-n]$. But, if $h \in [-n]$ we have

$$\sigma[h, 1] = \sigma[h, \tau(di) + 1] \leq \tau[h, \tau(di) + 1] \leq \tau[h, 1] - 1. \quad \square$$

We recall that the length of $\sigma \in B_n$ is given by

$$l_B(\sigma) = \frac{\text{inv}(\sigma) + \text{neg}(\sigma)}{2}.$$

Proposition 5.4. *The pair (H_{B_n}, l_B) is a covering system of B_n.*

Proof. By Proposition 5.3, H_{B_n} is an insertion system of B_n. The ρ-base property is trivial. It remains to prove the ρ-increasing property. Consider $(\sigma, \tau) \in H_{B_n}$ and let $\lambda(\sigma, \tau) = (i, j)$. We have

$$\text{inv}(\tau) = \begin{cases} \text{inv}(\sigma) + 2, & \text{if } (i, j) \text{ is non-central}, \\ \text{inv}(\sigma) + 1, & \text{if } (i, j) \text{ is central symmetric}, \end{cases}$$

and

$$\text{neg}(\tau) = \begin{cases} \text{neg}(\sigma), & \text{if } (i, j) \text{ is non-central}, \\ \text{neg}(\sigma) + 1, & \text{if } (i, j) \text{ is central symmetric}. \end{cases}$$

Thus in each case $l_B(\tau) = l_B(\sigma) + 1$. $\square$

We have found a covering system of B_n. So we have a characterization of the covering relation in B_n, which we state in the following.

Theorem 5.5. *Let $\sigma, \tau \in B_n$. Then $\sigma \lhd \tau$ if and only if either*

1. $\tau = \sigma(i, j)(-i, -j)$, *where (i, j) is a non-central free rise of σ, or*
2. $\tau = \sigma(i, j)$, *where (i, j) is a central symmetric free rise of σ.*

If $\sigma \in B_n$ and (i, j) is a suitable label of σ, then the covering transformation of σ with respect to (i, j) actually is a *covering transformation*, denoted by

$$ct^{B_n}_{(i,j)}(\sigma).$$

In the next proposition we prove that $i_{B_n}(\sigma, \tau)$ is the *minimal label* of σ with respect to τ, in the sense of Definition 3.5.

Proposition 5.6. *Let $\sigma, \tau \in B_n$, with $\sigma < \tau$. Then*

$$mi_\tau(\sigma) = i_{B_n}(\sigma, \tau) = \begin{cases} (di, ci) & \textit{if } (di, ci) \textit{ is non-central,} \\ (di, -di), & \textit{if } (di, ci) \textit{ is central.} \end{cases}$$

Proof. If (di, ci) is non-central (case 1) then $i_{B_n}(\sigma, \tau) = (di, ci)$; otherwise (case 2) $i_{B_n}(\sigma, \tau) = (di, -di)$. Let (i, j) be a suitable label of σ such that $ct^{B_n}_{(i,j)}(\sigma) \leq \tau$. We want to prove that

$$i_{B_n}(\sigma, \tau) \leq (i, j).$$

Necessarily $i \geq di$. If $i > di$ then $i_{B_n}(\sigma, \tau) < (i, j)$. So, suppose $i = di$.

In case 1, we have to prove that $j \geq ci$. Suppose, by contradiction, that $j < ci$ and set $\xi = \sigma(i, j)$. We have $\xi(di) = \sigma(j)$ and, since $\xi \leq \tau$, by Lemma 4.3, $\xi(di) \leq \tau(di)$. So $\sigma(j) \leq \tau(di)$ and this contradicts the minimality of ci.

In case 2, we have to prove that $j \geq -di$ (actually, the only possibility is $j = -di$). If we suppose $j < ci$, as in case 1 we get a contradiction. Thus $j \geq ci$. Since (di, j) is a suitable label of σ and it is central, it has to be symmetric, that is $j = -di$. □

Thus in the hyperoctahedral group the *minimal covering transformation* of σ with respect to τ is

$$mct^{B_n}_\tau(\sigma) = \begin{cases} \sigma(di, ci)(-di, -ci), & \text{if } (di, ci) \text{ is non-central,} \\ \sigma(di, -di), & \text{if } (di, ci) \text{ is central,} \end{cases}$$

and this gives a combinatorial description of the minimal chains in B_n.

6. The even-signed permutation group

Definition 6.1. Let $\sigma \in D_n$. A central rise (i, j) is *semi-free* if

$$\{k \in [i, j] : \sigma(k) \in [\sigma(i), \sigma(j)]\} = \{i, -j, j\},$$

that is if the only dots of the diagram of σ lying in the rectangle $[i, j] \times [\sigma(i), \sigma(j)]$ are those in the cells $(i, \sigma(i))$, $(-j, -\sigma(j))$ and $(j, \sigma(j))$.

An example of central semi-free rise is illustrated in Fig. 5(3).

As we did for B_n, we start defining a successor system of D_n.

