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Existence of frame SOLS of type $a^n b^1$ ☆

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Abstract

An SOLS (self-orthogonal latin square) of order v with n_i missing sub-SOLS (holes) of order h_i ($1 \leqslant i \leqslant k$), which are disjoint and spanning (i.e. $\sum_{i=1}^{k} n_i h_i = v$), is called a frame SOLS and denoted by FSOLS($h_1^{n_1} h_2^{n_2} \cdots h_k^{n_k}$). It has been proved that for $b \geqslant 2$ and n odd, an FSOLS($a^n b^1$) exists if and only if $n \geqslant 4$ and $n \geqslant 1+2b/a$. In this paper, we show the existence of FSOLS($a^n b^1$) for n even and FSOLS($a^n 1^1$) for n odd.

Keywords: Self-orthogonal Latin square; Frame; Room frame

1. Introduction

A *self-orthogonal* latin square of order v, or SOLS(v), is a latin square of order v which is orthogonal to its transpose. It is well known [6] that an SOLS(v) exists for all values of v, $v \neq 2$, 3 or 6.

Let S be a set and $\mathscr{H} = \{S_1, S_2, \ldots, S_k\}$ be a set of nonempty subsets of S. A *holey latin square* having *hole set* $\mathscr{H}$ is a $|S| \times |S|$ array, L, indexed by S, which satisfies the following properties:

(1) every cell of L is either empty or contains a symbol of S,
(2) every symbol of S occurs at most once in any row or column of L,
(3) the subarrays $S_i \times S_i$ are empty for $1 \leqslant i \leqslant k$ (these subarrays are referred to as *holes*),
(4) symbol $x \in S$ occurs in row or column y if and only if $(x, y) \in (S \times S) \setminus \bigcup_{i=1}^{k} (S_i \times S_i)$.

The *order* of L is $|S|$. Two holey latin squares on symbol set S and hole set $\mathscr{H}$, say L_1 and L_2, are said to be *orthogonal* if their superposition yields every or-

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dered pair in $(S\times S)\setminus\bigcup_{i=1}^{k}(S_i\times S_i)$. We shall use the notation IMOLS$(v;s_1,s_2,\ldots,s_k)$ to denote a pair of orthogonal holey latin squares on symbol set S and hole set $\mathscr{H}=\{S_1,S_2,\ldots,S_k\}$, where $v=|S|$ and $s_i=|S_i|$ for $1\leqslant i\leqslant k$. If $\mathscr{H}=\emptyset$, we obtain an MOLS(v). If $\mathscr{H}=\{S_1\}$, we simply write IMOLS(v,s_1) for the orthogonal pair of holey latin squares.

If L_1 and L_2 form an IMOLS$(v;s_1,s_2,\ldots,s_k)$ such that L_2 is the transpose of L_1, then we call L_1 a *holey* SOLS, denoted by ISOLS$(v;s_1,s_2,\ldots,s_k)$. If $\mathscr{H}=\emptyset$, or $\{S_1\}$, then a holey SOLS is an SOLS(v), or ISOLS(v,s_1) respectively.

If $\mathscr{H}=\{S_1,S_2,\ldots,S_k\}$ is a partition of S, then an IMOLS is called a *frame* MOLS. The *type* of the frame MOLS is defined to be the multiset $\{|S_i|\colon 1\leqslant i\leqslant k\}$. We shall use an "exponential" notation to describe types: Type $h_1^{n_1}h_2^{n_2}\cdots h_k^{n_k}$ denotes n_i occurrences of $h_i, 1\leqslant i\leqslant k$, in the multiset. We briefly denote a frame MOLS of type $h_1^{n_1}h_2^{n_2}\cdots h_k^{n_k}$ FMOLS$(h_1^{n_1}h_2^{n_2}\cdots h_k^{n_k})$.

If L_1 and L_2 form an FMOLS (frame MOLS) such that L_2 is the transpose of L_1, then we call L_1 an FSOLS.

We observe that the existence of an SOLS(v) is equivalent to the existence of an FSOLS(1^v), and the existence of an ISOLS(v,h) is equivalent to the existence of an FSOLS $(1^{v-h}h^1)$.

Most of our recursive constructions in the following sections rely on information regarding the location of (holey) transversals in certain latin squares. Suppose L is a holey latin square on symbol set S with hole S_1. A *holey transversal* with hole S_1 is a set T of $|S|-|S_1|$ (occupied) cells in L such that every symbol of $S\setminus S_1$ is contained in exactly one cell of T and the $|S|-|S_1|$ cells in T intersect each row and each column indexed by $S\setminus S_1$ in exactly one cell. $|S_1|$ is called the *size* of the hole of the holey transversal. A holey transversal T is *symmetric* if $(i,j)\in T$ implies $(j,i)\in T$. Two holey transversals T_1 and T_2 with the same hole are called a *symmetric pair of holey transversals* if $(i,j)\in T_1$ if and only if $(j,i)\in T_2$. If $S_1=\emptyset$, then we call the holey transversal a (*complete*) transversal. A set of holey transversals are said to be *disjoint* if they have no cell in common.

FSOLS has been very useful in recursive constructions of various combinatorial designs, such as 2-perfect m-cycle systems [18], edge-colored designs [11], holey Schröder designs [2], intersections of transversal designs [10], and skew Room frames [8]. The idea of frames in combinatorial designs has appeared for a long time, see for example [19]. The following are known results concerning FSOLS(a^n) and FSOLS(a^nb^1). The first two theorems give necessary conditions. The third theorem contains the updated known results on sufficiency, where the first two parts give complete solution for the uniform types, the next four parts give complete solution for special values of $a=2,6$, and almost complete solution for $a=1,3$.

Theorem 1.1 (Stinson and Zhu [21]). *If there exists an* FSOLS(a^nb^1), *then* $n\geqslant 1+2b/a$.

Theorem 1.2 (Xu and Lu [22, Theorem 1.3]). *If* $a\neq b$ *and there exists an* FSOLS (a^nb^1), *then* $n\geqslant 4$.

Theorem 1.3.

(1) [6] *There exists an* FSOLS(1^n) *if and only if* $n \geqslant 4$, $n \neq 6$.
(2) [21] *For* $a \geqslant 2$, *there exists an* FSOLS(a^n) *if and only if* $n \geqslant 4$.
(3) [1, Theorem 2.10] *There exists an* FSOLS($1^{v-n}n^1$) *for all values of* v *and* n *satisfying* $v \geqslant 3n+1$, *except for* $(v,n)=(6,1),(8,2)$ *and possibly for* $v=3n+2$, $n \in \{6,8,10\}$.
(4) [25] *For* $b \geqslant 0$ *and* $b \neq 2$, *there exists an* FSOLS($2^n b^1$) *if and only if* $n \geqslant 4$.
(5) [1, Theorem 4.6] For $b \geqslant 0$ *and* $b \neq 3$, *there exists an* FSOLS($3^n b^1$) *if and only if* $n \geqslant 4$, *with five possible exceptions* $(n,b)=(n,3n/2-2)$ for $n \in \{6,10,14,18,22\}$.
(6) [24] *For* $b \geqslant 0$ *and* $b \neq 6$, *there exists an* FSOLS($6^n b^1$) *if and only if* $n \geqslant 4$.
(7) [23] *For* $b \geqslant 2$ *and* n *odd*, FSOLS($a^n b^1$) *exists if and only if* $n \geqslant 5$ *and* $n \geqslant 1 + 2b/a$.

The last part of Theorem 1.3 solves the sufficiency for $b \geqslant 2$ and n odd, leaving the cases n even and $b=1$ unsolved. In this paper, we show the existence of FSOLS($a^n b^1$) for n even and FSOLS($a^n 1^1$) for n odd. Our main results are stated in Theorems 7.1 and 7.3. In Theorem 7.1, the existence of an FSOLS($a^n b^1$) is solved for $a \neq b$ with a few possible exceptions. For these missing cases, FMOLS are proved to exist in Theorem 7.3.

2. Direct constructions

Our first direct construction is a modification of the starter–adder type constructions. The idea has been described by several authors including Horton [17], Hedayat and Seiden [13], Zhu [26], and Heinrich and Zhu [16].

