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Research Article

The Cauchy Integral Operator on the Predual of a Morrey Space

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We show that the Cauchy integral operator is bounded from $H^{p,\varphi}(R^1)$ (the predual of a Morrey space) to $h^{p,\varphi}(R^1)$ (the local version of $H^{p,\varphi}(R^1)$). To prove our theorem we will introduce generalized atoms and consider a variant of "Tb theorem."

1. Introduction

Let R^n be the n-dimensional Euclidean space. The Cauchy integral operator C_A and Calderón's commutator T_A^k are defined, respectively, by

$$C_A f(x) = p.v. \int_{R^1} \frac{1}{x - y + i(A(x) - A(y))} f(y)\,dy,$$
$$T_A^k f(x) = p.v. \int_{R^1} \frac{(A(x) - A(y))^k}{(x-y)^{k+1}} f(y)\,dy, \tag{1.1}$$

where $A(x)$ is a real-valued function, and k is a positive integer. These operators are very important in real and complex analysis, and have attracted many mathematicians to investigate them; see, for example, [1–6].

The space, now called the Morrey space, was introduced by Morrey [7]. Since then, the space and the predual of it have been studied extensively; see, for instance, [8–11]. Recently, in [12], the author proved that Calderón's commutator T_A^1 is a bounded operator from $H^{p,\varphi}(R^1)$ to $h^{p,\varphi}(R^1)$, where $H^{p,\varphi}(R^1)$ is the predual of a Morrey space defined by using atoms and $h^{p,\varphi}(R^1)$ is a variant of $H^{p,\varphi}(R^1)$ which is similar to the local Hardy space (see Section 2). A natural question is whether C_A is bounded from $H^{p,\varphi}(R^1)$ to $h^{p,\varphi}(R^1)$. In Section 3, we will give the affirmative answer.

The relation between C_A and T_A^1 is written in [13]. Compared with Calderón's commutator, the Cauchy integral operator is difficult to study. Because we can calculate $T_A^1 1$ and apply "T1 theorem" by David and Journé [14], to prove the theorem, motivated by [6], we introduce a generalized atom and consider a variant of "Tb theorem."

2. Definitions and Notation

Throughout this paper, we always use the letter C to denote positive constants that may vary at each occurrence but is independent of the essential variables. And we assume that, unless otherwise stated, all given functions are complex valued.

We denote a Euclidean ball centered at x of radius r by $B(x,r)$ and the Lebesgue measure of a measurable set E by $|E|$.

First we recall some definitions of the ordinary atom and the ordinary Hardy space (see [4, 15, 16]).

Definition 2.1. Let $1 < p \le \infty$ and $n/(n+1) < q \le 1$. We say that a function $a(x)$ is an ordinary (H^q, p)-atom centered at x_0 if there exists a ball $B(x_0, r)$ such that the following conditions are satisfied:

$$\operatorname{supp}(a) \subset B(x_0, r), \tag{2.1}$$

$$\int_{R^n} a(x)dx = 0, \tag{2.2}$$

$$\|a\|_{L^p} \le \mathrm{r}^{n(1/p-1/q)}. \tag{2.3}$$

Definition 2.2. Let $\mathbf{H}^q(R^n)$ be Fefferman-Stein's Hardy space.

Remark 2.3. In general, the Hardy space is denoted by the symbol H^p. But in this paper, we will frequently use the symbol H^p to denote another Hardy space. So we use the symbol $\mathbf{H}^q$ for the ordinary Hardy space.

Definition 2.4. For $0 < \alpha \le 1$, the Lipschitz space $\Lambda^\alpha(R^n)$ and the local Lipschitz space $\Lambda^\alpha_{\mathrm{loc}}(R^n)$ are the set of all functions f satisfying the following conditions, respectively,

$$\begin{aligned} \|f\|_{\Lambda^\alpha} &= \sup_{0<|x-y|} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty, \\ \|f\|_{\Lambda^\alpha_{\mathrm{loc}}} &= \sup_{0<|x-y|<2} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty. \end{aligned} \tag{2.4}$$

