Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER Computers and Mathematics with Applications 52 (2006) 411–420
An International Journal computers & mathematics with applications
www.elsevier.com/locate/camwa

Existence Results for an Impulsive Abstract Partial Differential Equation with State-Dependent Delay

E. HERNÁNDEZ, M. PIERRI* AND G. GONCALVES†
ICMC, Universidade de São Paulo
São Carlos, Cx 668, 13560-970, SP, Brazil
<lalohm><michelle><gabriel>@icmc.sc.usp.br

(Received May 2005; revised and accepted March 2006)

Abstract—In this paper, we establish the existence of mild solutions for a class of impulsive abstract partial functional differential equation with state-dependent delay.

Keywords—Abstract functional differential equations, Impulsive differential equations, State dependent delay.

1. INTRODUCTION

In this paper, we establish the existence of mild solutions for an impulsive abstract functional differential equation with state-dependent delay described by

$$x'(t) = Ax(t) + f(t, x_{\rho(t,x_t)}), \qquad t \in I = [0, a], \tag{1.1}$$

$$x_0 = \varphi \in \mathcal{B}, \tag{1.2}$$

$$\Delta x(t_i) = I_i(x_{t_i}), \qquad i = 1, \dots, n, \tag{1.3}$$

where A is the infinitesimal generator of a compact C_0-semigroup of bounded linear operators $(T(t))_{t\geq 0}$ defined on a Banach space X; the functions $x_s : (-\infty, 0] \to X$, $x_s(\theta) = x(s+\theta)$, belongs to some abstract phase space $\mathcal{B}$ described axiomatically; $0 < t_1 < \cdots < t_n < a$ are pre-fixed numbers; $f : I \times \mathcal{B} \to X$, $\rho : I \times \mathcal{B} \to (-\infty, a]$, $I_i : \mathcal{B} \to X$, $i = 1, \dots, n$, are appropriate functions and the symbol $\Delta\xi(t)$ represents the jump of the function ξ at t, which is defined by $\Delta\xi(t) = \xi(t^+) - \xi(t^-)$.

The theory of impulsive differential equations has become an important area of investigation in recent years stimulated by their numerous applications to problems arising in mechanics, electrical engineering, medicine, biology, ecology, etc. Relative to ordinary impulsive differential equations,

*I wish to acknowledge the support of Fapep, Brazil, for this research.
†I acknowledge the support of Capes, Brazil, for this research.
The authors wish to thank to the referees for their comments, corrections, and suggestions.

0898-1221/06/$ - see front matter Typeset by AMS-TEX
doi:10.1016/j.camwa.2006.03.022

we cite among other works [1–5]. First-order abstract partial differential equations with impulses are treated in [6–9].

The literature related to ordinary and partial functional differential equations with delay for which $\rho(t,\psi) = t$ is very extensive and we refer the reader to [10,11] concerning this matter.

Functional differential equations with state-dependent delay appear frequently in applications as models of equations and for this reason the study of this type of equations has received great attention in the last years, see, for instance, [12–20] and the references therein. The literature related to partial functional differential equations with state-dependent delay is limited, to our knowledge, to the recent works [21,22]. The study of impulsive partial functional differential equations with state-dependent delay is an untreated topic and it is the motivation of our paper.

2. PRELIMINARIES

Throughout this paper, $A : D(A) \subset X \to X$ is the infinitesimal generator of a compact semigroup of linear operators $(T(t))_{t\geq 0}$ defined on a Banach spaces X and $\tilde{M}$ is a constant such that $\|T(t)\| \leq \tilde{M}$ for every $t \in I = [0,a]$. For related semigroup theory, we suggest [23].

To consider the impulsive condition (1.3), it is convenient to introduce some additional concepts and notations. We say that a function $u : [\sigma,\tau] \to X$ is a normalized piecewise continuous function on $[\sigma,\tau]$ if u is piecewise continuous and left continuous on $(\sigma,\tau]$. We denote by $\mathcal{PC}([\sigma,\tau];X)$ the space formed by the normalized piecewise continuous functions from $[\sigma,\tau]$ into X. In particular, we introduce the space $\mathcal{PC}$ formed by all functions $u : [0,a] \to X$ such that u is continuous at $t \neq t_i, u(t_i^-) = u(t_i)$ and $u(t_i^+)$ exists, for all $i = 1,\dots,n$. In this paper we always assume that $\mathcal{PC}$ is endowed with the norm $\|u\|_{\mathcal{PC}} = \sup_{s\in[0,a]} \|u(s)\|$. It is clear that $(\mathcal{PC}, \|\cdot\|_{\mathcal{PC}})$ is a Banach space.

To simplify the notations, we put $t_0 = 0$, $t_{n+1} = a$ and for $u \in \mathcal{PC}$ we denote by $\tilde{u}_i \in C([t_i,t_{i+1}];X)$, $i = 0,1,\dots,n$, the function given by

$$\tilde{u}_i(t) = \begin{cases} u(t), & \text{for } t \in (t_i,t_{i+1}], \\ u(t_i^+), & \text{for } t = t_i. \end{cases}$$

Moreover, for $B \subseteq \mathcal{PC}$, we denote by $\tilde{B}_i$, $i = 0,1,\dots,n$, the set $\tilde{B}_i = \{\tilde{u}_i : u \in B\}$.

LEMMA 2.1. *A set $B \subseteq \mathcal{PC}$ is relatively compact in $\mathcal{PC}$ if and only if, the set $\tilde{B}_i$ is relatively compact in $C([t_i,t_{i+1}];X)$, for every $i = 0,1,\dots,n$.*

In this work we will employ an axiomatic definition of the phase space $\mathcal{B}$ which is similar to that used in [24]. Specifically, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty,0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$ and we will assume that $\mathcal{B}$ satisfies the following axioms.

