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The homotopy analysis method for solving the Fornberg–Whitham equation and comparison with Adomian's decomposition method

Fayçal Abidi [a], Khaled Omrani [b,*]

[a] Ecole Polytechnique de Tunis, Rue El Khawarezmi, 2078 La Marsa, Tunisia

[b] Institut Supérieur des Sciences Appliquées et de Technologie de Sousse, 4003 Sousse Ibn Khaldoun, Tunisia

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ABSTRACT

In this work, an analytical technique, namely the homotopy analysis method (HAM), is applied to obtain an approximate analytical solution of the Fornberg–Whitham equation. A comparison is made between the HAM results and the Adomian's decomposition method (ADM) and the homotopy perturbation method (HPM). The results reveal that HAM is very simple and effective. The HAM contains the auxiliary parameter $\hbar$, which provides us with a simple way to adjust and control the convergence region of solution series.

1. Introduction

The Fornberg–Whitham equation [1] given as

$$u_t - u_{xxt} + u_x = uu_{xxx} - uu_x + 3u_x u_{xx}, \tag{1.1}$$

has a type of traveling wave solution called a kink-like wave solution and anti kink-like wave solutions. Such kinds of traveling wave solutions have never been found for the Fornberg–Whitham equation. Eq. (1.1) was used to study the qualitative behaviour of wave-breaking [2,3].

At the beginning of the 80s, a new method later called ADM for solving various kinds of nonlinear equations had been proposed by Adomian [4,5]. The convergence of Adomian's method has been investigated by several authors (see e.g., [6–9]). In recent years, a large amount of literature developed concerning the ADM [10–15] by applying it to a large number of applications in applied sciences.

The HAM is proposed in 1992 by Liao [16–20]. This method has been successfully applied to solve many types of nonlinear problems in science and engineering by many authors; see [21–31] and references therein.

In this article, we shall apply HAM to find the approximate analytical solution of the FW equation and compare it with the exact solution. With the present method, numerical results can be obtained by using a few iterations. The HAM contains the auxiliary parameter $\hbar$, which provides us with a simple way to adjust and control the convergence region of solution series for any values of x and t.

The paper is organized as follows. In Section 2, the analysis of ADM for the nonlinear FW equation is established. In Section 3, we apply the HAM for solving the nonlinear FW equation. In the last section, numerical experiments are given in order to assess the efficiency and convenience of the HAM.

* Corresponding author.
E-mail addresses: abidifaycel@yahoo.com (F. Abidi), Khaled.Omrani@issatso.rnu.tn (K. Omrani).

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2. Analysis of the Adomian decomposition method

2.1. The Adomian decomposition method

In this section, our attention will focus on the homogeneous Fornberg–Whitham equation. Let us consider the standard form of (1.1) in the operator form

$$L_t u - L_{xxt}(u) + L_x(u) + N(u) = 0, \tag{2.1}$$

where the notations $L_t = \frac{\partial}{\partial t}$, $L_x = \frac{\partial}{\partial x}$, $L_{xx} = \frac{\partial^2}{\partial x^2}$, $L_{xxt} = \frac{\partial^3}{\partial x^2 \partial t}$ symbolize the linear differential operators and the notation $N(u) = \frac{1}{2} L_x[u^2 - L_{xx}(u^2)]$ symbolizes the nonlinear operator. The inverse operator of L_t, noted L_t^{-1}, is defined by

$$L_t^{-1} = \int_0^t (\cdot) \mathrm{d}s.$$

Thus, applying the inverse operator L_t^{-1} to (2.1) yields

$$u(x,t) = u(x,0) + L_t^{-1}\left[L_{xxt}(u) - L_x(u) - N(u)\right]. \tag{2.2}$$

The ADM [4,5] assumes an infinite series solutions for unknown function $u(x,t)$ given by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{2.3}$$

The nonlinear operator $N(u)$ is decomposed as

$$N(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n), \tag{2.4}$$

where A_n is an appropriate Adomian's polynomial which can be calculated for all forms of nonlinearity according to specific algorithms constructed by Adomian [4,5]. For nonlinearity operator $N(u)$, these polynomials can be calculated using the basic formula:

$$A_n(u_0, u_1, \ldots, u_n) = \frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} N\left(\sum_{k=0}^{\infty} \lambda^k u_k\right)\right]_{\lambda=0}, \quad n \geq 0. \tag{2.5}$$