It is easy to see that $\Lambda^1(R^n) = \Lambda^1_{\mathrm{loc}}(R^n)$ and $\Lambda^\alpha(R^n) \subset \Lambda^\alpha_{\mathrm{loc}}(R^n)$ $(0 < \alpha < 1)$, where the inclusion is proper. A simple example is $f(x) = x$, for which we can check that it is in

$\Lambda^{\alpha}_{\text{loc}}(R^1)$ but not in $\Lambda^{\alpha}(R^1)$ $(0 < \alpha < 1)$. Furthermore, we know that the dual space of $\mathbf{H}^p(R^n)$ is $\Lambda^{n(1/p-1)}(R^n)$, that is, $(\mathbf{H}^p(R^n))^* = \Lambda^{n(1/p-1)}(R^n)$, where $n/(n+1) < p < 1$ (see [17]).

Following Alvarez [8], we define the space $H^{1,p,\varphi}(R^n)$ by using atoms.

Definition 2.5. Let $1 < p \leq \infty$ and a function $\varphi : (0,\infty) \to (0,\infty)$ be given. We say that a function $a(x)$ is an (H^1, p, φ)-atom centered at x_0 if there exists a ball $B(x_0, r)$, which satisfies the conditions (2.1), (2.2), and

$$\|a\|_{L^p} \leq r^{n(1/p-1)}\varphi(r^n)^{-1}. \tag{2.5$'$}$$

Definition 2.6. We assume that $1 < p \leq \infty$ and the function $\varphi(t)$ is non-increasing and the function $t^{p'}\varphi(t)$ is non-decreasing, where $1/p + 1/p' = 1$.

We denote by $H^{1,p,\varphi}(R^n)$ the family of distributions f which can be written as $f = \sum_{i=1}^{\infty}\lambda_i a_i$, where a_i' are (H^1, p, φ)-atoms and $\sum_{i=1}^{\infty}|\lambda_i| < \infty$. And let the norm $\|f\|_{H^{1,p,\varphi}}$ be the infimum of $\sum_{i=1}^{\infty}|\lambda_i|$ over all representations of f.

Remark 2.7. If a is an (H^1, p, φ)-atom, then a is in $H^{1,p,\varphi}(R^n)$ and $\|a\|_{H^{1,p,\varphi}} \leq C_{n,p}$. Furthermore $H^{1,p,\varphi}(R^n) \subset H^{1,q,\varphi}(R^n)$ if $q < p$.

Remark 2.8. The space $H^{1,p,\varphi}(R^n)$ is a Banach space and the dual of it is the Morrey space $L^{p',\varphi}(R^n)$ (see [8, 11]).

Following Goldberg [18], we define the local version of $H^{1,p,\varphi}(R^n)$.

Definition 2.9. Let $1 < p \leq \infty$ and a function $\varphi : (0,\infty) \to (0,\infty)$ be given. We say that a function $a(x)$ is a large (h^1, p, φ)-atom centered at x_0 if there exists a ball $B(x_0, r)$ of radius $r \geq 1$ such that the conditions (2.1), (2.5$'$) are satisfied.

Definition 2.10. We assume that $1 < p \leq \infty$ and the function $\varphi(t)$ is non-increasing and the function $t^{p'}\varphi(t)$ is non-decreasing.

We denote by $h^{1,p,\varphi}(R^n)$ the family of distributions f which can be written as $f = \sum_{i=1}^{\infty}\lambda_i a_i$, where a_i is an (H^1, p, φ)-atom or a large (h^1, p, φ)-atom and $\sum_{i=1}^{\infty}|\lambda_i| < \infty$. And let the norm $\|f\|_{h^{1,p,\varphi}}$ be the infimum of $\sum_{i=1}^{\infty}|\lambda_i|$ over all representations of f.

Remark 2.11. One has$\|f\|_{h^{1,p,\varphi}} \leq \|f\|_{H^{1,p,\varphi}}$.

Remark 2.12. When $\varphi \equiv 1, h^{1,p,\varphi}(R^n)$ is the same as the original local Hardy space $h^1(R^n)$ defined by Goldberg [18] for all $p > 1$.