(A) If $x : (-\infty,\sigma+b] \to X$, $b > 0$, is such that $x|_{[\sigma,\sigma+b]} \in \mathcal{PC}([\sigma,\sigma+b] : X)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma,\sigma+b]$ the following conditions hold:
  - (i) x_t is in $\mathcal{B}$,
  - (ii) $\|x(t)\| \leq H\|x_t\|_{\mathcal{B}}$,
  - (iii) $\|x_t\|_{\mathcal{B}} \leq K(t-\sigma)\sup\{\|x(s)\| : \sigma \leq s \leq t\} + M(t-\sigma)\|x_\sigma\|_{\mathcal{B}}$, where $H > 0$ is a constant; $K, M : [0,\infty) \to [1,\infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.

(B) The space $\mathcal{B}$ is complete.

EXAMPLE. THE PHASE SPACES $\mathcal{PC}_h(X)$, $\mathcal{PC}_g^0(X)$. Let $g : (-\infty,0] \to [1,\infty)$ be a continuous, nondecreasing function with $g(0) = 1$, which satisfies the conditions (g-1), (g-2) of [24]. This means that the function

$$G(t) := \sup_{-\infty<\theta\leq -t} \frac{g(t+\theta)}{g(\theta)}$$

is locally bounded for $t \geq 0$ and that $\lim_{\theta\to-\infty} g(\theta) = \infty$.

As usual, we said that $\varphi : (-\infty, 0] \to X$ is normalized piecewise continuous, if φ is left continuous and the restriction of φ to any interval $[-r, 0]$ is piecewise continuous.

Next, we modify slightly the definition of the spaces C_g, C_g^0 in [24]. We denote by $\mathcal{PC}_g(X)$ the space formed by the normalized piecewise continuous functions φ such that φ/g is bounded on $(-\infty, 0]$ and by $\mathcal{PC}_g^0(X)$ the subspace of $\mathcal{PC}_g(X)$ formed by the functions φ such that

$$\frac{\varphi(\theta)}{g(\theta)} \to 0$$

as $\theta \to -\infty$. It is easy to see that $\mathcal{PC}_g(X)$ and $\mathcal{PC}_g^0(X)$ endowed with the norm

$$\|\varphi\|_{\mathcal{B}} := \sup_{\theta \leq 0} \frac{\|\varphi(\theta)\|}{g(\theta)}$$

are phase spaces in the sense considered in this work. Moreover, in these cases $K(s) \equiv 1$ for $s \geq 0$.

EXAMPLE. THE PHASE SPACE $\mathcal{PC}_r \times L^2(g, X)$. Let $1 \leq p < \infty$, $0 \leq r < \infty$ and $g(\cdot)$ be a Borel nonnegative measurable function on $(-\infty, r)$ which satisfies the conditions (g-5)–(g-6) in the terminology of [24]. Briefly, this means that $g(\cdot)$ is locally integrable on $(-\infty, -r)$ and that there exists a nonnegative and locally bounded function G on $(-\infty, 0]$ such that $g(\xi + \theta) \leq G(\xi) g(\theta)$ for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure 0.

Let $\mathcal{B} := \mathcal{PC}_r \times L^p(g; X)$, $r \geq 0$, $p > 1$, be the space formed of all classes of functions $\varphi : (-\infty, 0] \to X$ such that $\varphi|_{[-r,0]} \in \mathcal{PC}([-r, 0], X)$, $\varphi(\cdot)$ is Lebesgue-measurable on $(-\infty, -r]$ and $g \mid \varphi \mid^p$ is Lebesgue integrable on $(-\infty, -r]$. The seminorm in $\|\cdot\|_{\mathcal{B}}$ is defined by

$$\|\varphi\|_{\mathcal{B}} := \sup_{\theta \in [-r,0]} \|\varphi(\theta)\| + \left(\int_{-\infty}^{-r} g(\theta) \|\varphi(\theta)\|^p \, d\theta \right)^{1/p}.$$

Proceeding as in the proof of [24, Theorem 1.3.8], it follows that $\mathcal{B}$ is a phase space which satisfies Axioms A and B. Moreover, for $r = 0$ and $p = 2$, this space coincides with $C_0 \times L^2(g, X)$, $H = 1$; $M(t) = G(-t)^{1/2}$ and $K(t) = 1 + (\int_{-t}^0 g(\tau)\, d\tau)^{1/2}$, for $t \geq 0$.

REMARK 1. In retarded functional differential equations without impulses, the axioms of the abstract phase space $\mathcal{B}$ include the continuity of the function $t \to x_t$, see [24,25] for details. Due to the impulsive effect, this property is not satisfied in impulsive delay systems and, for this reason, has been eliminated in our abstract description of $\mathcal{B}$.

REMARK 2. Let $\varphi \in \mathcal{B}$ and $t \leq 0$. The notation φ_t represents the function defined by $\varphi_t(\theta) = \varphi(t + \theta)$. Consequently, if the function $x(\cdot)$ in Axiom A is such that $x_0 = \varphi$, then $x_t = \varphi_t$. We observe that φ_t is well defined for $t < 0$ since the domain of φ is $(-\infty, 0]$. We also note that in general $\varphi_t \notin \mathcal{B}$; consider, for example, functions of the type $x^\mu(t) = (t - \mu)^{-\alpha} \mathcal{X}_{(\mu,0]}$, $\mu > 0$, where $\mathcal{X}_{(\mu,0]}$ is the characteristic function of $(\mu, 0]$, $\mu < -r$ and $\alpha p \in (0, 1)$, in the space $\mathcal{PC}_r \times L^p(g; X)$.