This formula is easy to set a computer code to get many polynomials as we need in the calculation of the numerical solution. We give here the first few Adomian polynomials for $N(u)$ given as:

$$\begin{aligned}
A_0 &= N(u_0),\\
A_1 &= N^{(1)}(u_0)\frac{u_1^1}{1!},\\
A_2 &= N^{(1)}(u_0)u_2 + N^{(2)}(u_0)\frac{u_1^2}{2!},\\
A_3 &= N^{(1)}(u_0)u_3 + N^{(2)}(u_0)u_1u_2 + N^{(3)}(u_0)\frac{u_1^3}{3!},\\
A_4 &= N^{(1)}(u_0)u_4 + N^{(2)}(u_0)\left(u_1u_3 + \frac{1}{2}u_2^2\right) + \frac{1}{2}N^{(3)}(u_0)u_1^2u_2 + N^{(4)}(u_0)\frac{u_1^4}{4!},\\
A_5 &= N^{(1)}(u_0)u_5 + N^{(2)}(u_0)(u_1u_4 + u_2u_3) + \frac{1}{2}N^{(3)}(u_0)(u_1^2u_3 + u_1u_2^2) + \frac{1}{6}N^{(4)}(u_0)u_1^3u_2 + N^{(5)}(u_0)\frac{u_1^5}{5!},\\
&\vdots
\end{aligned}$$

The rest of the polynomials can be constructed in a similar manner.

Substituting the initial condition into (2.2) identifying the zeroth component u_0 by terms arising from initial condition. Then, we obtain the subsequent components by the following recursive relationship:

$$u_0 = u(x,0) \tag{2.6}$$

$$u_{n+1} = L_t^{-1}\left[L_{xxt}(u_n) - L_x(u_n) - N(u_n)\right]. \tag{2.7}$$

The remaining components $u_n,\ n \geq 1$, can be completely determined such that each term is computed by using the previous term. As a result, the components $u_0, u_1, \ldots$ are identified and the series solutions thus entirely determined. For later numerical computation, let the expansion $\phi_n = \sum_{k=0}^{n-1} u_k(x,t)$ denote the n-term approximation to u.

The exact solution is $u(x,t) = \lim_{n\to\infty} \phi_n(x,t)$. We will show through several examples, that the number of terms required to obtain an accurate computable solution is small.

3. Homotopy analysis method

To describe the basic ideas of the HAM, we consider the following differential equation:

$$N[u(x,t)] = 0 \tag{3.1}$$

where N is a nonlinear operator, $u(x,t)$ is an unknown function and x and t denote spatial and temporal independent variables, respectively.

By means of generalizing the traditional homotopy method, (see Liao [17])

$$(1-p)L[\phi(x,t;p) - u_0(x,t)] = p\hbar N[\phi(x,t;p)] \tag{3.2}$$

where $p \in [0,1]$ is an embedding parameter, $\hbar$ is a nonzero auxiliary parameter, L is an auxiliary linear operator, $u_0(x,t)$ is an initial guess of $u(x,t)$ and $\phi(x,t;p)$ is an unknown function. It is important to note that we have great freedom to choose auxiliary objects such as $\hbar$ and L in HAM. Obviously, when $p=0$ and $p=1$, it holds

$$\phi(x,t;0) = u_0(x,t), \qquad \phi(x,t;1) = u(x,t) \tag{3.3}$$

respectively. Thus, as p increases from 0 to 1, the solution $\phi(x,t;p)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. Expanding $\phi(x,t;p)$ in Taylor series with respect to p, one has

$$\phi(x,t;p) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t)p^m, \tag{3.4}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;p)}{\partial p^m}\Big|_{p=0}. \tag{3.5}$$

If the auxiliary linear operator, the initial guess and the auxiliary parameter $\hbar$ and the auxiliary function are so properly chosen, then, as proved by Liao [17], the series (3.4) converges at $p=1$ and one has

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t), \tag{3.6}$$

which must be one of solutions of the original nonlinear equation, as proved by Liao [17]. As $\hbar = -1$, Eq. (3.2) becomes

$$(1-p)L[\phi(x,t;p) - u_0(x,t)] + pN[\phi(x,t;p)] = 0, \tag{3.7}$$

which is used in the homotopy perturbation method [32].