Definition 6.2. Let $\sigma, \tau \in D_n$. We say that (σ, τ) is a *good pair* in D_n if

$$\tau = \sigma(i, j)(-i, -j),$$

where (i, j) is

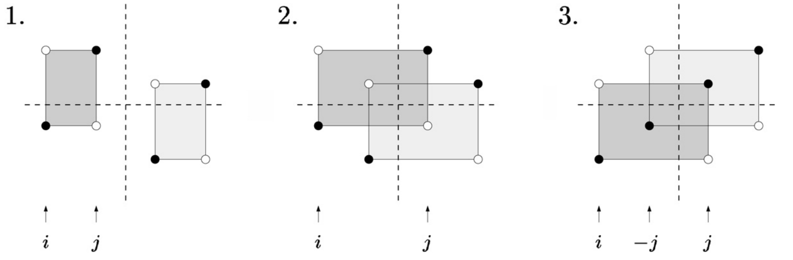

Fig. 5. The covering relation in D_n.

1. a non-central free rise of σ, or
2. a central non-symmetric free rise of σ, or
3. a central semi-free rise of σ.

Definition 6.2 is illustrated in Fig. 5, where we use the same notation as in Fig. 4.

We set

$$H_{D_n} = \{(\sigma, \tau) \in D_n^2 : (\sigma, \tau) \text{ is a good pair in } D_n\},$$

and define the *standard labelling* λ of D_n by associating with every good pair $(\sigma, \tau) \in H_{D_n}$, the pair $(i, j) \in [\pm n]^2$ mentioned in Definition 6.2, which is obviously unique.

It is easy to see that H_{D_n} is a successor system of D_n and, since τ is uniquely determined by σ and by the label (i, j), λ is a good labelling.

Given $\sigma \in D_n$, the *suitable labels* of σ are then the non-central free rises of σ, the central non-symmetric free rises of σ and the central semi-free rises of σ. If (i, j) is a suitable label of σ then the *transformation* of σ with respect to (i, j) is

$$t^{D_n}_{(i,j)}(\sigma) = \sigma(i, j)(-i, -j).$$

In order to prove that the insertion property holds, we need the following definition, which can be given in general for the symmetric group, and which the hyperoctahedral group and the even-signed permutation group inherit.

Definition 6.3. Let $\sigma, \tau \in S_n$, with $\sigma < \tau$. Suppose that the set

$$\{j \in [ci + 1, n] : \sigma(j) \in [\sigma(di) + 1, \sigma(ci) - 1]\}$$

is not empty. Then the *second covering index* of σ with respect to τ, denoted by $sci_\tau(\sigma)$ (or simply sci), is

$$sci_\tau(\sigma) = \min\{j \in [ci + 1, n] : \sigma(j) \in [\sigma(di) + 1, \sigma(ci) - 1]\}.$$

Definition 6.4. Let $\sigma, \tau \in D_n$, with $\sigma < \tau$. We say that (σ, τ) is a *D-special pair* if

1. $(di <)\ ci < 0$;
2. $(\sigma(di) <)\ \sigma(ci) < 0$;
3. $\tau(di) = -\sigma(ci)$;
4. $[ci + 1, -ci - 1] \times [\sigma(ci), -\sigma(ci)]$ is empty for σ.
   Moreover, a special pair (σ, τ) can be either *of the first kind*, if

5′. $[ci+1, -ci-1] \times [\sigma(di), -\sigma(di)]$ is not empty for σ,
or *of the second kind*, if
5″. $[ci+1, -ci-1] \times [\sigma(di), -\sigma(di)]$ is empty for σ.

Let $\sigma, \tau \in D_n$, with $\sigma < \tau$. We define the label

$$i_{D_n}(\sigma,\tau) = \begin{cases} (di, ci), & \text{if } (\sigma,\tau) \text{ is not a } D\text{-special pair,} \\ (di, sci), & \text{if } (\sigma,\tau) \text{ is a } D\text{-special pair of the first kind,} \\ (di, -ci), & \text{if } (\sigma,\tau) \text{ is a } D\text{-special pair of the second kind.} \end{cases}$$

Note that, if (σ, τ) is a D-special pair of the first kind, then, by 4 and 5′, sci necessarily exists. Also note that $i_{D_n}(\sigma, \tau)$ is always a suitable label of σ, so we can define the even-signed permutation

$$\chi_{D_n}(\sigma,\tau) = t^{D_n}_{i_{D_n}(\sigma,\tau)}(\sigma).$$

All cases are shown in Fig. 6, where σ, τ and $\chi = \chi_{D_n}(\sigma, \tau)$ are represented. Black circles denote σ, white squares τ and white circles χ. Only the dots in columns di and $-di$ of τ are represented, possibly with a gray rectangle around, denoting the range of variation $\tau(di)$. Inside the gray rectangles there are no dots of σ and χ other than those indicated and the diagrams of σ and χ are supposed to be the same anywhere else.