Additional terminologies and notations used in this paper are standard in functional analysis. In particular, for Banach spaces $(Z, \|\cdot\|_Z)$, $(W, \|\cdot\|_W)$, the notation $\mathcal{L}(Z, W)$ stands for the Banach space of bounded linear operators from Z into W and we abbreviate to $\mathcal{L}(Z)$ whenever $Z = W$. Moreover, $B_r(x, Z)$ denotes the closed ball with center at x and radius $r > 0$ in Z.

The paper has four sections. In Section 3 we establish the existence of mild solutions for system (1.1)–(1.3). Section 4 is reserved for examples.

To conclude this section, we recall the following well-known result for convenience.

THEOREM 2.1. *(See [26, Theorem 6.5.4].) Let D be a closed convex subset of a Banach space Z and assume that $0 \in D$. Let $\Gamma : D \to D$ be a completely continuous map. Then, either the map Γ has a fixed point in D or $\{z \in D : z = \lambda \Gamma(z),\ 0 < \lambda < 1\}$ is unbounded.*

3. EXISTENCE RESULTS

In this section, we establish the existence of mild solutions for the impulsive abstract Cauchy problem (1.1)–(1.3). To prove our results, we always assume that $\rho : I \times \mathcal{B} \to (-\infty, a]$ is continuous and that φ and f satisfies the following conditions.

$\mathbf{H}_\varphi$ Let $\mathcal{R}(\rho^-) = \{\rho(s,\psi) : (s,\psi) \in I \times \mathcal{B},\ \rho(s,\psi) \le 0\}$. The function $t \to \varphi_t$ is well defined from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $J^\varphi : \mathcal{R}(\rho^-) \to \mathbb{R}$ such that $\|\varphi_t\|_\mathcal{B} \le J^\varphi(t)\|\varphi\|_\mathcal{B}$ for every $t \in \mathcal{R}(\rho^-)$.

$\mathbf{H}_1$ The function $f : I \times \mathcal{B} \to X$ satisfies the following conditions.

(i) Let $x : (-\infty, a] \to X$ be such that $x_0 = \varphi$ and $x|_I \in \mathcal{PC}$. The function $t \to f(t, x_{\rho(t,x_t)})$ is measurable on $[0,a]$ and the function $t \to f(s, x_t)$ is continuous on $\mathcal{R}(\rho^-) \cup [0,a]$ for every $s \in [0,a]$.

(ii) For each $t \in I$, the function $f(t,\cdot) : \mathcal{B} \to X$ is continuous.

(iii) There exists an integrable function $m : I \to [0,\infty)$ and a continuous nondecreasing function $W : [0,\infty) \to (0,\infty)$ such that

$$\|f(t,\psi)\| \le m(t) W(\|\psi\|_\mathcal{B}), \qquad (t,\psi) \in I \times \mathcal{B}.$$

REMARK 3. We point out here that condition $\mathbf{H}_\varphi$ is frequently satisfied by functions that are continuous and bounded. In fact, assume that the space of continuous and bounded functions $C_b((-\infty,0], X)$ is continuously included in $\mathcal{B}$. Then, there exists L > 0 such that

$$\|\psi_t\|_\mathcal{B} \le L \frac{\sup_{\theta \le 0} \|\psi(\theta)\|}{\|\psi\|_\mathcal{B}} \|\psi\|_\mathcal{B}, \qquad t \le 0, \quad \psi \ne 0, \quad \psi \in C_b((-\infty,0] : X). \tag{3.1}$$

It is easy to see that the space $C_b((-\infty,0], X)$ is continuously included in $\mathcal{PC}_g(X)$ and $\mathcal{PC}_g^0(X)$. Moreover, if $g(\cdot)$ verifies (g-5)–(g-6) and $g(\cdot)$ is integrable on $(-\infty,-r]$, then the space $C_b((-\infty,0], X)$ is also continuously included in $\mathcal{PC}_r \times L^p(g;X)$. For complementary details related this matter, see Proposition 7.1.1 and Theorems 1.3.2 and 1.3.8 in [24].

REMARK 4. In delay differential equations without impulses, the function f is usually assumed to be continuous. This turns out to be a poor choice of a condition for an impulsive system since in general, the function $t \to x_t$ is discontinuous. This fact is the justification for condition $\mathbf{H}_1$-(i).

Let $x : (-\infty, a] \to X$ be a function such that $x, x' \in \mathcal{PC}$. If x is a solution of (1.1)–(1.3), from the semigroup theory, we get

$$x(t) = T(t)\varphi(0) + \int_0^t T(t-s) f\left(s, x_{\rho(s,x_s)}\right) ds, \qquad t \in [0, t_1),$$

which implies that

$$x(t_1^-) = T(t_1)\varphi(0) + \int_0^{t_1} T(t_1 - s) f\left(s, x_{\rho(s,x_s)}\right) ds.$$

By using that $x(t_1^+) = x(t_1^-) + I_1(x_{t_1})$, for $t \in (t_1, t_2)$ we find that

$$\begin{aligned}
x(t) &= T(t-t_1)x(t_1^+) + \int_{t_1}^t T(t-s) f\left(s, x_{\rho(s,x_s)}\right) ds \\
&= T(t-t_1)(x(t_1^-) + I_1(x_{t_1})) + \int_{t_1}^t T(t-s) f\left(s, x_{\rho(s,x_s)}\right) ds \\
&= T(t-t_1)\left[T(t_1)\varphi(0) + \int_0^{t_1} T(t_1-s) f\left(s, x_{\rho(s,x_s)}\right) ds + I_1(x_{t_1})\right] \\
&\quad + \int_{t_1}^t T(t-s) f\left(s, x_{\rho(s,x_s)}\right) ds \\
&= T(t)\varphi(0) + \int_0^t T(t-s) f(s, x_{\rho(s,x_s)})\, ds + T(t-t_1) I_1(x_{t_1}).
\end{aligned}$$

Reiterating these procedures, we can prove that

$$x(t) = T(t)\varphi(0) + \int_0^t T(t-s) f\left(s, x_{\rho(s,x_s)}\right) ds + \sum_{0<t_i<t} T(t-t_i) I_i(x_{t_i}), \qquad t \in I.$$

This expression motivates the following definition.