Construction 2.1. *Let* $\mathbf{e}=(\emptyset, a_{01}, a_{02}, \ldots, a_{0(n-1)}, \emptyset, a_{0(n+1)}, \ldots, a_{0(2n-1)}, \ldots, \emptyset, a_{0(hn-n+1)}, \ldots, a_{0(hn-1)})$ *be a vector of length hn with entries in* $(\mathbf{Z}_{hn} \backslash \{0, n, \ldots, (h-1)n\}) \cup X$, *where* $X=\{x_1, x_2, \ldots, x_u\}$ *is a set of u index symbols, and* $\emptyset$ *means that the cell it occupies is empty. Let* $\mathbf{f}=(a_{0x_1}, a_{0x_2}, \ldots, a_{0x_u})$ *and* $\mathbf{g}=(a_{x_1 0}, a_{x_2 0}, \ldots, a_{x_u 0})$ *be vectors of length u with entries in* $\mathbf{Z}_{hn} \backslash \{0, n, \ldots, (h-1)n\}$. *These vectors are used to construct an array* $A=(a_{ij})$ *of order* $hn+u$ *with n empty subarrays of order h and one empty subarray of order u having row and column indices and entries in* $\mathbf{Z}_{hn} \cup X$. *The array is constructed as follows, where all the elements including indices are calculated modulo hn, and* x_i*'s act as "infinite" elements.*

(1) *If* $a_{ij}=\emptyset$, $0 \leqslant i,j \leqslant hn-1$, *then* $a_{(i+1)(j+1)}=\emptyset$.
(2) If $a_{ij} \in \mathbf{Z}_{hn}$, $0 \leqslant i,j \leqslant hn-1$, *then* $a_{(i+1)(j+1)}=a_{ij}+1$.
(3) *If* $a_{ij} \in X$, $0 \leqslant i \leqslant hn-1$, *then* $a_{(i+1)(j+1)}=a_{i,j}$
(4) *If* $0 \leqslant i \leqslant hn-1$, *and* $j \in X$, *then* $a_{(i+1)j}=a_{ij}+1$.
(5) *If* $0 \leqslant j \leqslant hn-1$, *and* $i \in X$, *then* $a_{i(j+1)}=a_{ij}+1$.

	5		4	7	6	8	1	3
8		6		5	7	0	4	2
	7		6	1	4	3	8	5
5		4		8	2	7	0	6
1	6	8	7		3	2	5	0
3	2	7	8	0		4	6	1
4	8	5	0	3	1		2	7
6	0	3	5	2	8	1		4
7	4	1	2	6	0	5	3	

Fig. 1. An FSOLS($2^2 1^5$).

	5	7		6	8	2	9	1	4
9		6	5		7	8	2	3	0
4	3		1	0		7	6	9	8
	7	8		5	9	4	1	6	2
5		9	8		6	0	3	2	7
6	8		7	9		1	4	0	3
1	2	3	9	7	0		8	4	5
2	0	4	6	8	3	9		5	1
7	9	1	2	3	4	5	0		6
8	6	0	4	2	1	3	5	7	

Fig. 2. An FSOLS($2^3 1^4$).

	7	8	6		2	3	9	5	1
2		7	8	9		4	0	6	3
9	4		0	7	3		8	1	5
1	2	9		5	8	0		4	6
	3	1	9		6	8	5	7	2
3		0	4	8		9	6	2	7
5	9		1	3	7		4	0	8
6	8	5		2	4	1		9	0
7	0	3	2	6	9	5	1		4
8	6	4	5	1	0	7	2	3	

Fig. 3. An FSOLS($2^4 1^2$).

Conditions can be described for the vectors **e**, **f** and **g** so that the array as constructed is an FSOLS($h^n u^1$). However, we shall simply give the vectors and the reader can check for himself that they do yield the desired FSOLS($h^n u^1$).

Some FSOLS($a^n b^m$) of small size are very useful in the construction of FSOLS($a^n b^1$) and they can only be obtained by direct constructions. By computer search, we found several FSOLS($a^n b^m$) listed in Figs. 1–7.

	8		9	3	2	7	6	5	4
9		6		7	8	0	5	4	2
	7		4	5	6	9	8	1	3
4		8		0	2	5	9	7	6
6	2	1	7		9	8	3	0	5
7	6	4	8	2		3	1	9	0
8	9	5	0	1	4		2	3	7
5	4	3	6	9	0	1		2	8
3	0	9	5	6	7	2	4		1
1	5	7	2	8	3	4	0	6	

Fig. 4. An FSOLS($2^2 1^6$).

	4		6	5	7	8	9	a	1	3
a		9		2	6	0	8	4	7	5
	a		7	1	4	3	6	5	8	9
5		4		8	2	a	0	9	6	7
1	9	7	5		8	2	a	6	3	0
3	8	6	a	9		7	1	0	2	4
4	5	8	0	7	9		3	1	a	2
6	0	1	9	3	a	4		2	5	8
7	2	3	4	a	1	9	5		0	6
8	6	a	2	0	3	5	4	7		1
9	7	5	8	6	0	1	2	3	4	

Fig. 5. An FSOLS($2^2 1^7$).

	2	9		a	8	5	6	4	1	7
a		8	7		6	3	2	5	0	9
4	0		8	3		7	9	6	a	1
	a	4		9	1	2	5	7	8	6
5		7	6		9	a	8	3	2	0
6	7		a	0		8	1	9	4	3
1	5	0	9	8	7		a	2	3	4
2	3	1	4	5	0	9		a	6	8
7	9	a	1	6	3	4	0		5	2
8	6	3	2	7	a	1	4	0		5
9	8	6	5	2	4	0	3	1	7	

Fig. 6. An FSOLS($2^3 1^5$).

Lemma 2.2. *There exist* FSOLS *of types* $2^2 1^5$, $2^3 1^4$, $2^4 1^2$, $2^2 1^6$, $2^2 1^7$, $2^3 1^5$, *and* $2^4 1^3$.

By exhaustive computer search, we have the following nonexistence result.

Lemma 2.3. *There do not exist* FSOLS *of types* $1^4 2^2$, $1^2 2^3$, *and* $1^3 2^3$.

	a	7	1		9	3	2	5	8	6
3		a	6	8		0	4	7	2	9
1	0		a	9	8		5	3	7	4
2	4	8		1	6	9		a	0	5
	2	3	9		7	a	1	6	5	8
a		9	0	6		7	8	2	4	3
8	3		4	5	a		9	0	1	7
9	8	4		a	0	5		1	6	2
7	9	5	2	3	4	1	6		a	0
6	7	0	5	2	3	8	a	4		1
5	6	1	8	7	2	4	0	9	3	

Fig. 7. An FSOLS($2^4 1^3$).

0	6	w	7	z	3	8	x	9	y	5	4	2	1
x	1	7	z	5	y	2	4	w	3	9	6	8	0
8	y	2	9	7	w	x	6	z	1	4	5	0	3
w	7	x	3	6	4	9	1	y	z	8	0	5	2
9	z	0	y	4	6	3	2	x	w	7	8	1	5
6	8	1	x	w	5	z	y	7	2	0	3	9	4
z	9	8	w	y	0	6	5	3	x	2	1	4	7
5	w	z	8	x	2	y	7	4	0	1	9	3	6
y	5	3	0	1	x	w	z	8	4	6	2	7	9
4	x	y	1	2	z	0	w	5	9	3	7	6	8
7	0	6	2	9	8	4	3	1	5				
1	2	9	4	3	7	5	8	0	6				
3	4	5	6	0	1	7	9	2	8				
2	3	4	5	8	9	1	0	6	7				

Fig. 8. An ISOLS(14, 4) with a symmetric holey transversal.

The following lemmas indicate FSOLS with certain holey transversals, which are useful in the following sections.

Lemma 2.4. *There exists an* FSOLS($1^{10}4^1$) *with a symmetric holey transversal with a hole of size four.*

Proof. Fig. 8 is an ISOLS(14, 4) from [1], which has a symmetric holey transversal consisting of cells $(1,6),(6,1),(2,7),(7,2),(3,5),(5,3),(4,8),(8,4),(9,10),(10,9)$. □

Lemma 2.5. *For* $n \in \{6,10,14\}$, *there exists an* FSOLS(2^n) *with two symmetric transversals and* $n-2$ *pairs of symmetric transversals and all these transversals are disjoint.*

Proof. Fig. 9 is an FSOLS(2^6) which has 10 disjoint transversals occurring as two symmetric ones and four symmetric pairs. The two symmetric transversals are T_1:

	a	7	2	5	9		3	b	4	8	1
2		b	8	3	6	a		4	0	5	9
a	3		0	9	4	7	b		5	1	6
7	b	4		1	a	5	8	0		6	2
3	8	0	5		2	b	6	9	1		7
8	4	9	1	6		3	0	7	a	2	
	9	5	a	2	7		4	1	8	b	3
4		a	6	b	3	8		5	2	9	0
1	5		b	7	0	4	9		6	3	a
b	2	6		0	8	1	5	a		7	4
5	0	3	7		1	9	2	6	b		8
9	6	1	4	8		2	a	3	7	0	

Fig. 9. An FSOLS(2^6) having 10 disjoint transversals.