If (σ, τ) is not a D-special pair, we distinguish between the following cases:

1. $(di <)\ 0 < ci, \sigma(di) < 0 < \sigma(ci)$;
2. $(di <)\ 0 < ci, 0 < \sigma(di)\ (< \sigma(ci))$;
3. $(di <)\ 0 < ci, (\sigma(di) <)\ \sigma(ci) < 0$;
4. $(di <)\ ci < 0, 0 < \sigma(di)\ (< \sigma(ci))$;
5. $(di <)\ ci < 0, \sigma(di) < 0 < \sigma(ci), \sigma(ci) > -\sigma(di)$;
6. $(di <)\ ci < 0, \sigma(di) < 0 < \sigma(ci), \sigma(ci) < -\sigma(di)$;
7. $(di <)\ ci < 0, (\sigma(di) <)\ \sigma(ci) < 0, \tau(di) \neq -\sigma(ci)$;
8. $(di <)\ ci < 0, (\sigma(di) <)\ \sigma(ci) < 0, \tau(di) = -\sigma(ci)$,
   $[ci+1, -ci-1] \times [\sigma(ci), -\sigma(ci)]$ is not empty for σ.

Otherwise (σ, τ) can be either a D-special pair of the first kind:

9. $(di <)\ ci < 0, (\sigma(di) <)\ \sigma(ci) < 0, \tau(di) = -\sigma(ci)$,
   $[ci+1, -ci-1] \times [\sigma(ci), -\sigma(ci)]$ is empty for σ,
   but $[ci+1, -ci-1] \times [\sigma(di), -\sigma(di)]$ is not,
   and we distinguish between

   9a. $sci < 0$ and
   9b. $sci > 0$;

or a D-special pair of the second kind:

10. $(di <)\ ci < 0, (\sigma(di) <)\ \sigma(ci) < 0, \tau(di) = -\sigma(ci)$,
    $[ci+1, -ci-1] \times [\sigma(di), -\sigma(di)]$ is empty for σ.

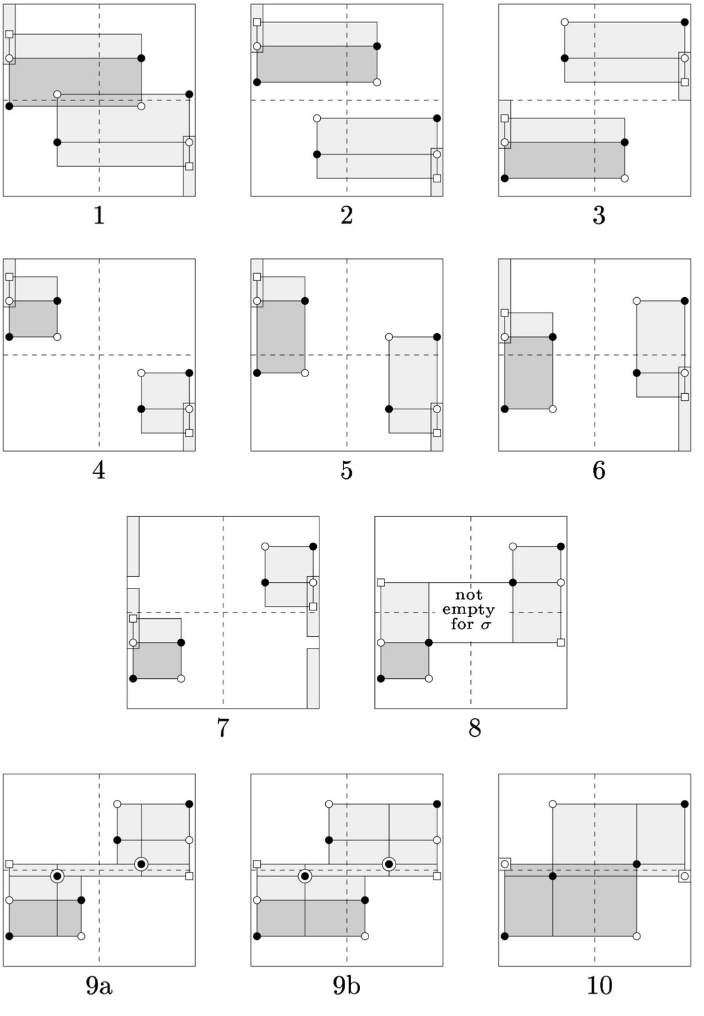

Fig. 6. The minimal covering transformation in D_n.

Theorem 6.5. *Let $\sigma, \tau \in D_n$, with $\sigma < \tau$. Then*

$$\chi_{D_n}(\sigma, \tau) \leq \tau.$$

The proof of Theorem 6.5 is rather technical and will be revealed in the last section.
We recall that the length of $\sigma \in D_n$ is given by

$$l_D(\sigma) = \frac{\mathrm{inv}(\sigma) - \mathrm{neg}(\sigma)}{2}.$$

Proposition 6.6. *The pair (H_{D_n}, l_D) is a covering system of D_n.*

Proof. By Theorem 6.5, H_{D_n} is an insertion system of D_n. The ρ-base property is trivial. It remains to prove the ρ-increasing property. Consider $(\sigma, \tau) \in H_{D_n}$. We refer to the cases as in Fig. 6. We have

$$\mathrm{inv}(\tau) = \begin{cases} \mathrm{inv}(\sigma) + 4, & \text{in cases 1 and 10,} \\ \mathrm{inv}(\sigma) + 2, & \text{in all other cases,} \end{cases}$$

and

$$\mathrm{neg}(\tau) = \begin{cases} \mathrm{neg}(\sigma) + 2, & \text{in cases 1 and 10,} \\ \mathrm{neg}(\sigma), & \text{in all other cases.} \end{cases}$$

Thus in each case $l_D(\tau) = l_D(\sigma) + 1$. □

We have found a covering system of D_n. So we have a characterization of the covering relation in D_n, which we state in the following.