DEFINITION 3.1. *A function $x : (-\infty, a] \to X$ is called a mild solution of the abstract Cauchy problem (1.1)–(1.3) if $x_0 = \varphi$, $x_{\rho(s,x_s)} \in \mathcal{B}$ for every $s \in I$ and*

$$x(t) = T(t)\varphi(0) + \int_0^t T(t-s) f\left(s, x_{\rho(s,x_s)}\right) ds + \sum_{0<t_i<t} T(t-t_i) I_i(x_{t_i}), \qquad t \in I. \tag{3.2}$$

The next lemma is proved using the phase spaces axioms.

LEMMA 3.1. *Let $x : (-\infty, a] \to X$ be a function such that $x_0 = \varphi$ and $x|_{[0,a]} \in \mathcal{PC}$. Then*

$$\|x_s\|_{\mathcal{B}} \le (M_a + J_0^{\varphi})\|\varphi\|_{\mathcal{B}} + K_a \sup\left\{\|x(\theta)\|;\ \theta \in [0, \max\{0, s\}]\right\}, \qquad s \in \mathcal{R}(\rho^-) \cup [0, a],$$

where $J_0^{\varphi} = \sup_{t \in \mathcal{R}(\rho^-)} J^{\varphi}(t)$, $M_a = \sup_{t \in I} M(t)$ and $K_a = \sup_{t\in I} K(t)$.

REMARK 5. In the sequel of this work, $M_a = \sup_{t\in[0,a]} M(t)$ and $K_a = \sup_{t\in[0,a]} K(t)$.

Now, we can establish our first existence result.

THEOREM 3.1. *Assume that there are constants L_i, $i = 1, 2, \dots, n$, such that*

$$\|I_i(\psi_1) - I_i(\psi_2)\| \le L_i \|\psi_1 - \psi_2\|_{\mathcal{B}}, \qquad \psi_j \in \mathcal{B}, \quad j = 1, 2, \quad i = 1, 2, \dots, n.$$

If

$$\tilde{M} K_a \left[\liminf_{\xi\to\infty} \frac{W(\xi)}{\xi} \int_0^a m(s)\, ds + \sum_{i=1}^n L_i\right] < 1, \tag{3.3}$$

then there exists a mild solution of (1.1)–(1.3).

PROOF. Let $Y = \{u \in \mathcal{PC} : u(0) = \varphi(0)\}$ endowed with the uniform convergence topology. On the space Y, we define the operator $\Gamma : Y \to Y$ by

$$\Gamma x(t) = T(t)\varphi(0) + \int_0^t T(t-s) f(s, \bar{x}_{\rho(s,\bar{x}_s)})\, ds + \sum_{0<t_i<t} T(t-t_i) I_i(\overline{x}_{t_i}), \qquad t \in I,$$

where $\bar{x} : (-\infty, a] \to X$ is such that $\bar{x}_0 = \varphi$ and $\bar{x} = x$ on I. From Axiom A, the strong continuity of $(T(t))_{t\ge 0}$ and our assumptions on φ and f, we infer that $\Gamma x \in \mathcal{PC}$.

Next, we prove that there exists $r > 0$ such that $\Gamma(B_r(0, Y)) \subset B_r(0, Y)$. If we assume this property is false, then for every $r > 0$ there exist $x^r \in B_r(0, Y)$ and $t^r \in I$ such that $r < \|\Gamma x^r(t^r)\|$. Then, by using Lemma 3.1 we find that

$$\begin{aligned}
r &< \|\Gamma x^r(t^r)\| \\
&\le \tilde{M} H \|\varphi\|_{\mathcal{B}} + \tilde{M} \int_0^{t^r} m(s) W(\|\overline{x^r}_{\rho(s,(\overline{x^r})_s)}\|_{\ fs\mathcal{B}})\, ds + \tilde{M} \sum_{i=1}^n \left(L_i \|\overline{x}_{t_i}\|_{\mathcal{B}} + \|I_i(0)\|\right) \\
&\le \tilde{M} H \|\varphi\|_{\mathcal{B}} + \tilde{M} W\left((M_a + J_0^{\varphi})\|\varphi\|_{\mathcal{B}} + K_a r\right) \int_0^{t^r} m(s)\, ds \\
&\quad + \tilde{M} \sum_{i=1}^n \left(L_i((M_a + J_0^{\varphi})\|\varphi\|_{\mathcal{B}} + K_a r) + \|I_i(0)\|\right),
\end{aligned}$$

and hence,

$$1 \le \tilde{M} K_a \left[\liminf_{\xi \to \infty} \frac{W(\xi)}{\xi} \int_0^a m(s)\, ds + \sum_{i=1}^{n} L_i \right],$$

which is contrary to our assumption.