$(i, i+3)$, $(i+1, i-2)$ and T_2: $(i, i-3)$, $(i+1, i+4)$ (for i even, $0 \leqslant i \leqslant 10$, the calculation is done modulo 12). The other eight transversals are all cyclic with (i, j) and $(i+1, j+1)$ in the same transversal, and they form four symmetric pairs.

Applying Construction 2.1 with the following vectors **e**, one can obtain an FSOLS(2^n) for $n = 10$ and 14 which have two symmetric transversals and $n-2$ pairs of symmetric transversals. The two symmetric transversals of the FSOLS(2^n) are T_1: $(i, i+1)$, $(i+1, i)$ and T_2: $(i, i-1)$, $(i-1, i)$ (for i even, $0 \leqslant i \leqslant 2(n-1)$, the calculation is done modulo $2n$).

2^{10} : $\mathbf{e} = (\emptyset, 19, 18, 17, 15, 11, 14, 12, 5, 8, \emptyset, 6, 13, 16, 7, 4, 3, 9, 2, 1)$.
2^{14} : $\mathbf{e} = (\emptyset, 27, 26, 25, 24, 23, 22, 12, 15, 20, 19, 21, 13, 16, \emptyset, 8, 11, 6, 17, 10, 5, 18, 9, 7, 4, 3, 2, 1)$. □

3. Recursive constructions

Construction 3.1 (Filling in holes) (Xu and Zhu [25]). *Suppose there exist* FSOLS *of type* $\{s_i\colon 1 \leqslant i \leqslant n\}$ *and for* $1 \leqslant i \leqslant n$, $s_i = \sum_{j=1}^{t_i} s_{ij}$.

(1) *If there exist* FSOLS *of type* $\{s_{nj}\colon 1 \leqslant j \leqslant t_n\}$, *then there exist* FSOLS *of type* $\{s_i\colon 1 \leqslant i \leqslant n-1\} \bigcup \{s_{nj}\colon 1 \leqslant j \leqslant t_n\}$.
(2) *Let* $a \geqslant 0$ *be an integer. If there exist* FSOLS *of type* $\{a\} \bigcup \{s_{ij}\colon 1 \leqslant j \leqslant t_i\}$ *for all* $1 \leqslant i \leqslant n-1$, *then there exist* FSOLS *of type* $\{a+s_n\} \bigcup (\bigcup_{i=1}^{n-1} \{s_{ij}\colon 1 \leqslant j \leqslant t_i\})$.

Given a set X of *points*, a *group divisible design* (GDD), or simply *divisible design* (DD), is a triple $(X, \mathcal{G}, \mathcal{A})$ which satisfies the following properties:

1. $\mathcal{G}$ is a partition of X and each member of $\mathcal{G}$ is called a *group* (also called a *point class*);

2. $\mathscr{A}$ is a set of subsets of X (each subset is called a *block*) such that a group and a block contain at most one common point;
3. Every pair of points from distinct groups occurs in a unique block.

The *group type* of the GDD is the multiset $\{|G|\colon G\in\mathscr{G}\}$. A GDD $(X,\mathscr{G},\mathscr{A})$ will be referred to as a K-GDD if $|A|\in$ K for every block A in $\mathscr{A}$. A TD(k,n) is a GDD of group type n^k and block size k. It is well known that the existence of a TD(k,n) is equivalent to the existence of $k-2$ MOLS(n).

Construction 3.2 (Weighting) (Stinson and Zhu [21, Lemma 2.5]). *Suppose* $(X,\mathscr{G},\mathscr{A})$ *is a* GDD *and let* w *be a map*: $X\rightarrow \mathbf{Z}^+\bigcup\{0\}$. *Suppose there exist* FSOLS *of type* $\{w(x)\colon x\in A\}$ *for every* $A\in\mathscr{A}$. *Then there exist* FSOLS *of type* $\{\sum_{x\in G} w(x)\colon G\in\mathscr{G}\}$.

The following recursive construction is referred to as *inflation construction*. It essentially "blows up" every occupied cell of an FSOLS into a latin square such that if one cell is filled with a certain latin square, then its symmetric cell is filled with the transpose of an orthogonal mate of the latin square. We mention the work of Brouwer and van Rees [7] and Stinson [20], which can be thought of as sources of inflation construction.

Construction 3.3 (Inflation construction). *Suppose there exists an* FSOLS$(h_1^{n_1}h_2^{n_2}\cdots h_k^{n_k})$ *and an* MOLS(h), *then there exists an* FSOLS$((hh_1)^{n_1}(hh_2)^{n_2}\cdots(hh_k)^{n_k})$. *In particular, the existence of* FSOLS(1^n) *and* MOLS(h) *implies the existence of an* FSOLS(h^n).

The following construction is a generalization of [27, Construction 3.3].

Construction 3.4. *Suppose there is an* FSOLS(t^n) *which has* $p+2q$ *disjoint transversals*, *p of them being symmetric and the rest being q symmetric pairs*, *where* $p,q\geqslant 0$. *Let h be a positive integer*, *where* $h\neq 2$ *or* 6 *if* $p+2q<t(n-1)$. *If* $p>0$, *let* $v_i\geqslant 0$ *be integers such that there exist* IMOLS$(h+v_i,v_i)$ *for* $1\leqslant i\leqslant p$. *If* $q>0$, *let* $w_j\geqslant 0$ *be integers such that there exist* IMOLS$(h+w_j,w_j)$ *for* $1\leqslant j\leqslant q$. *Then there exists an* FSOLS$((ht)^n(v+2w)^1)$, *where* $v=\sum v_i$ *and* $w=\sum w_j$ $(w=0$ *if* $q=0)$.

Construction 3.5 (Xu and Lu [22, Construction 2.5]). *Suppose there is an* FSOLS(t^n) *which has* $p+2q$ *disjoint transversals*, *p of them being symmetric and the rest being q symmetric pairs. For* $1\leqslant i\leqslant p$ *and* $1\leqslant j\leqslant q$, *let* $v_i\geqslant 0$ *and* $w_j\geqslant 0$ *be integers. Let s and h be positive integers*, *where there exists an* FMOLS(s^h) *if* $p+2q<t(n-1)$. *Suppose there exist* FMOLS$(s^hv_i^1)$ *for all* $1\leqslant i\leqslant p$, FMOLS$(s^hw_j^1)$ *for all* $1\leqslant j\leqslant q$ *and* FSOLS$((st)^nk^1)$ *for some* $k\geqslant 0$. *Then there exists an* FSOLS$((tsh)^n b^1)$, *where* $b=k+\sum v_i+2\sum w_j$.

The following is a modification of Construction 3.4, in which holey transversals are used.

Construction 3.6 (Zhu [27, Construction 3.4]). *Suppose there is an* FSOLS($t^n h^1$), *where* H *is the size* h *hole, having* $p+2q$ *disjoint holey transversals with hole* H *such that* p *of them are symmetric and the rest form* q *symmetric pairs. For* $1 \leqslant i \leqslant p$ *and* $1 \leqslant j \leqslant q$, *let* v_i *and* w_j *be non-negative integers. Let* m *be a positive integer, where* $m \neq 2$ *or* 6, *and suppose there exist* IMOLS($m+v_i, v_i$) *for* $1 \leqslant i \leqslant p$ *and* IMOLS($m+w_j, w_j$) *for* $1 \leqslant j \leqslant q$. *Then there exists an* FSOLS($(mt)^n(mh+v+2w)^1$), *where* $v=\sum v_i$ and $w=\sum w_j$.

Lemma 3.7. *Suppose there is an* SOLS(n) *with* $p+2q$ *disjoint transversals and occurring as* p *symmetric ones, one of which is on the main diagonal, and* q *symmetric pairs, where* $p>0$, $q \geqslant 0$. *For* $1 \leqslant i \leqslant p-1$ *and* $1 \leqslant j \leqslant q$, *let* $v_i \geqslant 0$, $w_j \geqslant 0$ *be integers, so that there exist* FMOLS($s^h v_i^1$) *for* $1 \leqslant i \leqslant p-1$ *and* FMOLS($s^h w_j^1$) *for* $1 \leqslant j \leqslant q$. *Moreover, suppose* $v \geqslant 0$, $s>0$ *and* $h>0$ *and there exists an* FSOLS($s^h v^1$) (*and there exists an* FMOLS(s^h) *if* $p+2q<n$). *Then there exists an* FSOLS($(sn)^h b^1$), *where* $b=\sum v_i + 2\sum w_j + v$.