According to the definition (3.5), the governing equation of can be deduced from the zero-order deformation equation (3.2). Define the vector

$$\overrightarrow{u_n} = \{u_0(x,t), u_1(x,t), u_2(x,t), \ldots, u_n(x,t)\}.$$

Differentiating Eq. (3.2) m times with respect to the embedding parameter p and then setting $p=0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation,

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar \Re_m[\overrightarrow{u}_{m-1}(x,t)], \tag{3.8}$$

where

$$\Re_m(\overrightarrow{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(x,t;p)]}{\partial p^{m-1}}\Big|_{p=0}, \tag{3.9}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m \ge 2. \end{cases} \tag{3.10}$$

It should be emphasized that $u_m(x,t)$ for $m \ge 1$ is governed by the linear equation (3.8) with the linear boundary conditions that come from the original problem, which can be easily solved by symbolic computation software such as Maple and Mathematica.

4. Numerical applications

In this section, we apply ADM and HAM to solve the Fornberg–Whitham equation. In our work, we use the Maple Package to calculate the numerical solutions obtained by these methods.

Consider the Fornberg–Whitham equation

$$u_t - u_{xxt} + u_x = uu_{xxx} - uu_x + 3u_x u_{xx}, \tag{4.1}$$

subject to the initial condition of

$$u(x,0) = \exp\left(\frac{1}{2}x\right). \tag{4.2}$$

Then, the exact solution is given by:

$$u(x,t) = \exp\left(\frac{1}{2}x - \frac{2}{3}t\right). \tag{4.3}$$

4.0.1. Implementation of ADM

Considering the given initial condition, we can assume $u_0(x,t) = \exp\left(\frac{1}{2}x\right)$ as an initial approximation. We next use the recursive relations (2.6)–(2.7) to obtain the rest of components of $u_n(x,t)$.

$$u_1(x,t) = L_t^{-1}\left[L_{xxt}(u_0) - L_x(u_0) - N(u_0)\right] = -\frac{1}{2}\exp\left(\frac{1}{2}x\right)t, \tag{4.4}$$

$$u_2(x,t) = L_t^{-1}\left[L_{xxt}(u_1) - L_x(u_1) - N(u_1)\right] = \frac{1}{8}\exp\left(\frac{1}{2}x\right)[-t + t^2], \tag{4.5}$$

$$u_3(x,t) = L_t^{-1}\left[L_{xxt}(u_2) - L_x(u_2) - N(u_2)\right] = -\frac{1}{96}\exp\left(\frac{1}{2}x\right)\left[3t - 6t^2 + 2t^3\right], \tag{4.6}$$

$$u_4(x,t) = L_t^{-1}\left[L_{xxt}(u_3) - L_x(u_3) - N(u_3)\right] = \frac{1}{384}\exp\left(\frac{1}{2}x\right)[-3t + 9t^2 - 6t^3 + t^4], \tag{4.7}$$

and the rest of the components of iteration formula (2.7) are obtained. The approximate solution which involves few terms is given by

$$\phi_5 = \sum_{i=0}^{4} u_i = \frac{1}{384}\exp\left(\frac{1}{2}x\right)[384 - 255t + 81t^2 - 14t^3 + t^4]. \tag{4.8}$$

4.1. Implementation of HAM

To solve Eq. (4.1) by means HAM, we choose the initial approximation

$$u_0(x,t) = u(x,0) = \exp\left(\frac{1}{2}x\right), \tag{4.9}$$

Eq. (4.1) suggests the nonlinear operator as

$$\begin{aligned} N[\phi(x,t;p)] = {} & \frac{\partial\phi(x,t;p)}{\partial t} - \frac{\partial^3\phi(x,t;p)}{\partial x^2\partial t} + \frac{\partial\phi(x,t;p)}{\partial x} - \phi(x,t;p)\frac{\partial^3\phi(x,t;p)}{\partial x^3} \\ & + \phi(x,t;p)\frac{\partial\phi(x,t;p)}{\partial x} - 3\frac{\partial\phi(x,t;p)}{\partial x}\frac{\partial^2\phi(x,t;p)}{\partial x^2}, \end{aligned} \tag{4.10}$$

and the linear operator

$$L[\phi(x,t;p)] = \frac{\partial\phi(x,t;p)}{\partial t}, \tag{4.11}$$

with the property

$$L(c_1) = 0,$$

where c_1 is the integration constant.