Let $r > 0$ be such that $\Gamma(B_r(0,Y)) \subset B_r(0,Y)$. Next, we will prove that Γ is a condensing map on $B_r(0,Y)$. Consider the decomposition $\Gamma = \Gamma_1 + \Gamma_2$ where

$$\begin{aligned} \Gamma_1 x(t) &= T(t)\varphi(0) + \int_0^t T(t-s) f\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right) ds, && t \in I, \\ \Gamma_2 x(t) &= \sum_{0<t_i<t} T(t-t_i) I_i(\bar{x}_{t_i}), && t \in I. \end{aligned}$$

STEP 1. The set $\Gamma_1(B_r(0,Y))(t) = \{\Gamma_1 x(t) : x \in B_r(0,Y)\}$ is relatively compact in X for every $t \in I$. The case $t = 0$ is obvious. Let $0 < \epsilon < t \le a$. If $x \in B_r(0,Y)$, from Lemma 3.1 it follows that

$$\|\bar{x}_{\rho(t,\bar{x}_t)}\|_{\mathcal{B}} \le r^* := (M_a + J_0^{\varphi})\|\varphi\|_{\mathcal{B}} + K_a r$$

and so

$$\left\| \int_0^\tau T(\tau - s) f(s, \bar{x}_{\rho(s,\bar{x}_s)})\, ds \right\| \le r^{**} := \tilde{M} W(r^*) \int_0^a m(s)\, ds, \qquad \tau \in I.$$

Consequently, for $x \in B_r(0,Y)$, we find that

$$\begin{aligned} \Gamma_1 x(t) &= T(t)\varphi(0) + T(\epsilon) \int_0^{t-\epsilon} T(t-\epsilon-s) f\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right) ds + \int_{t-\epsilon}^t T(t-s) f\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right) ds \\ &\in \{T(t)\varphi(0)\} + T(\epsilon) B_{r^{**}}(0,X) + C_\epsilon, \end{aligned}$$

where $\operatorname{diam}(C_\epsilon) \le 2\tilde{M} W(r^*) \int_{t-\epsilon}^t m(s)\, ds$, which proves that $\Gamma_1(B_r(0,Y))(t)$ is relatively compact in X.

STEP 2. The set of functions $\Gamma_1(B_r(0,Y))$ is equicontinuous on I. Let $0 < t < a$ and $\epsilon > 0$. Since the semigroup $(T(t))_{t \ge 0}$ is strongly continuous and $\Gamma_1(B_r(0,Y))(t)$ relatively compact in X, there exists $0 < \delta \le a - t$ such that

$$\|T(h)x - x\| < \epsilon, \qquad x \in \Gamma_1(B_r(0,Y))(t), \quad 0 < h < \delta.$$

Under these conditions, for $x \in B_r(0,Y)$ and $0 < h < \delta$, we get

$$\begin{aligned} \|\Gamma_1 x(t+h) - \Gamma_1 x(t)\| &\le \|(T(h) - I)\Gamma_1 x(t)\| + \int_t^{t+h} T(t-s) f(s, \bar{x}_{\rho(s,\bar{x}_s)})\, ds \\ &\le \epsilon + \tilde{M} W(r^*) \int_t^{t+h} m(s)\, ds, \end{aligned}$$

where $r^* = (M_a + J_0^{\varphi})\|\varphi\|_{\mathcal{B}} + K_a r$. This proves that $\Gamma_1(B_r(0,Y))$ is right equicontinuous at $t \in (0,a)$. Similarly, we can prove the right equicontinuity at zero and the left equicontinuity at $t \in (0,a]$. Thus, $\Gamma_1(B_r(0,Y))$ is equicontinuous on I.

STEP 3. The map $\Gamma_1(\cdot)$ is continuous on $B_r(0,Y)$. Let $(x^n)_{n \in \mathbb{N}}$ be a sequence in $B_r(0,Y)$ and $x \in B_r(0,Y)$ such that $x^n \to x$ in $\mathcal{PC}$. From Axiom A, it is easy to see that $(\overline{x^n})_s \to \overline{x}_s$ as $n \to \infty$ uniformly for $s \in (-\infty, a]$. From this fact, condition $\mathbf{H_1}$ and the inequality

$$\begin{aligned} \left\| f\left(s, \overline{x^n}_{\rho(s,(\overline{x^n})_s)}\right) - f(s, \overline{x}_{\rho(s,\overline{x}_s)}) \right\| &\le \left\| f\left(s, \overline{x^n}_{\rho(s,(\overline{x^n})_s)}\right) - f\left(s, \overline{x}_{\rho(s,(\overline{x^n})_s)}\right) \right\|_{\mathcal{B}} \\ &\quad + \left\| f\left(s, \overline{x}_{\rho(s,(\overline{x^n})_s)}\right) - f\left(s, \overline{x}_{\rho(s,\overline{x}_s)}\right) \right\|_{\mathcal{B}}, \end{aligned}$$

we infer that $f(s, \overline{x^n}_{\rho(s,(\overline{x^n})_s)}) \to f(\overline{x}_{\rho(s,\overline{x}_s)})$ as $n \to \infty$, for every $s \in I$. Now, a standard application of the Lebesgue dominated convergence theorem proves that $\Gamma_1 x^n \to \Gamma_1 x$ in Y. Thus, $\Gamma_1(\cdot)$ is continuous.

STEP 4. The map $\Gamma_2(\cdot)$ is a contraction on $B_r(0,Y)$.

The assertion follows directly from (3.3) and the estimate

$$\|\Gamma_2 x - \Gamma_2 y\|_{\mathcal{PC}} \le K_a \tilde{M} \sum_{i=1}^{n} L_i \|x - y\|_{\mathcal{PC}}.$$

The previous steps prove that Γ is a condensing operator from $B_r(0,Y)$ into $B_r(0,Y)$. Now, the existence of a mild solution is a consequence of [27, Theorem 4.3.2]. This completes the proof. ∎

THEOREM 3.2. *Assume that $\rho(t,\psi) \le t$ for every $(t,\psi) \in I \times \mathcal{B}$, the maps I_i are completely continuous, and that there are constants c_i^j, $i = 1,2,\dots,n$, $j = 1,2$, such that $\|I_i(\psi)\| \le c_i^1 \|\psi\|_{\mathcal{B}} + c_i^2$, for every $\psi \in \mathcal{B}$. If $\mu = 1 - K_a \tilde{M} \sum_{i=1}^{n} c_i^1 > 0$ and*

$$\frac{K_a \tilde{M}}{\mu} \int_0^a m(s)\, ds < \int_C^\infty \frac{ds}{W(s)},$$

where

$$C = \frac{1}{\mu} \left[\left(M_a + J_0^\varphi + \tilde{M} H K_a \right) \|\varphi\|_{\mathcal{B}} + K_a \tilde{M} \sum_{i=1}^{n} c_i^2 \right],$$

then there exists a mild solution of (1.1)–(1.3).