Remark 2.13. For the simplicity of notation, we write $H^{1,p,\varphi}(R^n) = H^{p,\varphi}(R^n), h^{1,p,\varphi}(R^n) = h^{p,\varphi}(R^n)$, and we denote the (H^1, p, φ)-atom by (H^p, φ)-atom, and the (h^1, p, φ)-atom by (h^p, φ)-atom.

Next we define Calderón-Zygmund operator. It can be referred to [4, 16]. But since we are interested in the Cauchy integral operator, our definitions will be presented as follows (see [6]).

Definition 2.14. Let $0 < \delta \leq 1$. A locally integrable function $K(x,y)$ defined on $\{(x,y) \in R^n \times R^n : x \neq y\}$ is called a Calderón-Zygmund kernel if it satisfies the following conditions:

$$\left|K(x,y)\right| \leq \frac{C}{|x-y|^n},$$
$$\left|K(x,y)-K(x,z)\right| + \left|K(y,x)-K(z,x)\right| \leq C\frac{|y-z|^\delta}{|x-z|^{n+\delta}}, \quad 2|y-z| < |x-z|. \tag{2.6}$$

We say that an operator T is a δ-Calderón-Zygmund operator associated with a Calderón-Zygmund kernel $K(x,y)$ if for every $f \in L^2(R^n)$,

$$Tf(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} K(x,y) f(y) dy \tag{2.7}$$

exists almost everywhere in R^n and T is bounded on $L^2(R^n)$, that is, $\|Tf\|_{L^2} \leq C\|f\|_{L^2}$.

Definition 2.15. The transpose of an operator T is denoted by

$${}^tTf(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} K(y,x) f(y) dy. \tag{2.8}$$

Definition 2.16. For a bounded function b, we define

$$\widetilde{{}^tT}b(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} \{K(y,x) - K(y,0)\chi_{|y|\geq 1}(y)\} b(y) dy. \tag{2.9}$$

Note that if $b \in L^2(R^n) \cap L^\infty(R^n)$, then $\widetilde{{}^tT}b(x) = {}^tTb(x) + C_b$ a.e. where C_b is a constant.

Definition 2.17. Let $\beta > 0$. A bounded function b is said to be β-accretive if $Reb(x) \geq \beta$ for almost all x.

3. Theorems

First we recall some known results. The L^p boundedness of C_A and T_A^k is well known, and the following theorem is the most essential (see [2, 4]).

Theorem 3.1. *If $A' \in L^\infty(R^1)$, then the Cauchy integral operator C_A and the Calderón's commutator T_A^k are both 1-Calderón-Zygmund operator.*

Recently, in [12], Komori showed that T_A^1 is bounded from $H^{p,\varphi}(R^1)$ to $h^{p,\varphi}(R^1)$, as follows.

Theorem 3.2. *Let $0 < \alpha < 1 < p \leq 1/(1-\alpha)$. If $A' \in L^\infty(R^1) \cap \bigwedge^\alpha(R^1)$, then T_A^1 is bounded from $H^{p,\varphi}(R^1)$ to $h^{p,\varphi}(R^1)$.*

The author proved this theorem as a corollary of the following theorem.

Theorem 3.3. *Let $0 < \alpha < 1 < p \le n/(n-\alpha)$. If T is a δ-Calderón-Zygmund operator such that $\widetilde{{}^tT}1 \in \bigwedge^{\alpha}(R^n)$, then T is a bounded operator from $H^{p,\varphi}(R^n)$ to $h^{p,\varphi}(R^n)$ and*

$$\|Tf\|_{h^{p,\varphi}} \le C\|f\|_{H^{p,\varphi}}. \tag{3.1}$$

Remark 3.4. Alvarez, in [8], proved that T is a bounded operator on $H^{p,\varphi}(R^n)$ provided $\widetilde{{}^tT}1 = C$.

Remark 3.5. Together with in [19, Theorem 2] and Theorem 3.3, we can check that the higher-order Calderón's commutator $T_A^k (k > 1)$ is also bounded from $H^{p,\varphi}(R^1)$ to $h^{p,\varphi}(R^1)$.

Next we turn to the Cauchy integral. Our main result is the following theorem.