Using the above definitions, we construct the zeroth-order deformation equation

$$(1-p)L[\phi(x,t;p) - u_0(x,t)] = p\hbar N[\phi(x,t;p)]. \tag{4.12}$$

Obviously, when $p = 0$ and $p = 1$,

$$\phi(x,t;0) = u_0(x,t), \qquad \phi(x,t;1) = u(x,t).$$

Therefore, as the embedding parameter p increases from 0 to 1, $\phi(x,t;p)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$.

Then, we obtain the mth-order deformation equation

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar \Re_m[\overrightarrow{u}_{m-1}], \tag{4.13}$$

subject to initial condition

$$u_m(x,0) = 0,$$

where

$$\begin{aligned}\Re_m(\overrightarrow{u}_{m-1}) = {} & \frac{\partial u_{m-1}(x,t)}{\partial t} - \frac{\partial^3 u_{m-1}(x,t)}{\partial x^2 \partial t} + \frac{\partial u_{m-1}(x,t)}{\partial x} \\ & + \sum_{k=0}^{m-1}\left[-u_k(x,t)\frac{\partial^3 u_{m-1-k}(x,t)}{\partial x^3} + u_k(x,t)\frac{\partial u_{m-1-k}(x,t)}{\partial x} - 3\frac{\partial u_k(x,t)}{\partial x}\frac{\partial^2 u_{m-1-k}(x,t)}{\partial x^2}\right].\end{aligned} \tag{4.14}$$

Now, the solution of the mth-order deformation equation (4.13) for $m \geq 1$ becomes

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar L^{-1}[\Re_m(\overrightarrow{u}_{m-1})]. \tag{4.15}$$

From (4.9) and (4.15), we now successively obtain

$$\begin{aligned}
u_0(x,t) &= \exp\left(\frac{1}{2}x\right), \\
u_1(x,t) &= \hbar t \frac{1}{2}\exp\left(\frac{1}{2}x\right), \\
u_2(x,t) &= \exp\left(\frac{1}{2}x\right)\left[-\left(\frac{4\hbar + 3\hbar^2}{8}\right)t + \frac{\hbar^2 t^2}{8}\right]. \\
u_3(x,t) &= \exp\left(\frac{1}{2}x\right)\left[t\left(\frac{4\hbar + 3\hbar^2}{8} + \frac{12\hbar^2 + 9\hbar^3}{2^5}\right) + t^2\left(\frac{\hbar^2}{2^3} + \frac{4\hbar^2 + 6\hbar^3}{2^5}\right) + \frac{\hbar^3}{3}\frac{t^3}{2^5}\right]. \\
u_4(x,t) &= \exp\left(\frac{1}{2}x\right)\left[t\frac{64\hbar + 144\hbar^2 + 108\hbar^3 + 27\hbar^4}{2^7} + t^2\frac{48\hbar^2 + 72\hbar^3 + 27\hbar^4}{2^7} + \frac{t^3}{3}\frac{12\hbar^3 + 9\hbar^4}{2^6} + \frac{\hbar^4}{3}\frac{t^4}{2^7}\right],
\end{aligned}$$

and so on. Therefore, we use five terms in evaluating the approximate solution

$$u_{app} = \sum_{i=0}^{4} u_i. \tag{4.16}$$

Then,

$$u_{app} = \exp\left(\frac{1}{2}x\right)\left[1 + t\frac{256\hbar + 240\hbar^2 + 108\hbar^3 + 27\hbar^4}{2^7} + t^2\frac{168\hbar^2 + 24\hbar^3 + 27\hbar^4}{2^7} + \frac{t^3}{3}\frac{16\hbar^3 + 9\hbar^4}{2^6} + \frac{t^4}{4}\frac{\hbar^4}{2^7}\right].$$

5. Comparison and discussion

As pointed by Liao [17], the auxiliary parameter $\hbar$ can be employed to adjust the convergence region of the homotopy analysis solution. To investigate the influence of $\hbar$ on the solution series, we plot the so-called $\hbar$-curve of $u_t(0,0)$ obtained from the 5th-order HAM approximation solution as shown in Fig. 1. According to this $\hbar$-curve, it is easy to discover the valid region of $\hbar$ which corresponds to the line segment nearly parallel to the horizontal axis. From Fig. 1 it is clear that the series of $u_t(0,0)$ is convergent when $-1.9 < \hbar < -0.8$.