Proof. Apply inflation construction with SOLS(n) as the initial square. Fill every occupied cell on the main diagonal with an FSOLS($s^h v^1$). Fill every symmetric pair of cells with FMOLS($s^h v_i^1$), or FMOLS($s^h w_j^1$), or FMOLS(s^h) if the cell is on the ith symmetric transversal, or on the jth symmetric pair of transversals, or not on any transversal and $p+2q<n$. When one cell is filled with a square, its symmetric cell is filled with the transpose of the orthogonal mate in the FMOLS. Place the size v_i or w_j hole at the lower right corner, forming a big hole of size b. Then we obtain the desired FSOLS. □

To apply the above constructions we need some "ingredients" provided in the following theorems and lemmas.

Theorem 3.8 (Bose et al. [5]). *There exists an* MOLS(v) *for any positive integer* v, $v \neq 2, 6$.

Theorem 3.9 (Heinrich and Zhu [15]). *There exists an* IMOLS(v, n) *for all values of* v *and* n *satisfying* $v \geqslant 3n$ *except that* IMOLS($6, 1$) *does not exist.*

Theorem 3.10 (Du [12], Bennett and Zhu [3,4]). *For all even* n, $n \notin \{2, 6, 10, 14\}$, *there exists an* FSOLS(1^n) *with* $n-1$ *disjoint symmetric transversals.*

Theorem 3.11 (Heinrich et al. [14, Theorem 2.5]). *If* $n \geqslant 5$ *is an odd prime power, then there exists an* FSOLS(1^n) *with* $n-1$ *disjoint transversals occurring as* $(n-1)/2$ *pairs of symmetric transversals.*

Theorem 3.12. *Suppose* n *is odd. If* $n \not\equiv 0 \pmod 3$ or $n \equiv 0 \pmod 9$, *then there exists an* SOLS(n), *or equivalently an* FSOLS(1^n), *with* $n-1$ *disjoint transversals occurring as* $(n-1)/2$ *pairs of symmetric transversals.*

Proof. From Theorem 3.11 the conclusion is true for n a prime power. Now, write $n = q_1 q_2 \cdots q_k$, where q_i's are distinct prime powers, $q_i \neq 3$, for $1 \leqslant i \leqslant k$. So, the conclusion holds for $k = 1$. Suppose the conclusion is true for $n = q_1 q_2 \cdots q_k$ and suppose $n' = nq$, where $q \neq 3$, is an odd prime power. For the main diagonal of the given FSOLS(1^n), the inflation construction provides n holes of size q each. By filling in these holes with an FSOLS(1^q) we get $q - 1$ disjoint transversals in an FSOLS($1^{n'}$), occurring as $(q - 1)/2$ pairs of symmetric transversals. For each pair of symmetric transversals in the FSOLS(1^n), the inflation construction provides q pairs of symmetric transversals in the FSOLS($1^{n'}$). These transversals are disjoint and altogether $(q - 1)/2 + q \times (n - 1)/2 = (n' - 1)/2$ in number. Then the conclusion comes from induction. □

Theorem 3.13 (Xu and Lu [22, Theorem 2.14]). *Suppose $n \geqslant 5$ is an odd prime, then there exists an* FSOLS(1^n) *with one symmetric holey transversal and $(n - 5)/2$ symmetric pairs of holey transversals with a common hole of size one, and all these holey transversals are disjoint.*

Lemma 3.14. *If $(n, u) \in \{(10, 3), (14, 4), (14, 5)\}$, then there exists an* FSOLS($1^n u^1$) *with $(n - 1) - 2u$ disjoint holey transversals with a common hole of size u and occurring as a symmetric one and $(n - 2)/2 - u$ symmetric pairs.*

Proof. For $(n, u) = (10, 3)$ and (14,5), the FSOLS($1^n u^1$) is from [22, Lemma 2.18]. FSOLS($1^{14} 4^1$) with five disjoint holey transversals with hole of size four and occurring as a symmetric one and two symmetric pairs can be constructed by Construction 2.1 with the following vectors: $\mathbf{e} = (\emptyset, 13, 12, 11, 10, 8, x_1, 6, 9, x_2, x_3, x_4, 5, 1)$, $\mathbf{f} = (2, 3, 4, 7)$, $\mathbf{g} = (4, 9, 5, 11)$. □

A *transversal design* TD(k, n) is a GDD with kn points, k groups of size n, and n^2 blocks of size k. It is well known that a TD(k, n) is equivalent to $k - 2$ MOLS (mutually orthogonal latin squares) of order n and that for any prime power p, there exist $p - 1$ MOLS of order p. From [9] (Tables 2.68 and 2.72; the cases of 39 and 54 are from http://www.emba.uvm.edu/∼dinitz/newresults.html), we have the following theorem.

Theorem 3.15. *There is a* TD($6, m$) *if $m \geqslant 5$ and $m \notin E_6 = \{6, 10, 14, 18, 22\}$. There is a* TD($7, m$) *if $m \geqslant 7$ and $m \notin E_7 = E_6 \cup \{15, 20, 26, 30, 34, 38, 46, 60, 62\}$.*

4. Existence of FSOLS($a^n b^1$) for n and a both even

Lemma 4.1. *Suppose n and a are both even, $n \notin \{2, 6, 10, 14\}$ and $a \notin \{2, 6\}$. Then there exists an* FSOLS($a^n b^1$) *for $0 \leqslant b \leqslant a(n - 1)/2$.*

Proof. From Theorem 3.10 we know that there exists an FSOLS(1^n) with $n-1$ disjoint symmetric transversals. Applying Construction 3.4 with $t=1$, $p=n-1$, $q=0$, $h=a$, $0\leqslant v_i\leqslant a/2$ $(1\leqslant i\leqslant n-1)$, the input designs are from Theorems 3.8 and 3.9, we obtain an FSOLS(a^nb^1), where $0\leqslant b\leqslant (n-1)a/2$. Note that Construction 3.4 requires $h\notin\{2,6\}$, which is satisfied since $a\notin\{2,6\}$ and $h=a$. □

Lemma 4.2. *Suppose* $a\geqslant 8$ *is even and* $n\in\{6,10,14\}$, *Then there exists an* FSOLS (a^nb^1) *for* $0\leqslant b\leqslant a(n-1)/2$.

Proof. From Lemma 2.5 we know that there exists an FSOLS(2^n) with $2n-2$ disjoint transversals occurring as two symmetric ones and $n-2$ symmetric pairs for $n\in\{6,10,14\}$.

(i) If $a\equiv 0 \pmod 4$, applying Construction 3.4 with $t=2$, $p=2$, $q=n-2$, $h=a/2$, and $0\leqslant v_i,w_j\leqslant a/4$, the input designs are from Theorems 3.8 and 3.9, then we obtain FSOLS(a^nb^1) for $n\in\{6,10,14\}$, and $0\leqslant b\leqslant (n-1)a/2$.

(ii) If $a\equiv 2 \pmod 4$, $a\neq 10$, applying Construction 3.5 with $t=2$, $p=2$, $q=n-2$, $s=1$, $h=a/2$, $0\leqslant v_i,w_j\leqslant (a-2)/4$, $0\leqslant k\leqslant n-1$, the input designs are from Theorem 1.3 (3) and (4), then we obtain FSOLS(a^nb^1) for $n\in\{6,10,14\}$, and $0\leqslant b\leqslant (n-1)a/2$.

For $a=10$, we consider the same construction. Although v_i and w_j can no longer take the value 1 since an FSOLS(1^6) does not exist, this construction still works since we can take $v_i,w_j\in\{0,2\}$ and $0\leqslant k\leqslant n-1$. This still covers the whole interval $0\leqslant b\leqslant 5(n-1)$. □

Lemma 4.3. *There exists an* FSOLS(4^nb^1) *for* $n\in\{6,10,14\}$ *and* $1\leqslant b\leqslant 2(n-1)$.

Proof. FSOLS(4^6b^1) for $1\leqslant b\leqslant 3$ can be constructed by Construction 2.1:

4^61^1: $\mathbf{e}=(\emptyset,23,22,20,15,13,\emptyset,14,7,19,11,x,\emptyset,10,17,5,8,21,\emptyset,4,9,16,3,1)$, $\mathbf{f}=(2)$, $\mathbf{g}=(15)$.

4^62^1: $\mathbf{e}=(\emptyset,23,22,20,19,15,\emptyset,10,16,13,9,8,\emptyset,x_1,x_2,2,17,7,\emptyset,11,5,4,3,1)$, $\mathbf{f}=(21,14)$, $\mathbf{g}=(13,19)$.