Theorem 3.6. *Let $0 < \alpha \le 1 < p \le 1/(1-\alpha)$ and $p < \infty$. If $A' \in L^{\infty}(R^1) \cap \bigwedge^{\alpha}_{\mathrm{loc}}(R^1)$, then C_A is bounded from $H^{p,\varphi}(R^1)$ to $h^{p,\varphi}(R^1)$.*

In order to prove Theorem 3.6, we introduce a variant of "Tb theorem".

Theorem 3.7. *Let $0 < \alpha \le 1 < p \le n/(n-\alpha)$ and $p < \infty$. Assume that T is a δ-Calderón-Zygmund operator. If there exists a β-accretive function b such that $b, \widetilde{{}^tT}b \in \bigwedge^{\alpha}_{\mathrm{loc}}(R^n)$, then T is a bounded operator from $H^{p,\varphi}(R^n)$ to $h^{p,\varphi}(R^n)$ and*

$$\|Tf\|_{h^{p,\varphi}} \le C\|f\|_{H^{p,\varphi}}. \tag{3.2}$$

4. Some Lemmas

In this section, we define atoms and molecules on $h^{p,\varphi}(R^n)$ and study some properties of them.

Definition 4.1. Let $1 < p \le \infty$ and a function $\varphi : (0,\infty) \to (0,\infty)$ be given. We say that a function $a(x)$ is a small (h^p,φ)-atom centered at x_0 if there exists a ball $B(x_0,r)$ of radius $r < 1$, which satisfies the conditions (2.1), (2.5′), and

$$\left|\int_{R^n} a(x)dx\right| \le r^{n(1-1/p)}. \tag{4.1′}$$

Lemma 4.2. *If a is a small (h^p,φ)-atom, then $a \in h^{p,\varphi}(R^n)$ and*

$$\|a\|_{h^{p,\varphi}} \le C. \tag{4.2}$$

Proof. Assuming a supported in $B = B(x_0, r)$, we write

$$\begin{aligned} a(x) &= \left(a(x) - a_B\chi_B(x)\right) + a_B\chi_B(x) \\ &= a_1(x) + a_2(x), \end{aligned} \tag{4.3}$$

where $a_B = (1/|B|)\int_B a(y)dy$.

It is easy to check that $a_1(x)/2$ is an (H^p, φ)-atom, so

$$\|a_1\|_{h^{p,\varphi}} \le \|a_1\|_{H^{p,\varphi}} \le C. \tag{4.4}$$

Since $\operatorname{supp}(a_2) \subset B \subset B(x_0, 1)$ and

$$\begin{aligned} \|a_2(x)\|_{L^p} &= Cr^{n(1/p-1)}r^{n(1-1/p)} \\ &\le C\varphi(1)\varphi(1)^{-1}, \end{aligned} \tag{4.5}$$

a_2 is a constant multiple of a large (h^p, φ)-atom and we have

$$\|a\|_{h^{p,\varphi}} \le C. \tag{4.6}$$

□

Definition 4.3. Let b be β-accretive and $1 < p < \infty$. A function $a(x)$ is a small (h^p, φ, b)-atom centered at x_0 if there exists a ball $B(x_0, r)$ of radius $r < 1$, which satisfies the conditions (2.1), (2.5′), and

$$\left|\int_{R^n} a(x)b(x)dx\right| \le r^{n(1-1/p)}. \tag{4.6″}$$

Lemma 4.4. *Let $0 < \alpha \le 1 < p \le n/(n-\alpha)$ and $p < \infty$. Assume that b is β-accretive and $b \in \bigwedge_{\mathrm{loc}}^{\alpha}(R^n)$. If $a(x)$ is a small (h^p, φ, b)-atom, then $a \in h^{p,\varphi}(R^n)$ and*

$$\|a\|_{h^{p,\varphi}} \le C. \tag{4.7}$$

Proof. According to Lemma 4.2, we only need to show that a is a small (h^p, φ)-atom:

$$\begin{aligned}\left|\int_{B(x_0,r)} a(x)dx\right| &\le \left|\frac{1}{b(x_0)}\int_{B(x_0,r)} a(x)(b(x)-b(x_0))dx\right| \\ &\quad + \left|\frac{1}{b(x_0)}\int_{B(x_0,r)} a(x)b(x)dx\right| \\ &\le C\frac{r^\alpha}{\beta}\left(\int_{B(x_0,r)}|a(x)|^p dx\right)^{1/p}|B(x_0,r)|^{1/p'} + \frac{1}{\beta}r^{n(1-1/p)} \\ &\le C\frac{1}{\beta}\left(r^{\alpha+n(1/p-1)}\varphi(r^n)^{-1}+1\right)r^{n(1-1/p)} \\ &\le C\left(\varphi(1)^{-1}+1\right)r^{n(1-1/p)}.\end{aligned} \tag{4.8}$$

Note that we have used the fact that $p \le n/(n-\alpha)$, $p < \infty$ and $r < 1$ in the last inequality. □

Definition 4.5. Let $1 < p < \infty$ and $0 < \gamma$. A function $M(x)$ is called a large (h^p, φ, γ)-molecule centered at x_0 if there exists $r \ge 1$ such that the following conditions are satisfied:

$$\left(\int_{|x-x_0|\le 2r}|M(x)|^p dx\right)^{1/p} \le r^{n(1/p-1)}\varphi(r^n)^{-1}, \tag{4.9}$$

$$\left(\int_{|x-x_0|\ge 2r}|M(x)|^p|x-x_0|^\gamma dx\right)^{1/p} \le r^{n(1/p-1)+\gamma/p}\varphi(r^n)^{-1}. \tag{4.10}$$

Definition 4.6. Let $1 < p < \infty, 0 < \gamma$, and b be β-accretive. A function $M(x)$ is called a small $(h^p, \varphi, \gamma, b)$-molecule centered at x_0 if there exists $r < 1$ which satisfies (4.9), (4.10), and the following condition:

$$\left|\int_{R^n} \mathrm{M}(x)b(x)dx\right| \le r^{n(1-1/p)}. \tag{4.11}$$

Lemma 4.7. *Let $1 < p < \infty$ and $n(p-1) < \gamma$. If a function $M(x)$ is a large (h^p, φ, γ)-molecule, then $M \in h^{p,\varphi}(R^n)$ and*

$$\|M\|_{h^{p,\varphi}} \le C. \tag{4.12}$$

Lemma 4.8. *Let $0 < \alpha \leq 1 < p < \infty$ and $n(p-1) < \gamma$. Suppose that b is β-accretive and $b \in \bigwedge_{\text{loc}}^{\alpha}(R^n)$. If a function $M(x)$ is a small $(h^p, \varphi, \gamma, b)$-molecule, then $M \in h^{p,\varphi}(R^n)$ and*

$$\|M\|_{h^{p,\varphi}} \leq C. \tag{4.13}$$

Lemmas 4.7 and 4.8 are the key lemmas to prove our theorems. The proofs of two lemmas are similar in nature. So we will only prove Lemma 4.8. The idea of our proof comes from Komori in [6, 12, 20].

Proof of Lemma 4.8. Let $E_0 = \{x : |x - x_0| < 2r\}$ and $E_i = \{x : 2^i r \leq |x - x_0| < 2^{i+1} r\}$, $i = 1, 2, 3, \dots$, $b(E_i) = \int_{E_i} b(x)dx$. Since $b(E_i) \neq 0$, we denote that $\chi_i = \chi_{E_i}(x)$, $\widetilde{\chi}_i = \chi_i / b(E_i)$ and $m_i = (1/b(E_i)) \int_{E_i} b(x)M(x)dx$, $\widetilde{m}_i = \int_{E_i} b(x)M(x)dx$.