PROOF. Let $y : (-\infty, a] \to X$ the function defined by $y(t) = \varphi(t)$ on $(-\infty, 0]$ and $y(t) = T(t)\varphi(0)$ on $[0,a]$. On the space

$$\mathcal{BPC} = \{u : (-\infty, a] \to X; u_0 = 0, u|_{[0,a]} \in \mathcal{PC}\}$$

endowed with the norm $\|\cdot\|_{\mathcal{PC}}$, we define the operator $\Gamma : \mathcal{BPC} \to \mathcal{BPC}$ by

$$\Gamma x(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ \int_0^t T(t-s) f\left(s, \overline{x}_{\rho(s,\overline{x}_s)}\right) ds + \sum\limits_{0<t_i<t} T(t-t_i) I_i(\overline{x}_{t_i}), & t \in [0,a], \end{cases}$$

where $\overline{x} = y + x$ on $(-\infty, a]$. In order to use Theorem 2.1, we will establish *a priori* estimates for the solutions of the integral equation $z = \lambda \Gamma z$, $\lambda \in (0,1)$. Assume that x^λ, $\lambda \in (0,1)$, is a solution of $z = \lambda \Gamma z$. If $\alpha^\lambda(s) = \sup_{\theta \in [0,s]} \|x^\lambda(\theta)\|$, then from Lemma 3.1 we have that

$$\begin{aligned} \|x^\lambda(t)\| &\le \tilde{M} \int_0^t m(s) W \left((M_a + J_0^\varphi) \|\varphi\|_{\mathcal{B}} + K_a \sup_{\theta \in [0,s]} \|\overline{x^\lambda}(\theta)\| \right) ds \\ &\quad + \tilde{M} \sum_{0<t_i \le t} c_i^1 \left((M_a + J_0^\varphi) \|\varphi\|_{\mathcal{B}} + K_a \sup_{\theta \in [0,t]} \|\overline{x^\lambda}(\theta)\| \right) + \tilde{M} \sum_{i=1}^{n} c_i^2 \\ &\le \tilde{M} \int_0^t m(s) W \left((M_a + J_0^\varphi + \tilde{M} H K_a) \|\varphi\|_{\mathcal{B}} + K_a \alpha^\lambda(s) \right) ds \\ &\quad + \tilde{M} \sum_{0<t_i \le t} c_i^1 \left((M_a + J_0^\varphi + \tilde{M} H K_a) \|\varphi\|_{\mathcal{B}} + K_a \alpha^\lambda(t) \right) + \tilde{M} \sum_{i=1}^{n} c_i^2, \end{aligned}$$

since $\rho(s, (\overline{x^\lambda})_s) \le s$ for each $s \in I$. If $\xi^\lambda(t) = (M_a + J_0^\varphi + \tilde{M} H K_a) \|\varphi\|_{\mathcal{B}} + K_a \alpha^\lambda(t)$, we obtain that

$$\begin{aligned} \xi^\lambda(t) \le \left(M_a + J_0^\varphi + \tilde{M} H K_a \right) \|\varphi\|_{\mathcal{B}} + K_a \tilde{M} \sum_{i=1}^{n} c_i^2 + K_a \tilde{M} \int_0^t m(s) W(\xi^\lambda(s))\, ds \\ + K_a \tilde{M} \sum_{i=1}^{n} c_i^1 \xi^\lambda(t), \end{aligned}$$

and so,

$$\xi^{\lambda}(t) \leq \frac{1}{\mu}\left[\left(M_a + J_0^{\varphi} + \tilde{M}HK_a\right)\|\varphi\|_{\mathcal{B}} + K_a\tilde{M}\sum_{i=1}^{n} c_i^2\right] + \frac{K_a\tilde{M}}{\mu}\int_0^t m(s)W(\xi^{\lambda}(s))\,ds.$$

Denoting by $\beta_\lambda(t)$ the right-hand side of the last inequality, it follows that

$$\beta'_\lambda(t) \leq \frac{K_a\tilde{M}}{\mu} m(t)W(\beta_\lambda(t)),$$

and hence,

$$\int_{\beta_\lambda(0)=C}^{\beta_\lambda(t)} \frac{ds}{W(s)} \leq \frac{K_a\tilde{M}}{\mu}\int_0^a m(s)\,ds < \int_C^{\infty}\frac{ds}{W(s)},$$

which implies that the set of functions $\{\beta_\lambda(\cdot) : \lambda \in (0,1)\}$ is bounded in $C(I : \mathbb{R})$. Thus, $\{x^\lambda(\cdot) : \lambda \in (0,1)\}$ is bounded in $\mathcal{BPC}$.