We get the same value as in ADM (4.8) or HPM when $\hbar = -1$. Therefore, the HAM is rather general and contains the ADM and HPM.

In Table 1, we compute the absolute errors for differences between the exact solution (4.3) and the approximate solution (4.16) obtained by the HAM ($\hbar = -1$) at some points. Besides, the behavior of the exact and approximate solutions are shown in Fig. 2.

In Fig. 3, we study the diagrams of the results obtained by HAM for $\hbar = -1$ and $\hbar = -1.01$ in comparison with exact solution (4.3). We can see that the best value of $\hbar$ in this case is not -1. On the other hand, the HAM solution has the same shape as the exact solution for large range of t, i.e., $t = 5$ as shown in Fig. 3 and Table 3, when we take $\hbar = -1.01$. We can notice hight errors for $\hbar = -1$ (see Table 2). Therefore, based on these present results, we can say that HAM is more effective than ADM.

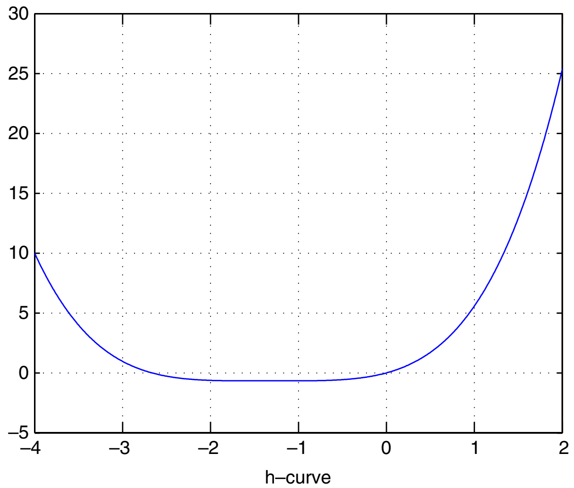

Fig. 1. The $\hbar$-curve of $u_t(0,0)$ given by the 5th-order HAM approximate solution.

Table 1
Absolute errors for differences between the exact solution and 5th-order HAM approximate given by HAM for $\hbar = -1$.

x_i/t_i	0.2	0.4	0.6	0.8	1
−4	2.22193E−5	9.47416E−6	4.83886E−5	6.71560E−5	5.36314E−5
−2	6.03987E−5	2.57532E−5	1.31533E−4	1.82549E−4	1.45785E−4
0	1.64180E−5	7.00049E−5	3.57546E−4	4.96219E−4	3.96285E−4
2	4.46289E−4	1.90293E−4	9.71910E−4	1.34886E−3	1.07721E−3
4	1.21314E−4	5.17269E−4	2.64192E−3	3.66659E−3	2.92817E−3

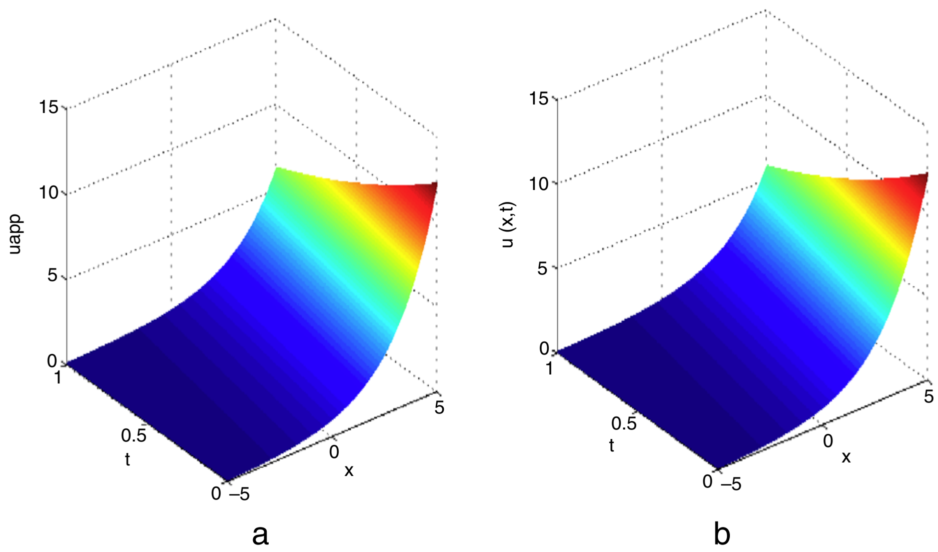

Fig. 2. The behavior of the solutions obtained by (a) HAM for $\hbar = -1$ (or ADM) (b) exact solution.