Theorem 6.7. *Let $\sigma, \tau \in D_n$. Then $\sigma \lhd \tau$ if and only if*

$$\tau = \sigma(i, j)(-i, -j),$$

where (i, j) is

1. *a non-central free rise of σ, or*
2. *a central non-symmetric free rise of σ, or*
3. *a central semi-free rise of σ.*

If $\sigma \in D_n$ and (i, j) is a suitable label of σ, then the transformation of σ with respect to (i, j) actually is a *covering transformation*, denoted by

$$ct^{D_n}_{(ij)}(\sigma) = \sigma(i, j)(-i, -j).$$

We now prove that $i_{D_n}(\sigma, \tau)$ is the *minimal label* of σ with respect to τ.

Proposition 6.8. *Let $\sigma, \tau \in D_n$, with $\sigma < \tau$. Then*

$$mi_\tau(\sigma) = i_{D_n}(\sigma, \tau).$$

Proof. Let $\chi = \chi_{D_n}(\sigma, \tau)$. If (σ, τ) is not a D-special pair (case 1), then $i_{D_n}(\sigma, \tau) = (di, ci)$, if (σ, τ) is a D-special pair of the first kind (case 2), then $i_{D_n}(\sigma, \tau) = (di, sci)$, and if (σ, τ) is a D-special pair of the second kind (case 3), then $i_{D_n}(\sigma, \tau) = (di, -ci)$. Let (i, j) be a suitable label of σ such that $ct^{D_n}_{(i,n)}(\sigma) \leq \tau$ and let $\xi = ct^{D_n}_{(i.j)}(\sigma)$. We want to prove that

$$i_{D_n}(\sigma, \tau) \leq (i, j).$$

Necessarily $i \geq di$. If $i > di$ then $i_{D_n}(\sigma, \tau) < (i, j)$. So suppose $i = di$.

In case 1, we have to prove that $j \geq ci$. Suppose, by contradiction, that $j < ci$. By the definition of ci, we have $\sigma(j) > \tau(di)$. But $\xi(di) = \sigma(j)$, so $\xi(di) > \tau(di)$, which contradicts $\xi \leq \tau$.

In cases 2 and 3, we have to prove, respectively, that $j \geq sci$ and $j \geq -ci$. Suppose that the contrary is true. Looking at Fig. 6 (9a, 9b, 10), it is easy to see that, in both cases, the only possibilities are $j = ci$ or $j < ci$ and $\sigma(j) > \tau(di)$. But if $j = ci$, then $\xi = \sigma(di, ci)(-di, -ci)$ and the pair (ξ, τ) does not satisfy the D-condition, since

$(di + 1, -\sigma(ci) + 1)$ is a non-valid D-cell of (ξ, τ), contradicting $\xi \leq \tau$. On the other hand, if $j < ci$ and $\sigma(j) > \tau(di)$, then the conclusion is the same as in case 1. $\square$

Thus in the even-signed permutation group the *minimal covering transformation* of σ with respect to τ is

$$mct_\tau^{D_n}(\sigma) = \begin{cases} \sigma(di, ci)(-di, -ci), & \text{if } (\sigma, \tau) \text{ is not a } D\text{-special pair,} \\ \sigma(di, sci)(-di, -sci), & \text{if } (\sigma, \tau) \text{ is a } D\text{-special pair of the first kind,} \\ \sigma(di, -ci)(-di, ci), & \text{if } (\sigma, \tau) \text{ is a } D\text{-special pair of the second kind,} \end{cases}$$

and this completes the descriptions of the minimal chains in classical Weyl groups.