We write

$$M(x) = \sum_{i=0}^{\infty}(M(x) - m_i)\chi_i(x) + \sum_{i=0}^{\infty} m_i \chi_i(x) = \sum_{i=0}^{\infty} M_i(x) + \sum_{i=0}^{\infty} \widetilde{m}_i \widetilde{\chi}_i(x), \tag{4.14}$$

where $M_i(x) = (M(x) - m_i)\chi_i(x)$. Letting $N_j = \sum_{i=j}^{\infty} \widetilde{m}_i$, we have

$$\begin{aligned} M(x) &= \sum_{i=0}^{\infty} M_i(x) + \sum_{i=1}^{\infty} N_i\big(\widetilde{\chi}_i(x) - \widetilde{\chi}_{i-1}(x)\big) + N_0 \widetilde{\chi}_0(x) \\ &= I + II + III. \end{aligned} \tag{4.15}$$

Next we will calculate the above three terms.

(a) It is clear that $\text{supp}(M_i) \subset B(x_0, 2^{i+1} r)$ and $\int M_i(x) b(x) dx = 0$. So

$$\begin{aligned} \left(\int |M_0(x)|^p dx\right)^{1/p} &\leq \left(\int_{E_0} |M(x)|^p dx\right)^{1/p} + |m_0| |B(x_0, 2r)|^{1/p} \\ &\leq r^{n(1/p-1)} \varphi(r^n)^{-1} + C|m_0| r^{n/p}. \end{aligned} \tag{4.16}$$

By the definition of m_0 and condition (4.9), we have

$$\begin{aligned} |m_0| &= \frac{1}{|b(E_0)|} \left| \int_{E_0} M(y) b(y) dy \right| \\ &\leq C \frac{\|b\|_{L^\infty}}{\beta |E_0|} \left(\int_{E_0} |M(y)|^p dy \right)^{1/p} |E_0|^{1/p'} \\ &\leq C r^{n(1/p-1)} \varphi(r^n)^{-1} r^{-n/p}. \end{aligned} \tag{4.17}$$

Therefore we get

$$\left(\int |M_0(x)|^p dx\right)^{1/p} \leq C r^{n(1/p-1)} \varphi(r^n)^{-1}. \tag{4.18}$$

So $M_0(x)$ is a multiple constant of large (h^p,φ)-atom or small (h^p,φ,b)-atom and

$$\|M_0\|_{h^{p,\varphi}} \le C. \tag{4.19}$$

When $i \ge 1$, we have

$$\begin{aligned}\left(\int |M_i(x)|^p dx\right)^{1/p} &\le \left(\int_{E_i} |M(x)|^p dx\right)^{1/p} + |m_i \mid E_i|^{1/p}\\ &= \widetilde{I} + \widetilde{II}.\end{aligned} \tag{4.20}$$

By condition (4.10), we have

$$\begin{aligned}\widetilde{I} &\le \left(\int_{2^i r\le |x-x_0|<2^{i+1}r} \frac{|M(x)|^p|x-x_0|^\gamma}{(2^i r)^\gamma} dx\right)^{1/p}\\ &\le \left(2^i r\right)^{-\gamma/p}\left(\int_{|x-x_0|\ge 2^i r} |M(x)|^p|x-x_0|^\gamma dx\right)^{1/p}\\ &\le \left(2^i r\right)^{-\gamma/p} r^{n(1/p-1)+\gamma/p}\varphi(r^n)^{-1}\\ &\le C\left(2^{i+1}r\right)^{n(1/p-1)}\varphi\left(\left(2^{i+1}r\right)^n\right)^{-1}2^{-i(n(1/p-1)+\gamma/p)}.\end{aligned} \tag{4.21}$$

The last inequality was obtained for φ being non-increasing.

Using the estimate of $\widetilde{I}$, we have

$$\begin{aligned}\widetilde{II} &\le |E_i|^{1/p}\frac{\|b\|_{L^\infty}}{\beta|E_i|}\int_{E_i}|M(y)|dy\\ &\le C\left(\int_{E_i}|M(y)|^p dy\right)^{1/p}\\ &\le C\left(2^{i+1}r\right)^{n(1/p-1)}\varphi\left(\left(2^{i+1}r\right)^n\right)^{-1}2^{-i(n(1/p-1)+\gamma/p)}.\end{aligned} \tag{4.22}$$