Table 2
The absolute errors for differences between the exact solution and 5th-order ADM when $t = 5$.

x_i	u_{exact}	u_{ADM}	$\lvert u_{\text{exact}} - u_{\text{ADM}}\rvert$
−4	0.0048279499	0.0031719207	1.65602E−3
−2	0.0131237287	0.0086221743	4.50155E−3
0	0.0356739933	0.0234374990	1.22364E−2
2	0.0969719679	0.0637097231	3.32622E−2
4	0.2635971382	0.1731809521	9.04161E−2

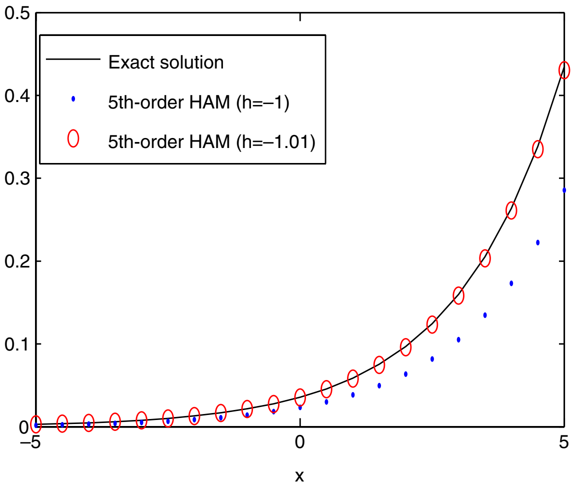

Fig. 3. The results obtained by HAM for various $\hbar$ by 5th-order HAM approximate solution (4.16) in comparison with the exact solution, when $-5 < x < 5$, and $t = 5$.

Table 3
The absolute errors for differences between the exact solution and 5th-order HAM approximate given by HAM for $\hbar = -1.01$ when $t = 5$.

x_i	u_{exact}	u_{HAM}	$\lvert u_{\text{exact}} - u_{\text{HAM}} \rvert$
−4	0.0048279499	0.0048752608	4.73109E−5
−2	0.0131237287	0.0132523327	1.28604E−4
0	0.0356739933	0.0360235773	3.49584E−4
2	0.0969719679	0.0979222379	9.50270E−4
4	0.2635971382	0.2661802682	2.58313E−3

6. Conclusion

In this work, the homotopy analysis method has been applied for finding the approximate solutions of the nonlinear Fornberg–Whitham equation. The numerical results showed that this method has very good accuracy and reductions in the size of calculations compared with the ADM and HPM. In addition, the results of the ADM and HPM can be obtained as a special case of the HAM, when $\hbar = -1$.

The HAM contains the auxiliary parameter $\hbar \neq 0$, which provides us with a simple way to adjust and control the convergence region of solution series for large values of t. Unlike, other numerical methods are given low degree of accuracy for large values of t. It is obvious to see that the HAM is a very powerful, easy and efficient technique for solving various kinds of nonlinear problems in science and engineering without any assumptions and restrictions.

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References

[1] J. Zhou, L. Tian, A type of bounded traveling wave solutions for the Fornberg–Whitham equation, J. Math. Anal. Appl. 346 (2008) 255–261.
[2] G.B. Whitham, Variational methods and applications to water wave, Proc. R. Soc. Lond. A 299 (1967) 6–25.
[3] B. Fornberg, G.B. Whitham, A numerical and theoretical study of certain nonlinear wave phenomena, Phil. Trans. R. Soc. Lond. A 289 (1978) 373–404.
[4] G. Adomian, A review of the decomposition method in applied mathematics, J. Math. Anal. Appl. 135 (1988) 44–501.
[5] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Academic Publishers, Boston, 1999.
[6] Y. Cherruault, Convergence of Adomian's method, Kybernetes 18 (2) (1989) 8–31.
[7] Y. Cherruault, G. Adomian, Decomposition methods: A new proof of convergence, Math. Comput. Modelling 18 (1993) 103–106.
[8] K. Abbaoui, Y. Cherruault, Convergence of Adomian's method applied to differential equations, Comput. Math. Appl. 28 (1994) 8–31.
[9] T. Mavoungou, Y. Cherruault, Convergence of Adomian's method and applications to non-linear partial differential equation, Kybernetes 21 (6) (1992) 13–25.
[10] A.M. Wazwaz, A reliable modification of Adomian decomposition method, Appl. Math. Comput. 102 (1999) 77–86.
[11] A.M. Wazwaz, The existence of noise terms for systems of inhomogeneous differential and integral equations. 146 (1) (2003) pp. 81–92.
[12] A.M. Wazwaz, A new algorithm for calculating Adomian polynomials for nonlinear operators, Appl. Math. Comput. 111 (2000) 53–69.
[13] T. Achouri, K. Omrani, Numerical solutions for the damped generalized regularized long-wave equation with a variable coefficient by Adomian decomposition method, Commun. Nonlinear Sci. Numer. Simul. 14 (2009) 869–877.
[14] D. Kaya, S.M. El-Sayed, An application of the decomposition method for the generalized KdV and RLW equations, Chaos Solitons Fractals 17 (2003) 869–877.