7. Proof of Theorem 6.5

To prove Theorem 6.5 we need two preliminary lemmas. We first introduce further notation: for $\sigma \in D_n$ and $(h, k) \in [-n] \times [n]$, we set

$$\sigma_N[h, k] = \sigma_{[\pm|h|]\times[k+1,n]}.$$

Lemma 7.1. *Let $\sigma, \tau \in D_n$ be such that (σ, τ) satisfies the B-condition. Let $(h, k) \in [-n] \times [n]$ be such that $\sigma_{NW}[h, k] = \tau_{NW}[h, k]$. If (h, k) is free for σ, then (h, k) is also free for τ, hence it is a D-cell of (σ, τ).*

Proof. Consider the equality $\sigma_{[h]\times[\pm n]} = \tau_{[h]\times[\pm n]} (=h)$. We have $\sigma_{[h]\times[\pm n]} = \sigma_N[h, k]$ (since (h, k) is free for σ) and $\tau_{[h]\times[\pm n]} = \tau_N[h, k] + \tau_{[h]\times[\pm k]}$. So

$$\tau_{[h]\times[\pm k]} = \sigma_N[h, k] - \tau_N[h, k],$$

and

$$\sigma_N[h, k] \geq \tau_N[h, k].$$

On the other hand, by the B-condition, we have $\sigma[|h|, k + 1] \leq \tau[|h|, k + 1]$, that is

$$\sigma_{NW}[h, k] + \sigma_N[h, k] \leq \tau_{NW}[h, k] + \tau_N[h, k].$$

So, by $\sigma_{NW}[h, k] = \tau_{NW}[h, k]$, we have

$$\sigma_N[h, k] \leq \tau_N[h, k].$$

Thus $\sigma_N[h, k] = \tau_N[h, k]$ and $\tau_{[h]\times[\pm k]} = 0$, that is, (h, k) is free for τ. $\square$

For the second lemma, we introduce the following notation: for $\sigma \in D_n$, $(h, k) \in [-n] \times [n]$ and $k_1 \in [k]$, we set

$$\sigma_{\text{left}}[h; k_1, k] = \sigma_{[-n,h-1]\times[k_1,k]},$$
$$\sigma_{\text{right}}[h; k_1, k] = \sigma_{[h,n]\times[k_1,k]}.$$

Lemma 7.2. *Let $\sigma, \tau \in D_n$ be such that (σ, τ) satisfies the B-condition. Let $(h, k) \in [-n] \times [n]$ be such that $\sigma_{NW}[h, k] = \tau_{NW}[h, k]$. Let $k_1 \in [k]$. Set*

$$\sigma_{\text{left}} = \sigma_{\text{left}}[h; k_1, k],$$

and similarly for σ_{right}, τ_{left} *and* τ_{right}. *Then*

$$\begin{cases}\tau_{\text{right}} \le \sigma_{\text{right}},\\ \sigma_{\text{left}} \le \tau_{\text{left}} \le \sigma_{\text{left}} + \sigma_{\text{right}}.\end{cases}$$

Moreover, if $\sigma_{\text{right}} \le \tau_{\text{right}}$, *in particular if* $\sigma_{\text{right}} = 0$, *then*

$$\begin{cases}\tau_{\text{right}} = \sigma_{\text{right}},\\ \tau_{\text{left}} = \sigma_{\text{left}},\end{cases}$$

and if $k_1 \in [2, k]$ *we have*

$$\sigma_{NW}[h, k_1 - 1] = \tau_{NW}[h, k_1 - 1];$$

otherwise, if $k_1 = 1$, *we have*

$$\sigma_{W_{\text{up}}}[h, k] = \tau_{W_{\text{up}}}[h, k].$$

Proof. By the B-condition, we have $\sigma[h-1, k_1] \le \tau[h-1, k_1]$, that is

$$\sigma_{NW}[h, k] + \sigma_{\text{left}} \le \tau_{NW}[h, k] + \tau_{\text{left}}.$$

So, by $\sigma_{NW}[h, k] = \tau_{NW}[h, k]$, we have $\sigma_{\text{left}} \le \tau_{\text{left}}$.

Consider the equality $\sigma_{[\pm n]\times[k_1,k]} = \tau_{[\pm n]\times[k_1,k]} (= k - k_1 + 1)$, that is

$$\sigma_{\text{left}} + \sigma_{\text{right}} = \tau_{\text{left}} + \tau_{\text{right}}.$$

It follows that

$$\sigma_{\text{right}} - \tau_{\text{right}} = \tau_{\text{left}} - \sigma_{\text{left}} \ge 0.$$

So $\tau_{\text{right}} \le \sigma_{\text{right}}$ and $\sigma_{\text{left}} \le \tau_{\text{left}} \le \sigma_{\text{left}} + \sigma_{\text{right}}$.

If $\sigma_{\text{right}} \le \tau_{\text{right}}$ then obviously $\tau_{\text{right}} = \sigma_{\text{right}}$ and $\tau_{\text{left}} = \sigma_{\text{left}}$.

In this case, if $k_1 \in [2, k]$ we have

$$\sigma_{NW}[h, k_1 - 1] = \sigma_{NW}[h, k] + \sigma_{\text{left}} = \tau_{NW}[h, k] + \tau_{\text{left}} = \tau_{NW}[h, k_1 - 1],$$

and if $k_1 = 1$ we have

$$\sigma_{W_{\text{up}}}[h, k] = \sigma_{\text{left}} = \tau_{\text{left}} = \tau_{W_{\text{up}}}[h, k]. \quad \square$$

We can now prove Theorem 6.5.