4^63^1: $\mathbf{e}=(\emptyset,23,22,20,19,21,\emptyset,10,16,x_1,11,15,\emptyset,x_2,x_3,14,13,7,\emptyset,8,5,4,3,1)$, $\mathbf{f}=(9,2,17)$, $\mathbf{g}=(10,11,19)$.

For $n=10$, 14 and $0\leqslant b\leqslant 2(n-4)$, applying inflation construction with an ISOLS(4), filling every cell with an FSOLS(1^nk^1) where $0\leqslant k\leqslant (n-4)/2$, and taking the needed input designs from Theorem 1.3 (3), we obtain an FSOLS(4^nb^1).

From Theorem 3.13 we know that for $n=6,10$, there exists an FSOLS(1^{n+1}) with $(n-4)/2$ symmetric pair of holey transversals and one symmetric holey transversal, each of which has a hole of size one and all of them are disjoint. Applying Construction 3.6 with $t=h=1$, $p=1$, $q=(n-4)/2$, $m=4$, $0\leqslant v_1\leqslant 2$, $0\leqslant w_j\leqslant 2$, we obtain an FSOLS($4^nb^1$), for $n=6,10$ and $4\leqslant b\leqslant 2(n-1)$, because of the existence of IMOLS($4+v_1,v_1$) and IMOLS($4+w_j,w_j$) by Theorem 3.9.

Applying Lemma 3.14 and Construction 3.6 with $t=1$, $n=14$, $h=5$, $p=q=1$, $m=4$, $0 \leqslant v_1, w_1 \leqslant 2$, the input designs are from Theorems 3.8 and 3.9, we obtain the FSOLS($4^{14}b^1$) for $20 \leqslant b \leqslant 26$.

Combine Lemmas 4.1–4.3 and Theorem 1.3 (2), (4), (6) we have the main result of this section.

Theorem 4.4. *Suppose $n \geqslant 4$ and $a \geqslant 2$ are even. Then there exists an* FSOLS($a^n b^1$) *for* $0 \leqslant b \leqslant a(n-1)/2$.

5. Existence of FSOLS($a^n b^1$) for n even and a odd

Lemma 5.1. *Suppose $n \geqslant 4$ is even, $n \notin \{6, 10, 14\}$, and $a \geqslant 5$ is odd. Then there exists an* FSOLS($a^n b^1$) *for* $0 \leqslant b \leqslant a(n-1)/2$, *with possible exceptions* $(n, b) \in \{(18, (17a-1)/2), (22, (21a-1)/2)\}$. *Furthermore, if there exist both* FMOLS($1^{18}8^1$) *and* FMOLS($1^{22}10^1$), *then there exists an* FMOLS($a^n b^1$) *for* $0 \leqslant b \leqslant a(n-1)/2$.

Proof. For $a \geqslant 5$, from Theorem 3.10 we know that there is an FSOLS(1^n) with $n-1$ disjoint symmetric transversals. Applying Construction 3.5 with $t=1$, $p=n-1$, $q=0$, $s=1$, $h=a$, $0 \leqslant v_i$ even $\leqslant (a-1)/2$, $0 \leqslant k \leqslant n/2-1$ (for $n=18, 22, 0 \leqslant k \leqslant n/2-2$), the input designs are from Theorem 1.3(3), we obtain an FSOLS($a^n b^1$), where $0 \leqslant b \leqslant n/2 - 1 + (n-1)(a-1)/2 = a(n-1)/2 - \frac{1}{2} = \lfloor a(n-1)/2 \rfloor$ for $n \neq 18, 22$ and $0 \leqslant b \leqslant ((n-1)a-3)/2$ for $n = 18, 22$. The missing cases come from the fact that we do not have FSOLS($1^{18}8^1$) and FSOLS($1^{22}10^1$) as input designs in the construction. However, since we have both FMOLS($1^{18}8^1$) and FMOLS($1^{22}10^1$), if we use FMOLS instead of FSOLS as input designs, we obtain an FMOLS($a^n b^1$), where $0 \leqslant b \leqslant \lfloor a(n-1)/2 \rfloor$. □

Lemma 5.2. *For $a = 5, 7, 9, 11$, there exists an* FSOLS($a^6 b^1$) *for* $0 \leqslant b \leqslant a-1$.

Proof. For $a=5$, applying Construction 2.1 with **e**, **f** and **g** listed as follows.

$5^6 1^1$: $\mathbf{e} = (\emptyset, 29, 28, 26, 25, 27, \emptyset, 23, 16, 22, 13, 21, \emptyset, 2, x, 19, 15, 14, \emptyset, 20, 7, 11, 17, 8, \emptyset, 9, 5, 4, 3, 1)$, $\mathbf{f} = (10)$, $\mathbf{g} = (11)$.

$5^6 2^1$: $\mathbf{e} = (\emptyset, 29, 28, 26, 25, 27, \emptyset, 22, 16, 19, 23, 15, \emptyset, x_1, x_2, 14, 2, 20, \emptyset, 8, 21, 11, 17, 10, \emptyset, 9, 5, 4, 3, 1)$, $\mathbf{f} = (7, 13)$, $\mathbf{g} = (27, 11)$.

$5^6 3^1$: $\mathbf{e} = (\emptyset, 29, 28, 26, 25, 27, \emptyset, 23, 21, 10, 20, x_1, \emptyset, x_2, 13, 19, x_3, 14, \emptyset, 22, 15, 2, 11, 8, \emptyset, 9, 5, 4, 3, 1)$, $\mathbf{f} = (7, 16, 17)$, $\mathbf{g} = (17, 20, 8)$.

$5^6 4^1$: $\mathbf{e} = (\emptyset, 29, 28, 26, 25, 27, \emptyset, 23, 21, 19, 14, 22, \emptyset, x_1, 11, 16, x_2, 7, \emptyset, x_3, x_4, 20, 17, 8, \emptyset, 9, 5, 4, 3, 1)$, $\mathbf{f} = (2, 10, 13, 5)$, $\mathbf{g} = (3, 19, 8, 17)$.

For $a=7$, from [9, p. 199, Example 2.47] we know there exists a $\{5, 7\}$–GDD($7^6 3^1$), applying Construction 3.2 with weight one to each point we get an FSOLS of the same type.

Start with a TD(7,7) and apply Construction 3.2. Take two disjoint blocks B_1 and B_2. Give weight zero to the intersecting points with group seven. For other points of group seven, give weight one each. For points in the first 6 groups, give weight zero to points in B_1 and weight two to points in B_2, and give weight one to other points. The input designs, FSOLS of types 2^6, 1^7, 1^5 and $1^5 2^1$, are from Theorem 1.3. Then we obtain an FSOLS($7^6 5^1$).

FSOLS of types $7^6 1^1$, $7^6 2^1$, $7^6 4^1$ and $7^6 6^1$ can be obtained by applying Construction 2.1 with $\mathbf{e}, \mathbf{f}$ and $\mathbf{g}$ listed as follows.

$7^6 1^1$: $\mathbf{e} = (\emptyset, 41, 40, 38, 37, 39, \emptyset, 35, 34, 29, 21, 26, \emptyset, 32, 28, 23, 15, 27, \emptyset, 2, x, 25, 19, 10, \emptyset, 20, 7, 17, 31, 14, \emptyset, 11, 33, 22, 13, 9, \emptyset, 8, 5, 4, 3, 1)$, $\mathbf{f} = (16)$, $\mathbf{g} = (17)$.

$7^6 2^1$: $\mathbf{e} = (\emptyset, 41, 40, 38, 37, 39, \emptyset, 35, 34, 32, 31, 22, \emptyset, 28, 17, 29, 33, 14, \emptyset, 27, x_1, 25, 21, x_2, \emptyset, 26, 16, 10, 23, 7, \emptyset, 20, 19, 13, 11, 9, \emptyset, 8, 5, 4, 3, 1)$, $\mathbf{f} = (2, 15)$, $\mathbf{g} = (27, 10)$.

$7^6 4^1$: $\mathbf{e} = (\emptyset, 41, 40, 38, 37, 39, \emptyset, 35, 34, 32, 31, 26, \emptyset, 27, 15, 25, x_1, x_2, \emptyset, 16, 7, 10, x_3, x_4, \emptyset, 33, 29, 2, 23, 19, \emptyset, 14, 17, 13, 11, 9, \emptyset, 8, 5, 4, 3, 1)$, $\mathbf{f} = (20, 21, 22, 28)$, $\mathbf{g} = (11, 4, 20, 41)$.