Hence $M_i(x)$ is a multiple constant of large (h^p,φ)-atom or small (h^p,φ,b)-atom and

$$\|M_i\|_{h^{p,\varphi}} \le C2^{-i(n(1/p-1)+\gamma/p)}. \tag{4.23}$$

Since $p < (n+\gamma)/n$, we finally concluded

$$\sum_{i=0}^{\infty}\|M_i\|_{h^{p,\varphi}}^{p} \le C, \qquad \|I\|_{h^{p,\varphi}} \le C. \tag{4.24}$$

(b) Let $A_i = N_i(\widetilde{\chi}_i(x) - \widetilde{\chi}_{i-1}(x))$. It is easy to see that $\operatorname{supp}(A_i) \subset B(x_0, 2^{i+1}r)$ and $\int_{R^n} A_i(x)b(x)dx = 0,\ i = 1,2,3,\dots$.

Using the estimate of $\widetilde{I}$ again, we have

$$\begin{aligned}
\left(\int |A_i(x)|^p dx\right)^{1/p} &\le \left(\frac{|E_i|^{1/p}}{|b(E_i)|} + \frac{|E_{i-1}|^{1/p}}{|b(E_{i-1})|}\right)|N_i| \\
&\le C\frac{|E_i|^{1/p}}{\beta|E_i|}\|b\|_{L^\infty}\sum_{k=i}^{\infty}\int_{E_k}|M(y)|dy \\
&\le C|E_i|^{1/p-1}\sum_{k=i}^{\infty}\left(\int_{E_k}|M(y)|^p dy\right)^{1/p}|E_k|^{1/p'} \\
&\le C\left(2^i r\right)^{n(1/p-1)}\sum_{k=i}^{\infty}\left(2^k r\right)^{-\gamma/p} r^{n(1/p-1)+\gamma/p}\varphi(r^n)^{-1}\left(2^k r\right)^{n/p'} \\
&\le C2^{-i(\gamma/p)} r^{n(1/p-1)}\varphi(r^n)^{-1} \\
&\le C2^{-i(n(1/p-1)+\gamma/p)}\left(2^{i+1}r\right)^{n(1/p-1)}\varphi\left(\left(2^{i+1}r\right)^n\right)^{-1}.
\end{aligned} \tag{4.25}$$

So $A_i(x)$ is a multiple constant of large (h^p,φ)-atom or small (h^p,φ,b)-atom and

$$\|A_i\|_{h^{p,\varphi}} \le C2^{-i(n(1/p-1)\gamma/p)}. \tag{4.26}$$

Using condition $p < (n+\gamma)/n$ again, we get

$$\sum_{i=0}^{\infty}\|A_i\|_{h^{p,\varphi}}^{p} \le C, \qquad \|II\|_{h^{p,\varphi}} \le C. \tag{4.27}$$

(c) Condition (4.9) and estimate of $\widetilde{I}$ together with the fact $\operatorname{supp}(N_0\widetilde{\chi}_0(x)) \subset B(x_0, 2r)$ imply

$$\begin{aligned}
\left(\int \left|N_0\widetilde{\chi}_0(x)\right|^p dx\right)^{1/p} &\le \frac{|E_0|^{1/p}}{\beta|E_0|}\left|\int_{R^n} M(x)b(x)dx\right| \\
&\le C|E_0|^{1/p-1}\left(\int_{|x-x_0|\le 2r}|M(x)|dx + \int_{|x-x_0|\ge 2r}|M(x)|dx\right) \\
&\le Cr^{n(1/p-1)}\left(\int_{|x-x_0|\le 2r}|M(x)|^p dx\right)^{1/p} r^{n/p'} \\
&\quad + Cr^{n(1/p-1)}\sum_{i=1}^{\infty}\left(\int_{2^i r\le|x-x_0|<2^{i+1}r}|M(x)|^p dx\right)^{1/p}\left(2^{i+1}r\right)^{n/p'} \\
&\le Cr^{n(1/p-1)}\varphi(r^n)^{-1} \\
&\quad + Cr^{n(1/p-1)}\sum_{i=1}^{\infty}\left(2^i r\right)^{n(1-1/p)}\left(2^i r\right)^{-\gamma/p} r^{n(1/p-1)+\gamma/p}\varphi(r^n)^{-1} \\
&\le Cr^{n(1/p-1)}\varphi(r^n)^{-1}\left(1+\sum_{i=1}^{\infty}2^{-i(\gamma/p-n(1-1/p))}\right) \\
&\le Cr^{n(1/p-1)}\varphi(r^n)^{-1}.
\end{aligned} \tag{4.28}$$