Proof. Let $\chi = \chi_{D_n}(\sigma, \tau)$. We recall that

$$\chi = \begin{cases}\sigma(di, ci)(-di, -ci), & \text{if } (\sigma, \tau) \text{ is not a } D\text{-special pair},\\ \sigma(di, sci)(-di, -sci), & \text{if } (\sigma, \tau) \text{ is a } D\text{-special pair of the first kind},\\ \sigma(di, -ci)(-di, ci), & \text{if } (\sigma, \tau) \text{ is a } D\text{-special pair of the second kind}.\end{cases}$$

We refer to the cases as in Fig. 6. Let us show, case by case, that $\chi \le \tau$. In every case we may assume, without loss of generality, that $di = -n$.

In all cases, except 1, 9 and 10, we have

$$\chi = mct_\tau^{B_n}(\sigma),$$

so by Proposition 3.6 and Theorem 2.7, the pair (χ, τ) satisfies the B-condition.

In case 1, in order to prove that (χ, τ) satisfies the B-condition, we only have to show that $\sigma[h, k] \leq \tau[h, k] - 2$, when $h \in [-ci]$ and $k = -\sigma(di)$. We have

$$\begin{aligned}\sigma[h, k] &= \sigma[h, \tau(di) + 1] \\ &\leq \tau[h, \tau(di) + 1] \\ &= \tau[h, k] - 1 - \tau_{[di,h]\times[k,\tau(di)-1]} \\ &\leq \tau[h, k] - 1.\end{aligned}$$

Suppose, by contradiction, that $\sigma[h, k] = \tau[h, k] - 1$. It follows that $\sigma[h, \tau(di) + 1] = \tau[h, \tau(di) + 1]$ and $\tau_{[di,h]\times[k,\tau(di)-1]} = 0$. The pair (σ, τ) satisfies the B-condition, $\sigma_{NW}[h + 1, \tau(di)] = \tau_{NW}[h + 1, \tau(di)]$ and $(h + 1, \tau(di))$ is free for σ, thus, by Lemma 7.1, $(h + 1, \tau(di))$ is a D-cell of (σ, τ). Since (σ, τ) satisfies the D-condition, it has to be valid, that is, $\sigma_{W_{\mathrm{up}}}[h+1, \tau(di)] \equiv \tau_{W_{\mathrm{up}}}[h+1, \tau(di)]$. But $\sigma_{W_{\mathrm{up}}}[h+1, \tau(di)] = 0$ and, since $\tau_{[di,h]\times[k,\tau(di)-1]} = 0$, we have $\tau_{W_{\mathrm{up}}}[h + 1, \tau(di)] = 1$, a contradiction. Thus $\sigma[h, k] \leq \tau[h, k] - 2$.

In case 10, for the B-condition of (χ, τ), we have again to show that $\sigma[h, k] \leq \tau[h, k]-2$, when $h \in [ci]$ and $k = -\sigma(ci)$. As before, we have $\sigma[h, k] \leq \tau[h, k]-1$ and, supposing by contradiction that $\sigma[h, k] = \tau[h, k]-1$, we get $\sigma_{NW}[h+1, \tau(di)] = \tau_{NW}[h+1, \tau(di)]$. Now $(h + 1, \tau(di))$ is obviously free for both σ and τ. So $(h + 1, \tau(di))$ is a D-cell of (σ, τ) and the conclusion is the same as before.

In case 9a, the B-condition of (χ, τ) is proved if we show that $\sigma[h, k] \leq \tau[h, k] - 1$, when $(h, k) \in [ci, sci - 1] \times [\sigma(di) + 1, \sigma(sci)]$. If we suppose, by contradiction, that $\sigma[h, k] = \tau[h, k]$, we get $\sigma_{NW}[h + 1, \tau(di)] = \tau_{NW}[h + 1, \tau(di)]$, with the same conclusion as in previous cases.

Finally, in case 9b, we have to show that $\sigma[h, k] \leq \tau[h, k] - 1$, when $(h, k) \in [ci] \times [\sigma(di) + 1, \sigma(sci)]$ or $(h, k) \in [-sci]\times[-\sigma(sci) + 1, -\sigma(di)]$. If $(h, k) \in [ci]\times[\sigma(di)+1, \sigma(sci)]$ this is proved as in case 9a. If $(h, k) \in [-sci]\times[-\sigma(sci) + 1, -\sigma(di)]$, suppose by contradiction that $\sigma[h, k] = \tau[h, k]$, that is, $\sigma_{NW}[h + 1, k - 1] = \tau_{NW}[h + 1, k - 1]$. Then, by the D-condition of (σ, τ), we get $\sigma_{W_{\mathrm{up}}}[h+1, k-1] \equiv \tau_{W_{\mathrm{up}}}[h+1, k-1]$. On the other hand, since $\sigma_{\mathrm{right}}[h+1; \tau(di)+1, k] = 0$, by Lemma 7.2 we get $\sigma_{\mathrm{left}}[h+1; \tau(di)+1, k] = \tau_{\mathrm{left}}[h+1; \tau(di)+1, k]$, which implies $\tau_{W_{\mathrm{up}}}[h+1, k-1] = 1+\sigma_{W_{\mathrm{up}}}[h+1, k-1]$, a contradiction.