[15] S. Saha Ray, K.S. Chaudhuri, R.K. Bera, Application of modified decomposition method for the analytical solution of space fractional diffusion equation, Appl. Math. Comput. 196 (2008) 2025–2033.
[16] S.J. Liao, The proposed homotopy analysis method techniques for the solution of nonlinear problems, Ph.D. Thesis, Shanghai Jiao Tong University, Shanghai, 1992.
[17] S.J. Liao, Beyond Perturbation: Introduction to the Homotopy Analysis Method, Chapman and Hall/CRC Press, Boca Raton, 2003.
[18] S.J. Liao, An approximate solution technique which does not depend upon small parameters: A special example, Internat. J. Non-Linear Mech. 30 (1995) 371–380.
[19] S.J. Liao, On the homotopy analysis method for nonlinear problems, Appl. Math. Comput. 147 (2004) 499–513.
[20] S.J. Liao, Comparison between the homotopy analysis method and the homotopy perturbation method, Appl. Math. Comput. 169 (2005) 1186–1194.
[21] H. Tari, D.D. Ganji, M. Rostamian, Approximate solutions of K(2, 2), KdV and the modified KdV equations by Hes methods and Liao's method, Int. J. Nonlinear Sci. Numer. Simul. 8 (2) (2007).
[22] S. Abbasbandy, The application of homotopy analysis method to nonlinear equations arising in heat transfer, Phys. Lett. A 360 (2006) 109–113.
[23] S. Abbasbandy, The application of homotopy analysis method to solve a generalized Hirota Satsuma coupled KdV equation, Phys. Lett. A 36 (2007) 478–483.
[24] A. Fakhari, G. Domairry, Ebrahimpour, Approximate explicit solutions of nonlinear BBMB equations by homotopy analysis method and comparison with exact solution, Phys. Lett. A 368 (2007) 64–68.
[25] A. Sami Bataineh, M.S.M. Nourani, I. Hashim, Solving systems of ODEs by homotopy analysis method, Commun. Nonlinear Sci. Numer. Simul. (2007) 5–26.
[26] M. Ayab, A. Rasheed, T. Hayat, Exact flow of a third grade fluid past a porous plate using homotopy analysis method, Internat. J. Engrg. Sci. 41 (2003) 2091–2103.
[27] T. Hayat, M. Khan, Homotopy solutions for a generalized second-grade fluid past a porous plate, Nonlinear Dynam. 42 (2005) 395–405.
[28] M.M. Rashidi, G. Domairry, S. Dinarvand, The homotopy analysis method for explicit analytical solutions of Jaulent–Miodek equations, Numer. Methods Partial Differential Equations 25 (2) (2009) 430–439.
[29] A. Yildirim, H. Koçak, Homotopy perturbation method for solving the space–time fractional advection–dispersion equation, Adv. Water Resour. 32 (2009) 1711–1716.
[30] M.E. Berberler, A. Yildirim, He's homotopy perturbation method for solving the shock wave equation, Appl. Anal. 88 (2009) 997–1004.
[31] H. Koçak, A. Yildirim, Series solution for a delay differential equation arising in electrodynamics, Comm. Numer. Methods Engrg. 25 (2009) 1084–1096.
[32] J.H. He, Homotopy perturbation technique, Comput. Methods Appl. Mech. Engrg. 178 (1999) 257–262.