To prove that the map Γ is completely continuous, we introduce the decomposition $\Gamma x = \Gamma_1 x + \Gamma_2 x$ where $(\Gamma_i x)_0 = 0$, $i = 1, 2$, and

$$\begin{aligned} \Gamma_1 x(t) &= \int_0^t T(t-s) f\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right) ds, && t \in I, \\ \Gamma_2 x(t) &= \sum_{0<t_i<t} T(t-t_i) I_i(\bar{x}_{t_i}), && t \in I. \end{aligned}$$

From the proof of Theorem 3.1, we deduce that Γ_1 is completely continuous. Next, by using Lemma 2.1, we prove that Γ_2 is also completely continuous. The continuity of Γ_2 can be proven using the phase space axioms. From the definition of Γ_2, for $r > 0$, $t \in [t_i, t_{i+1}] \cap (0, a]$, $i \geq 1$, and $u \in B_r = B_r(0, \mathcal{BPC})$, we find that

$$\widetilde{\Gamma_2 u}(t) \in \begin{cases} \sum_{j=1}^{i} T(t-t_j) I_j(B_{r^*}(0;X)), & t \in (t_i, t_{i+1}), \\ \sum_{j=0}^{i} T(t_{i+1}-t_j) I_j(B_{r^*}(0;X)), & t = t_{i+1}, \\ \sum_{j=1}^{i-1} T(t_i - t_j) I_j(B_{r^*}(0;X)) + I_i(B_{r^*}(0;X)), & t = t_i, \end{cases}$$

where $r^* := (M_a + H\tilde{M})\|\varphi\|_{\mathcal{B}} + K_a r$, which proves that $[\widetilde{\Gamma_2(B_r)}]_i(t)$ is relatively compact in X, for every $t \in [t_i, t_{i+1}]$, since the maps I_j are completely continuous. Moreover, using the compactness of the operators I_i and the strong continuity of $(T(t))_{t\geq 0}$, we can prove that $[\widetilde{\Gamma_2(B_r)}]_i$ is equicontinuous at t, for every $t \in [t_i, t_{i+1}]$. Now, from Lemma 2.1 we conclude that Γ_2 is completely continuous.

These remarks, in conjunction with Theorem 2.1, show that Γ has a fixed point $x \in \mathcal{BPC}$. Clearly, the function $u = x + y$ is a mild solution of (1.1)–(1.3). The proof is ended. ∎

4. EXAMPLE

In this section, $X = L^2([0,\pi])$ and $A : D(A) \subset X \to X$ is the operator $Af = f''$ with domain $D(A) := \{f \in X : f'' \in X, f(0) = f(\pi) = 0\}$. It is well known that A is the infinitesimal generator of a compact C_0-semigroup of bounded linear operators $(T(t))_{t\geq 0}$ on X. Moreover, A has discrete spectrum, the eigenvalues are $-n^2$, $n \in \mathbb{N}$, with corresponding normalized eigenvectors

$$z_n(\xi) := \left(\frac{2}{\pi}\right)^{1/2} \sin(n\xi),$$

the set $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of X and

$$T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} \langle x, z_n \rangle z_n,$$

for every $x \in X$.

Consider the differential system,

$$\frac{\partial}{\partial t} u(t,\xi) = \frac{\partial^2}{\partial \xi^2} u(t,\xi) + \int_{-\infty}^{t} \beta(s-t) u(s - \sigma(\|u(t)\|), \xi)\, ds, \tag{4.1}$$

$$u(t,0) = u(t,\pi) = 0, \tag{4.2}$$

$$u(\tau,\xi) = \varphi(\tau,\xi), \qquad \tau \leq 0, \tag{4.3}$$

$$\Delta u(t_j,\xi) = \int_{-\infty}^{t_j} \gamma_j(t_j - s) u(s,\xi)\, ds, \qquad j = 1,\ldots,n, \tag{4.4}$$

for $(t,\xi) \in [0,a] \times [0,\pi]$, where $\varphi \in \mathcal{B} = \mathcal{PC}_0 \times L^2(g,X)$ and $0 < t_1 < \cdots < t_n < a$ are pre-fixed. To study this system we impose the following conditions.

(i) The functions $\beta : \mathbb{R} \to \mathbb{R}$, $\sigma : \mathbb{R} \to \mathbb{R}^+$ are continuous, bounded and

$$L_f = \left(\int_{-\infty}^{0} \frac{\beta^2(s)}{g(s)}\, ds \right)^{1/2} < \infty.$$

(ii) The functions $\gamma_j : \mathbb{R} \to \mathbb{R}$ are continuous and

$$L_j := \left(\int_{-\infty}^{0} \frac{(\gamma_j(-s))^2}{g(s)}\, ds \right)^{1/2} < \infty,$$

for every $j = 1, 2, \ldots, n$.

By defining the functions $\rho, f, I_j : \mathcal{B} \to X$ by $\rho(t,\psi) = t - \sigma(\|\psi(0)\|)$,

$$f(\psi)(\xi) = \int_{-\infty}^{0} \beta(s) \psi(s,\xi)\, ds$$

$$I_j(\psi)(\xi) = \int_{-\infty}^{0} \gamma_j(-s) \psi(s,\xi)\, ds, \qquad j = 1, 2, \ldots, n,$$

we can represent system (4.1)–(4.4) by the abstract impulsive Cauchy problem (1.1)–(1.3). Moreover, the maps f, I_j, $j = 1,\ldots,n$, are bounded linear operators, $\|f\|_{\mathcal{L}(\mathcal{B},X)} \leq L_f$ and $\|I_j\|_{\mathcal{L}(\mathcal{B},X)} \leq L_j$ for every $j = 1,\ldots,n$.

PROPOSITION 4.1. *Let $\varphi \in \mathcal{B}$ be such that $\mathbf{H}_\varphi$ is valid and $t \to \varphi_t$ is continuous on $\mathcal{R}(\rho^-)$. If*

$$\left(1 + \left(\int_{-a}^{0} g(\tau)\, d\tau \right)^{1/2} \right) \left(a L_f + \sum_{i=1}^{n} L_i \right) < 1,$$

then there exists a mild solution of (4.1)–(4.3).