It remains to prove that (χ, τ) satisfies the D-condition. If (h, k) is a D-cell of (χ, τ) which is also a D-cell of (σ, τ), then it has to be valid for (σ, τ), and this necessarily implies that it is also valid for (χ, τ), as can be easily checked in every case. So, case by case, we have to look for the D-cells of (χ, τ) which are *not* D-cells of (σ, τ) (we call them *new* D-cells) and show that they are valid for (χ, τ).

In case 1, if (h, k) is a new D-cell, then $(h, k) \in [-ci + 1] \times [-\sigma(di), \sigma(ci) - 1]$ and $\sigma_{NW}[h, k] = \tau_{NW}[h, k] - 1$. From

$$\sigma_{NW}[h, k] = \sigma_{NW}[h, \tau(di)] \leq \tau_{NW}[h, \tau(di)] \leq \tau_{NW}[h, k] - 1,$$

it follows that $\sigma_{NW}[h, \tau(di)] = \tau_{NW}[h, \tau(di)]$. Moreover $(h, \tau(di))$ is free for σ. So, by Lemma 7.1, $(h, \tau(di))$ is a D-cell of (σ, τ). By the D-condition of (σ, τ), it has to be valid, that is, $\sigma_{N\text{left}}[h, \tau(di)] \equiv \tau_{N\text{left}}[h, \tau(di)]$. Since $(h, \tau(di))$ is free for both σ and τ, this implies $\chi_{N\text{left}}[h, k] = \sigma_{N\text{left}}[h, k] \equiv \tau_{N\text{left}}[h, k]$. Thus (h, k) is valid for (σ, χ).

In case 2, if (h, k) is a new D-cell, then $(h, k) \in [-ci + 1] \times [\sigma(di), \sigma(ci) - 1]$ and $\sigma_{NW}[h, k] = \tau_{NW}[h, k] - 1$. In this case the reasoning is the same as in case 1.

In case 3, if (h, k) is a new D-cell, then there are two subcases: either $(3')(h, k) \in [di + 1, -ci] \times [-\sigma(ci), -\sigma(di) - 1]$, or $(3'')(h, k) \in [-ci + 1] \times [-\sigma(ci), -\sigma(di) - 1]$ and $\sigma_{NW}[h, k] = \tau_{NW}[h, k] - 1$.

In subcase $3'$ we have $\sigma_{NW}[h, k] = \tau_{NW}[h, k]$ and $\sigma_{\text{right}}[h; -\tau(di), k] = 1 \leq \tau_{\text{right}}[h; -\tau(di), k]$. So, by Lemma 7.2, we have

$$\sigma_{\text{left}}[h; -\tau(di), k] = \tau_{\text{left}}[h; -\tau(di), k] \tag{3}$$

and $\sigma_{NW}[h, -\tau(di) - 1] = \tau_{NW}[h, -\tau(di) - 1]$. Thus $(h, -\tau(di) - 1)$ is a D-cell of (σ, τ), which has to be valid, that is, $\sigma_{W_{\text{up}}}[h, -\tau(di) - 1] \equiv \tau_{W_{\text{up}}}[h, -\tau(di) - 1]$. This, together with (3), implies $\chi_{W_{\text{up}}}[h, k] = \sigma_{W_{\text{up}}}[h, k] \equiv \tau_{W_{\text{up}}}[h, k]$.

In subcase $3''$ we have $\chi_{NW}[h, k] = \tau_{NW}[h, k]$ and $\chi_{\text{right}}[h; -\tau(di), k] = 1 \leq \tau_{\text{right}}[h; -\tau(di), k]$. So, by Lemma 7.2, we have

$$\chi_{\text{left}}[h; -\tau(di), k] = \tau_{\text{left}}[h; -\tau(di), k] \tag{4}$$

and $\chi_{NW}[h, -\tau(di) - 1] = \tau_{NW}[h, -\tau(di) - 1]$. But $\sigma_{NW}[h, -\tau(di) - 1] = \chi_{NW}[h, -\tau(di) - 1]$; thus $(h, -\tau(di) - 1)$ is a D-cell of (σ, τ), which has to be valid, so $\chi_{W_{\text{up}}}[h, -\tau(di) - 1] = \sigma_{W_{\text{up}}}[h, -\tau(di) - 1] \equiv \tau_{W_{\text{up}}}[h, -\tau(di) - 1]$. This, together with (4), implies $\chi_{W_{\text{up}}}[h, k] \equiv \tau_{W_{\text{up}}}[h, k]$.

In cases 4 and 5 there are no new D-cells.

In case 6, if (h, k) is a new D-cell, then $(h, k) \in [di + 1, ci] \times [\sigma(ci), -\sigma(di) - 1]$. The pair (χ, τ) satisfies the B-condition, we have $\chi_{NW}[h, k] = \tau_{NW}[h, k]$ and $\chi_{\text{right}}[h; 1, k] = 0$. So, by Lemma 7.2, $\chi_{W_{\text{up}}}[h, k] = \tau_{W_{\text{up}}}[h, k]$.