$7^6 6^1$: $\mathbf{e} = (\emptyset, 41, 40, 38, 37, 39, \emptyset, 35, 34, 32, 31, 28, \emptyset, 27, 29, 25, x_1, 20, \emptyset, 2, 10, 22, x_2, x_3, \emptyset, x_4, 23, x_5, x_6, 16, \emptyset, 26, 17, 13, 11, 9, \emptyset, 8, 5, 4, 3, 1)$, $\mathbf{f} = (7, 14, 15, 19, 21, 33)$, $\mathbf{g} = (8, 31, 20, 11, 41, 4)$.

For $a = 9$, start with a TD(7,7) and apply Construction 3.2, take two disjoint blocks B_1 and B_2. Give weight zero or two to the intersecting points with group seven. For other points of group seven, give weight one each. For points in the first 6 groups, give weight two to points in B_1 or B_2, and give weight one to other points. The input designs, FSOLS of types 2^6, 2^7, 1^7, $1^5 2^2$ and $1^5 2^1$ are from Theorem 1.3 and Lemma 2.2. Then we obtain an FSOLS($9^6 b^1$) for $b = 5$ and 7.

Other desired FSOLS($9^6 b^1$) can be obtained by applying Construction 2.1:

$9^6 1^1$: $\mathbf{e} = (\emptyset, 53, 52, 50, 49, 51, \emptyset, 47, 46, 44, 43, 38, \emptyset, 45, 40, 26, 39, 32, \emptyset, 35, 41, 22, 19, 31, \emptyset, 14, x, 37, 27, 33, \emptyset, 34, 17, 20, 29, 25, \emptyset, 2, 21, 16, 11, 15, \emptyset, 23, 10, 13, 9, 7, \emptyset, 8, 5, 4, 3, 1), \mathbf{f} = (28)$, $\mathbf{g} = (29)$.

$9^6 2^1$: $\mathbf{e} = (\emptyset, 53, 52, 50, 49, 51, \emptyset, 47, 46, 44, 43, 45, \emptyset, 41, 37, 40, 35, 32, \emptyset, 22, 17, 29, 38, 34, \emptyset, x_1, 25, 31, x_2, 39, \emptyset, 21, 33, 28, 23, 2, \emptyset, 20, 11, 16, 27, 26, \emptyset, 15, 10, 13, 9, 7, \emptyset, 8, 5, 4, 3, 1), \mathbf{f} = (14, 19)$, $\mathbf{g} = (29, 32)$.

$9^6 3^1$: $\mathbf{e} = (\emptyset, 53, 52, 50, 49, 51, \emptyset, 47, 46, 44, 43, 45, \emptyset, 41, 40, 26, 37, 33, \emptyset, 34, 39, 31, 17, 22, \emptyset, 28, x_1, 35, 29, x_2, \emptyset, x_3, 21, 23, 38, 32, \emptyset, 20, 11, 16, 25, 19, \emptyset, 14, 10, 13, 9, 7, \emptyset, 8, 5, 4, 3, 1), \mathbf{f} = (2, 15, 27)$, $\mathbf{g} = (41, 23, 29)$.

$9^6 4^1$: $\mathbf{e} = (\emptyset, 53, 52, 50, 49, 51, \emptyset, 47, 46, 44, 43, 45, \emptyset, 41, 40, 34, 37, 33, \emptyset, 22, 28, 32, 23, x_1, \emptyset, x_2, x_3, 31, 25, 39, \emptyset, 26, x_4, 16, 21, 20, \emptyset, 15, 11, 38, 17, 2, \emptyset, 14, 10, 13, 9, 7, \emptyset, 8, 5, 4, 3, 1), \mathbf{f} = (19, 27, 29, 35)$, $\mathbf{g} = (23, 29, 44, 43)$.

$9^6 6^1$: $\mathbf{e} = (\emptyset, 53, 52, 50, 49, 51, \emptyset, 47, 46, 44, 43, 45, \emptyset, 41, 40, 38, 37, 28, \emptyset, 29, 39, x_1, 19, 27, \emptyset, 26, x_2, 16, 23, x_3, \emptyset, x_4, x_5, 32, x_6, 25, \emptyset, 15, 11, 22, 17, 2, \emptyset, 14, 10, 13, 9, 7, \emptyset, 8, 5, 4, 3, 1), \mathbf{f} = (20, 21, 31, 33, 34, 35)$, $\mathbf{g} = (3, 8, 29, 41, 39, 16)$.

$9^6 8^1$: $\mathbf{e}=(\emptyset,53,52,50,49,51,\emptyset,47,46,44,43,45,\emptyset,41,40,38,37,x_1,\emptyset,x_2,x_3,32,11,33,\emptyset,x_4,x_5,26,23,19,\emptyset,x_6,x_7,x_8,35,39,\emptyset,34,16,22,17,2,\emptyset,14,10,13,9,7,\emptyset,8,5,4,3,1)$, $\mathbf{f}=(15,20,21,25,27,28,29,31)$, $\mathbf{g}=(19,3,29,27,8,41,39,16)$.

For $a=11$, applying Construction 3.2 with a TD$(7,7)$, fix two blocks, say B_1 and B_2, which have no point in common. Give weight zero to each point of B_1 and weight one to each point of B_2. For the other five points in the group seven, we give weight zero, or one, or two to one point and weight zero or two to the rest. Give weight two to each of the remaining points of the TD$(7,7)$. The input designs are from Theorem 1.3 and Lemma 2.2. Then we obtain an FSOLS$(11^6 b^1)$ for $0 \leqslant b \leqslant 10$. □

Lemma 5.3. *Suppose there is a* TD$(7,t)$. *Then there exists an* FSOLS$(a^6 b^1)$ *for* $a=2t+1$ *and* $0 \leqslant b \leqslant a-1$.

Proof. Applying weighting construction with a TD$(7,t)$, fix a block B and a group G. Let x be the common point in both B and G. Give weight three to each point of $B\setminus\{x\}$. Give weight zero, or one, or two to x. Give weight zero or two to each point of $G\setminus\{x\}$. Give weight two to each of the remaining points. The input designs are from Theorem 1.3, then we obtain an FSOLS$(a^6 b^1)$ for $a=2t+1$ and $0 \leqslant b \leqslant a-1$. □

Lemma 5.4. *Suppose there is a* TD$(7,t)$. *Then there exists an* FSOLS$(a^6 b^1)$ *for* $a=2t-1$ *and* $0 \leqslant b \leqslant a-1$.

Proof. Start with a TD$(7,t)$, fix two blocks, say B_1 and B_2, which have no common point. Give weight zero to each point of B_1. In B_2, give weight zero, or one, or two to the point of group seven and weight three to other points. For the other $t-2$ points in the group seven, we give each of them weight zero or two. Give weight two to each of the remaining points of the TD$(7,t)$. The input designs are from Theorem 1.3, then we obtain an FSOLS$(a^6 b^1)$ for $a=2t-1$ and $0 \leqslant b \leqslant a-1$. □

Lemma 5.5. *There exists an* FSOLS$(a^6 b^1)$ *for odd* $a \geqslant 13$ *and* $0 \leqslant b \leqslant a-1$.

Proof. Combine Theorem 3.15 with Lemma 5.3 or Lemma 5.4 we obtain an FSOLS $(a^6 b^1)$ for $0 \leqslant b \leqslant a-1$ except for $a=29$.

For $a=29$, applying Construction 3.2 with a TD$(7,11)$, fix two blocks, say B_1 and B_2, which have no point in common. Give weight zero to each point of B_1. In B_2, give weight zero, or one, or two to the point of group seven and weight two to other points. For the other nine points in the group seven, we give each of them weight zero or three. Give weight three to each of the remaining points of the TD$(7,11)$. The input designs are from Theorem 1.3, then we obtain an FSOLS$(29^6 b^1)$ for $0 \leqslant b \leqslant 28$. □

Lemma 5.6. *There exists an* FSOLS$(a^6 b^1)$ *for odd* a, $a \geqslant 5$, *and* $a \leqslant b \leqslant (5a-3)/2$.

Proof. From Theorem 3.13 we know that there exists an FSOLS(1^7) with one symmetric pair of holey transversals and one symmetric holey transversal, each of which has a hole of size one as a common hole and all of them are disjoint elsewhere.

Applying Construction 3.6 with $t=h=1,\ n=6,\ p=q=1,\ m=a,\ 0\leqslant v_1\leqslant (a-1)/2, 0\leqslant w_1\leqslant (a-1)/2$ (if $a=5$, then $v_1\neq 1$, $w_1\neq 1$) and taking the input IMOLS from Theorem 3.9, we obtain an FSOLS(a^6b^1) for $a\leqslant b\leqslant (5a-3)/2,\ a\geqslant 7$ and an FSOLS(5^6b^1) for odd b, $5\leqslant b\leqslant 11$. FSOLS of types 5^66^1, 5^68^1 and 5^610^1 can be constructed by Construction 2.1:

5^66^1: $\mathbf{e}=(\emptyset,29,28,26,25,27,\emptyset,22,x_1,20,x_2,21,\emptyset,2,x_3,14,19,x_4,\emptyset,x_5,17,x_6,23,10,\emptyset,9,5,4,3,1)$, $\mathbf{f}=(7,8,11,13,15,16)$, $\mathbf{g}=(8,16,13,4,25,20)$.