PROOF. Let $x : (-\infty, a] \to X$ be such that $x_0 = \varphi$ and $x|_I \in \mathcal{PC}$. A straightforward estimation permit to prove that $t \to f(x_t)$ is continuous on $\mathcal{R}(\rho^-) \times [0,a]$ and that $t \to f(x_{\rho(t,x_t)})$ is continuous on $[0,a]$. Now, the existence of a mild solution for (4.1)–(4.3) is a consequence of Theorem 3.1. ∎

From Remark 3, we obtain the next result.

COROLLARY 4.1. *Assume that $\varphi \in \mathcal{B}$ is continuous and bounded on $(-\infty, 0]$. Then, there exists a mild solution of (4.1)–(4.3).*

REFERENCES

1. V. Lakshmikantham, D.D. Bainov and P.S. Simeonov, Theory of impulsive differential equations, In *Series in Modern Applied Mathematics, Volume 6*, World Scientific, Teaneck, NJ, (1989).
2. D. Franco and J. Nieto, Maximum principles for periodic impulsive first order problems. Positive solutions of nonlinear problems, *J. Comput. Appl. Math.* **88** (1), 149–159, (1998).
3. Z. He and J. Yu, Periodic boundary value problem for first-order impulsive functional differential equations, *J. Comput. Appl. Math.* **138** (2), 205–217, (2002).
4. Z. Luo and J. Shen, Stability results for impulsive functional differential equations with infinite delays, *J. Comput. Appl. Math.* **131** (1/2), 55–64, (2001).
5. J. Sun and Y. Zhang, Impulsive control of a nuclear spin generator, *J. Comput. Appl. Math.* **157** (1), 235–242, (2003).
6. V. Lakshmikantham and Z. Drici, Positivity and boundedness of solutions of impulsive reaction-diffusion equations. Positive solutions of nonlinear problems, *J. Comput. Appl. Math.* **88** (1), 175–184, (1998).
7. J.H. Liu, Nonlinear impulsive evolution equations, *Dynam. Contin. Discrete Impuls. Systems* **6** (1), 77–85, (1999).
8. Y.V. Rogovchenko, Impulsive evolution systems: Main results and new trends, *Dynam. Contin. Discrete Impuls. Systems* **3** (1), 57–88, (1997).
9. Y.V. Rogovchenko, Nonlinear impulse evolution systems and applications to population models, *J. Math. Anal. Appl.* **207** (2), 300–315, (1997).
10. J.K. Hale and S.M. Verduyn Lunel, *Introduction to Functional-Differential Equations, Appl. Math. Sci, Volume 99*, Springer-Verlag, New York, (1993).
11. J. Wu, *Theory and Applications of Partial Functional-Differential Equations, Applied Mathematical Sciences, Volume 119*, Springer-Verlag, New York, (1996).
12. O. Arino, K. Boushaba and A. Boussouar, A mathematical model of the dynamics of the phytoplankton-nutrient system. Spatial heterogeneity in ecological models, *Alcal'a de Henares*, (1998); *Nonlinear Analysis RWA* **1** (1), 69–87, (2000).
13. W.G. Aiello, H.I. Freedman and J. Wu, Analysis of a model representing stage-structured population growth with state-dependent time delay, *SIAM J. Appl. Math.* **52** (3), 855–869, (1992).
14. Y. Cao, J. Fan and T.C. Gard, The effects of state-dependent time delay on a stage-structured population growth model, *Nonlinear Analysis TMA* **19** (2), 95–105, (1992).
15. A. Domoshnitsky, M. Drakhlin and E. Litsyn, On equations with delay depending on solution, *Nonlinear Analysis TMA* **49** (5), 689–701, (2002).
16. F. Chen, D. Sun and J. Shi, Periodicity in a food-limited population model with toxicants and state dependent delays, *J. Math. Anal. Appl.* **288** (1), 136–146, (2003).
17. F. Hartung, Linearized stability in periodic functional differential equations with state-dependent delays, *J. Comput. Appl. Math.* **174** (2), 201–211, (2005).
18. F. Hartung, T.L. Herdman and J. Turi, Parameter identification in classes of neutral differential equations with state-dependent delays, *Nonlinear Analysis TMA, Ser. A: Theory Methods* **39** (3), 305–325, (2000).
19. Z. Yang and J. Cao, Existence of periodic solutions in neutral state-dependent delays equations and models, *J. Comput. Appl. Math.* **174** (1), 179–199, (2005).
20. D.R. Willé and C.T.H. Baker, Stepsize control and continuity consistency for state-dependent delay-differential equations, *J. Comput. Appl. Math.* **53** (2), 163–170, (1994).
21. A.V. Rezounenko and J. Wu, A non-local PDE model for population dynamics with state-selective delay: Local theory and global attractors, *Journal of Computational and Applied Mathematics* **190** (1-2), 99–113, (2006).
22. E. Hernández, A. Prokopczyk and L. Ladeira, A note on partial functional differential equations with state-dependent delay, *Nonlinear Analysis: Real World Applications* **7** (4), 510–519, (2006).
23. A. Pazy, Semigroups of linear operators and applications to partial differential equations, In *Applied Mathematical Sciences, Volume 44*, Springer-Verlag, New York, (1983).
24. Y. Hino, S. Murakami and T. Naito, Functional differential equations with infinite delay, In *Lecture Notes in Mathematics, Volume 1473*, Springer-Verlag, Berlin, (1991).
25. J.K. Hale and J. Kato, Phase space for retarded equations with infinite delay, *Funkcial. Ekvac.* **21** (1), 11–41, (1978).
26. A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, (2003).
27. R.H. Martin, *Nonlinear Operators and Differential Equations in Banach Spaces*, Robert E. Krieger Publ., Malabar, FL, (1987).