In case 7, if $\tau(di) > 0$ then there are no new D-cells. In fact, if $\tau(di) \in [-\sigma(ci) - 1]$, then the presence of a new D-cells implies that $(di + 1, \tau(di))$ is a non-valid D-cell of (σ, τ), contradicting $\sigma < \tau$. If $\tau(di) \in [-\sigma(ci) + 1, n]$ and (h, k) is a new D-cell, then $(h, k) \in [di + 1, ci] \times [-\sigma(ci), -\sigma(di) - 1]$. In particular, if $\tau(di) \in [-\sigma(ci) + 1, -\sigma(di) - 1]$, since $\sigma^{-1}(\tau(di)) \in [ci + 1, -ci - 1]$, then $k \in [-\sigma(ci), \tau(di) - 1]$. So

$$\sigma_{NW}[h, k] = \sigma[h - 1, \tau(di) + 1] \leq \tau[h - 1, \tau(di) + 1] \leq \tau_{NW}[h, k] - 1,$$

contradicting $\sigma_{NW}[h, k] = \tau_{NW}[h, k]$. It remains to consider case 7, when $\tau(di) < 0$, that is, when $\tau(di) \in [\sigma(ci)]$. If (h, k) is a new D-cell, then $(h, k) \in [di + 1, ci] \times [-\sigma(ci), -\sigma(di) - 1]$. We have $\sigma_{\text{right}}[h; -\tau(di), k] = 1 \leq \tau_{\text{right}}[h; -\tau(di), k]$, and the conclusion is the same as in subcase $3'$.

In case 8 there are no new D-cells.

In case 9a, if (h, k) is a new D-cell, then $(h, k) \in [ci + 1, sci] \times [-\sigma(sci), -\sigma(di) - 1]$. We have $\sigma_{\text{right}}[h; \tau(di) + 1, k] = 1$, so, by Lemma 7.2, either $\tau_{\text{left}} = \sigma_{\text{left}}$ or $\tau_{\text{left}} = \sigma_{\text{left}} + 1$. If we suppose, by contradiction, that $\tau_{\text{left}} = \sigma_{\text{left}}$, then $(h, \tau(di) + 1)$ is a D-cell of (σ, τ), which has to be valid. But $\sigma_{W_{\text{up}}}[h, \tau(di) + 1] = 0 \not\equiv 1 = \tau_{W_{\text{up}}}[h, \tau(di) + 1]$, a contradiction. So $\tau_{\text{left}} = \sigma_{\text{left}} + 1$ and $\sigma_{W_{\text{up}}}[h, k] = \tau_{W_{\text{up}}}[h, k] + 2$. Thus $\chi_{W_{\text{up}}}[h, k] = \sigma_{W_{\text{up}}}[h, k] \equiv \tau_{W_{\text{up}}}[h, k]$.

In case 9b, if (h, k) is a new D-cell, then either $(9\text{b}')(h, k) \in [ci + 1, -sci] \times [-\sigma(sci), -\sigma(di) - 1]$, or $(9\text{b}'')(h, k) \in [-sci + 1] \times [-\sigma(sci), -\sigma(di) - 1]$. In subcase 9b′ the reasoning is the same as in case 9a. In subcase 9b″ we have $\chi_{NW}[h, k] = \tau_{NW}[h, k]$ and $\sigma_{\text{right}}[h; \tau(di)+1, k] = 1$. So, by Lemma 7.2, either $\tau_{\text{left}} = \chi_{\text{left}}$ or $\tau_{\text{left}} = \chi_{\text{left}}+1$. If we suppose, by contradiction, that $\tau_{\text{left}} = \chi_{\text{left}}$, then $\sigma_{NW}[h, \tau(di)+1] = \chi_{NW}[h, \tau(di)+1] = \tau_{NW}[h, \tau(di) + 1]$. So, by the D-condition of (σ, τ), we get $\sigma_{W_{\text{up}}}[h, \tau(di) + 1] \equiv \tau_{W_{\text{up}}}[h, \tau(di)+1]$. But $\sigma_{W_{\text{up}}}[h, \tau(di)+1] = 0 \not\equiv 1 = \tau_{W_{\text{up}}}[h, \tau(di)+1]$, a contradiction. So $\tau_{\text{left}} = \chi_{\text{left}} + 1$, which implies $\tau_{W_{\text{up}}}[h, k] = \chi_{W_{\text{up}}}[h, k] + 2$, that is, (h, k) is valid for (χ, τ).

In case 10, if (h, k) is a new D-cell, then $(h, k) \in [di + 1] \times [-\sigma(di) - 1]$. The pair (χ, τ) satisfies the B-condition; we have $\chi_{NW}[h, k] = \tau_{NW}[h, k]$ and $[h, n] \times [k]$ is empty for χ. Thus, as in case 6, by Lemma 7.2 we get $\chi_{W_{\text{up}}}[h, k] = \tau_{W_{\text{up}}}[h, k]$, that is, once again, (h, k) is valid for (χ, τ).

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