5^68^1: $\mathbf{e}=(\emptyset,29,28,26,25,27,\emptyset,23,22,20,x_1,19,\emptyset,17,x_2,10,15,x_3,\emptyset,x_4,21,x_5,x_6,x_7,\emptyset,x_8,5,4,3,1)$, $\mathbf{f}=(2,7,8,9,11,13,14,16)$, $\mathbf{g}=(10,3,13,19,20,15,27,17)$.

5^610^1: $\mathbf{e}=(\emptyset,29,28,26,x_1,25,\emptyset,23,22,20,19,21,\emptyset,17,x_2,10,15,x_3,\emptyset,x_4,x_5,x_6,x_7,x_8,\emptyset,x_9,27,x_{10},3,1)$, $\mathbf{f}=(2,4,5,7,8,9,11,13,14,16)$, $\mathbf{g}=(3,13,19,17,21,7,15,8,22,27)$. □

Lemma 5.7. *There exists an* FSOLS($a^{10}b^1$) *for odd a and* $4a\leqslant b\leqslant (9a-1)/2$.

Proof. From Lemma 2.4 we know that there exists an FSOLS($1^{10}4^1$) with a symmetric holey transversal with a hole of size four. Applying a variation of Construction 3.6 with $t=1,\ n=10,\ h=4,\ p=1,\ q=0,\ m=a,\ 0\leqslant v_1\leqslant (a-1)/2$ (if $a=5$, then $v_1\neq 1$), the input designs are from Theorems 3.8 and 3.9, then we obtain an FSOLS($a^{10}b^1$) for $4a\leqslant b\leqslant (9a-1)/2,\ a\neq 5$ and FSOLS($5^{10}b^1$) for $b\in\{20,22\}$.

From Lemma 3.14 we know that there exists an FSOLS($1^{10}3^1$) with three symmetric holey transversal with a common hole of size three and occurring as a symmetric one and a symmetric pair. Applying a variation of Construction 3.6 with $t=1,\ n=10,\ h=3,\ p=1,\ q=1,\ m=5,\ v_1=2,\ w_1=2$, then we obtain an FSOLS($5^{10}21^1$). □

Lemma 5.8. *There exists an* FSOLS($a^{10}b^1$) *for odd* $a\geqslant 5$ *and* $0\leqslant b\leqslant 4a$.

Proof. (i) If $a\not\equiv 0 \pmod 3$ or $a\equiv 0 \pmod 9$, then from Theorem 3.12 we know that there is an FSOLS(1^a) with $a-1$ disjoint transversals occurring as $(a-1)/2$ symmetric pairs. Filling in size one holes provides another symmetric transversal in the resulting SOLS(a).

Applying Lemma 3.7 with $n=a,\ p=1,\ q=(n-1)/2,\ s=1,\ h=10,\ 0\leqslant v,w_j\leqslant 4,\ 0\leqslant j\leqslant (n-1)/2$, the input designs are from Theorem 1.3 (3), then we get an FSOLS($a^{10}b^1$) for $0\leqslant b\leqslant 4a$.

(ii) If $a\equiv 0 \pmod 3$ and $a\not\equiv 0 \pmod 9$, then we write $a=3n$, and there exists an SOLS(n) with n disjoint transversals occurring, a symmetric one on the main diagonal and $(n-1)/2$ symmetric pairs. Applying Lemma 3.7 with $p=1,\ q=(n-1)/2,\ s=3,\ h=10,\ 0\leqslant v,w_j\leqslant 13,\ 0\leqslant j\leqslant q$, the input designs are from Theorem 1.3 (5) and Lemma 5.7, then we get an FSOLS($(3n)^{10}b^1$) for $0\leqslant b\leqslant 13n$. □

Lemma 5.9. *There exists an* FSOLS($a^{14}b^1$) *for odd* $a\geqslant 5$ *and* $0\leqslant b\leqslant 5a$.

Proof. (i) If $a \not\equiv 0 \pmod 3$ or $a \equiv 0 \pmod 9$, then from Theorem 3.12 we know that there is an SOLS(a) with a disjoint transversals occurring, a symmetric one on the main diagonal and $(a-1)/2$ symmetric pairs. Applying Lemma 3.7 with $n=a$, $p=1$, $q=(a-1)/2$, $s=1$, $h=14$, $0 \leqslant v$, $w_j \leqslant 5$, $0 \leqslant j \leqslant (a-1)/2$, the input designs are from Theorem 1.3(3), then we get an FSOLS($a^{14}b^1$) for $0 \leqslant b \leqslant 5a$.

(ii) If $a \equiv 0 \pmod 3$ and $a \not\equiv 0 \pmod 9$, then we write $a=3n$, and there exists an SOLS(n) with n disjoint transversals occurring, a symmetric one on the main diagonal and $(n-1)/2$ symmetric pairs. Applying Lemma 3.7 with $p=1$, $q=(n-1)/2$, $s=3$, $h=14$, $0 \leqslant v$, $w_j \leqslant 18$, $0 \leqslant j \leqslant q$, the input designs are from Theorem 1.3 (5), then we get an FSOLS($(3n)^{14}b^1$) for $0 \leqslant b \leqslant 18n$. □

Lemma 5.10. *There exists an* FSOLS($a^{14}b^1$) *for odd a and* $5a \leqslant b \leqslant (13a-3)/2$.

Proof. From Lemma 3.14 we know that there exists an FSOLS($1^{14}5^1$) with three disjoint holey transversals with a common hole of size five occurring as a symmetric one and a symmetric pair, and an FSOLS($1^{14}4^1$) with five disjoint holey transversals with a common hole of size four occurring as a symmetric one and two symmetric pairs.

Start with the FSOLS($1^{14}5^1$), applying Construction 3.6 with $t=1$, $n=14$, $h=5$, $p=q=1$, $m=a$, $0 \leqslant v_1$, $w_1 \leqslant (a-1)/2$ (for $a=5$, $v_1 \neq 1$, $w_1 \neq 1$), and taking the input designs from Theorems 3.8 and 3.9, we obtain an FSOLS($a^{14}b^1$) for $5a \leqslant b \leqslant (13a-3)/2$, $a \neq 5$ and an FSOLS($5^{14}b^1$) for $b \in \{25, 27, 29, 31\}$.

Start with the FSOLS($1^{14}4^1$), applying Construction 3.6 with $t=1$, $n=14, h=4$, $p=1$, $q=2$, $m=5$, $v_1=w_1=w_2=0$ or 2, we can obtain an FSOLS($5^{14}b^1$) for $b \in \{26, 28, 30\}$. □

Combine Lemmas 5.1, 5.2, 5.5–5.10, Theorem 1.3 (5) we have the main result of this section.

Theorem 5.11. *Suppose* $n \geqslant 4$ *is even and* $a > 1$ *is odd. Then there exists an* FSOLS($a^n b^1$) *for* $0 \leqslant b \leqslant a(n-1)/2$, *with possible exceptions* $(n,b) \in \{(6,(5a-1)/2), (14,(13a-1)/2), (18,(17a-1)/2), (22,(21a-1)/2)\}$.

6. Existence of FSOLS($a^n 1^1$) for n odd

In this section, we extend Theorem 1.3 (7) and give the existence of FSOLS($a^n 1^1$) for n odd.

Lemma 6.1. *Suppose n is odd and* $n \geqslant 7$. *Then there exists an* FSOLS($a^n 1^1$).

Proof. From Theorem 1.3 (1) we know that there is an FSOLS(1^n). Applying Construction 3.5 with $=q=0$, $t=s=k=1$, $h=a$ $(a \neq 2,6)$ we obtain an FSOLS($a^n 1^1$).

FSOLS($2^n 1^1$) and FSOLS($6^n 1^1$) are from Theorem 1.3. □

Lemma 6.2. *Suppose there is a* TD$(6,t)$. *Then there exists an* FSOLS$(a^5 1^1)$ *for* $a=2t-1$ *and* $a=2t+1$.

Proof. Applying weighting construction with a TD$(6,t)$, take a block B and its intersecting point x with the last group. Give weight zero to x and weight one to other points in B. Give each point of the last group weight zero except one point y different from x, y receives weight one. Give weight two to each of the remaining points. The input designs are from Theorem 1.3 and Lemma 2.2, then we obtain an FSOLS$(a^5 1^1)$ for $a=2t-1$.

If we give weight three instead of weight one to each point except x in B, we obtain an FSOLS$(a^5 1^1)$ for $a=2t+1$. □

Lemma 6.3. *There exists an* FSOLS$(a^5 1^1)$ *for odd* $a>1$.

Proof. FSOLS$(a^5 1^1)$ for $a\geqslant 9$ can be obtained by applying Lemma 6.2 and Theorem 3.15.

FSOLS$(3^5 1^1)$ is from Theorem 1.3 (5). FSOLS$(5^5 1^1)$ and FSOLS$(7^5 1^1)$ can be constructed by Construction 2.1:

$5^5 1^1$: $\mathbf{e}=(\emptyset,24,23,4,18,\emptyset,9,16,x,21,\emptyset,17,6,12,22,\emptyset,13,8,11,7,\emptyset,3,14,2,1)$, $\mathbf{f}=(19)$, $\mathbf{g}=(11)$.

$7^5 1^1$: $\mathbf{e}=(\emptyset,34,33,22,28,\emptyset,18,16,19,17,\emptyset,24,29,12,32,\emptyset,9,4,6,11,\emptyset,27,23,x,21,\emptyset,7,13,31,8,\emptyset,3,26,2,1)$, $\mathbf{f}=(14)$, $\mathbf{g}=(26)$. □

Lemma 6.4. *There exists an* FSOLS$(a^5 1^1)$ *for* a *even*.

Proof. We know that an SOLS(n) exists if $n\neq 2,3,6$. Note that the main diagonal is a symmetric transversal. Applying Lemma 3.7 with $p=1$, $q=0$, $s=2$, $h=5$, $v=1$, the input designs FSOLS$(2^5 1^1)$ and FSOLS(2^5) are from Theorem 1.3 (4), then we obtain an FSOLS$((2n)^5 1^1)$.

Start with an SOLS(4), applying Lemma 3.7 with $p=1$, $q=0$, $s=3$, $h=5$, $v=1$ we obtain an FSOLS$(12^5 1^1)$.

FSOLS$(6^5 1^1)$ is from Theorem 1.3 (6).

Start with a TD$(6,5)$ (Theorem 3.15) and delete six points from one block to get a 5–GDD(4^6). Delete 3 points from a group of the GDD, this gives a $\{4,5\}$–GDD$(4^5 1^1)$. Applying Construction 3.2 and giving weight one to each point lead to an FSOLS$(4^5 1^1)$. □

From the above discussion, we can extend Theorem 1.3 (7) as follows.

Theorem 6.5. *For every* n *odd, there exists an* FSOLS$(a^n b^1)$ *if and only if* $n\geqslant 5$ *and* $n\geqslant 1+2b/a$, *except for* $(a,n,b)=(1,5,1)$.

7. Concluding remarks

The following theorem provides a summary of the main results of this article.

	6	8	7	4	5	3	2
5		7	8	6	3	4	1
6	1		2	8	7	5	4
2	5	1		7	8	6	3
8	7	2	3		4	1	6
7	8	4	1	3		2	5
4	3	6	5	2	1		
3	4	5	6	1	2		

	3	4	2	7	8	5	6
4		1	3	8	7	6	5
7	8		5	6	4	1	2
8	7	6		3	5	2	1
2	6	7	8		1	3	4
5	1	8	7	2		4	3
3	4	5	6	1	2		
6	5	2	1	4	3		

Fig. 10. An FMOLS($1^6 2^1$) with a holey transversal.

Theorem 7.1. *Suppose a, n and b are positive integers and $a \neq b$. Then there exists an* FSOLS($a^n b^1$) *if and only if $n \geqslant 4$ and $n \geqslant 1 + 2b/a$, except for $(a,n,b) = (1,6,2)$ and except possibly for* $(a,n,b) \in \{(t+2, 6, (5a-1)/2),\ (t, 14, (13a-1)/2),\ (t, 18, (17a-1)/2), (t, 22, (21a-1)/2)$: t is odd$\}$.

Proof. The necessity comes from Theorems 1.1 and 1.2. The sufficiency comes from Theorems 1.3 (3), 4.4, 5.11 and 6.5. □

For frame mutually orthogonal latin squares (FMOLS), we have the following.

Lemma 7.2. *There exists* FMOLS($1^n u^1$) *if and only if $n \geqslant 2u + 1$, where $u > 1$.*

Proof. From Theorems 1.1 and 1.3 (3), we need only to show the existence of FMOLS($1^n u^1$) for $(n,u) \in \{(6,2), (14,6), (18,8), (22,10)\}$.

An FMOLS($1^6 2^1$) is shown in Fig. 10.

FMOLS of type $1^{14} 6^1$ with a holey transversal with a hole of size six can be obtained by a modification of Construction 2.1 with the following vectors: $\mathbf{e}_1 = (\emptyset, 13, 12, 11, 10, 9, 8, 6, x_1,\ x_2,\ x_3,\ x_4, x_5,\ x_6)$, $\mathbf{f}_1 = (5,7,4,3,2,1)$, $\mathbf{g}_1 = (3,9,1,11,7,5)$, $\mathbf{e}_2 = (\emptyset,\ x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6, 13, 12, 11, 9, 8, 10, 6)$, $\mathbf{f}_2 = (1,2,3,4,5,7)$, $\mathbf{g}_2 = (1,3,5,8,9,10)$.

The modification of Construction 2.1 is: Construct an array $A = (a_{ij})$ by Construction 2.1 with vectors $\mathbf{e}_1$, $\mathbf{f}_1$ and $\mathbf{g}_1$ and array $B = (b_{ij})$ with $\mathbf{e}_2$, $\mathbf{f}_2$ and $\mathbf{g}_2$. Then A and B form an FMOLS.

Start with an FSOLS(6^4), filling the first three of the four holes of size six with three FMOLS($1^6 2^1$) we get an FMOLS of type $1^{18} 8^1$.

Start with an ISOLS(5, 1) shown in Fig. 11, which has two disjoint symmetric holey transversals with a common hole of size one, one is on cells (1, 1), (2, 2), (3, 3), (4, 4), the other is on cells (1, 4), (2, 3), (3, 2), (4, 1), fill each of the occupied cells on the holey transversals with an FSOLS($1^5 2^1$), for the first transversal and an ISOLS(7; 2,1) for the second, which exists because of the existence of an ISOLS(7,2), then we get a holey MOLS with a sub-MOLS of order eight in the lower right corner missing. Filling the missing sub-MOLS with an FOLS($1^4 2^1 1^2$) we get an FMOLS of type ($1^{22} 10^1$). □

0	4	3	2	1
2	1	0	4	3
4	3	2	1	0
1	0	4	3	2
3	2	1	0	

Fig. 11. An ISOLS(5, 1) with two holey transversals.

Theorem 7.3. *Suppose a, n and b are positive integers and* $a \neq b$. *Then there exists an* FMOLS($a^n b^1$) *if and only if* $n \geqslant 4$ *and* $n \geqslant 1 + 2b/a$.

Proof. For necessity, Theorems 1.1 and 1.2 are also true for FMOLS. For sufficiency, we need only to give the existence of the exception and possible exceptions of Theorem 7.1 for FMOLS.

Fig. 10 is an FMOLS of type $1^6 2^1$ with a holey transversal on cells $(1,2),(2,1),(3,4),(4,3),(5,6),(6,5)$, having a size two hole. Filling every pair of symmetric occupied cells not on the transversal into an MOLS(a) ($a > 1$ is odd) and that on the transversal into an IMOLS($a + (a-1)/2, (a-1)/2$). If one cell is filled with a square, then the symmetric cell is filled with the transpose of the orthogonal mate of the square. Then we obtain an FMOLS of type $a^6((5a-1)/2)^1$, where the size $(5a-1)/2$ hole contains the new size $(a-1)/2$ hole and the inflated size $2a$ hole from the size two hole in the FMOLS($1^6 2^1$).

From the proof of Lemma 7.2 we know that there exists an FMOLS of type $1^{14}6^1$ with a holey transversal with a hole of size six. Using the same way as in the construction of FMOLS($a^6((5a-1)/2)^1$), we obtain an FMOLS($a^{14}((13a-1)/2)^1$).

Lemma 5.1 provides FMOLS($a^{18}((17a-1)/2)^1$) and FMOLS($a^{22}((21a-1)/2)^1$), because the existence of FMOLS($1^{18}8^1$) and FMOLS($1^{22}10^1$) comes from Lemma 7.2, This completes the proof. □

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