By condition (4.11), we have

$$\left|\int_{R^n} N_0\widetilde{\chi}_0(x)b(x)dx\right| = \frac{1}{|b(E_0)|}\left|\int_{E_0} b(x)dx\right|\left|\int_{R^n} M(x)b(x)dx\right| \le r^{n(1-1/p)}. \tag{4.29}$$

By Lemma 4.4, we have

$$\left\|N_0\widetilde{\chi}_0\right\|_{h^{p,\varphi}} \le C. \tag{4.30}$$

Finally by (a), (b), (c) we end the proof of Lemma 4.8. □

5. Proof of the Theorems

First, we consider Theorem 3.7.

Proof of Theorem 3.7. To prove the theorem it suffices to show that there is a constant $C > 0$ such that $\|Ta\|_{h^{p,\varphi}} \le C$ for every (H^p, φ)-atom $a(x)$.

Assuming (H^p, φ)-atom $a(x)$ supported in $B(x_0, r)$, we show that $Ta(x)$ is a constant multiple of a large (h^p, φ, γ)-molecule with $r \ge 1$ or a constant multiple of a small $(h^p, \varphi, \gamma, b)$-molecule with $r < 1$.

Since T is bounded on $L^p(R^n)$ (see [4]), we have

$$\left(\int_{|x-x_0|\le 2r}|Ta(x)|^p dx\right)^{1/p} \le C\|a\|_{L^p} \le Cr^{n(1/p-1)}\varphi(r^n)^{-1}. \quad (5.1)$$

If $|x-x_0|\ge 2r$, then

$$\begin{aligned}|Ta(x)| &= \left|\int_{B(x_0,r)}\left[K(x,y)-K(x,x_0)\right]a(y)dy\right| \\ &\le C\frac{r^\delta}{|x-x_0|^{n+\delta}}\left(\int_{B(x_0,r)}\left|a(y)\right|^p dy\right)^{1/p}|B(x_0,r)|^{1/p'} \\ &\le C\frac{r^\delta\varphi(r^n)^{-1}}{|x-x_0|^{n+\delta}}.\end{aligned} \quad (5.2)$$

Since $(n+\delta)p-n>n(p-1)$, we can choose some γ satisfying $n(p-1)<\gamma<(n+\delta)p-n$. Therefore we obtain

$$\begin{aligned}\left(\int_{|x-x_0|\ge 2r}|Ta(x)|^p|x-x_0|^\gamma dx\right)^{1/p} &\le Cr^\delta\varphi(r^n)^{-1}\left(\int_{|x-x_0|\ge 2r}\frac{|x-x_0|^\gamma}{|x-x_0|^{(n+\delta)p}}\right)^{1/p} \\ &\le C\varphi(r^n)^{-1}r^{n(1/p-1)+\gamma/p}.\end{aligned} \quad (5.3)$$

If $r\ge 1$, by Lemma 4.7, we have

$$\|Ta\|_{h^{p,\varphi}}\le C. \quad (5.4)$$

If $r<1$, by (2.2), we get

$$\begin{aligned}\left|\int_{R^n}Ta(x)b(x)dx\right| &= \left|\left\langle a,\ \widetilde{{}^tT}b\right\rangle\right| \\ &= \left|\int_{B(x_0,r)}a(x)\left[\widetilde{{}^tT}b(x)-\widetilde{{}^tT}b(x_0)\right]dx\right| \\ &\le C\left\|\widetilde{{}^tT}b\right\|_{\Lambda^\alpha_{\text{loc}}}r^\alpha\int_{B(x_0,\mathrm{r})}|a(x)|dx \\ &\le Cr^{n(1-1/p)}r^{\alpha+n(1/p-1)}\varphi(r^n)^{-1} \\ &\le C\varphi(1)^{-1}r^{n(1-1/p)},\end{aligned} \quad (5.5)$$

which is obtained for $p\le n/(n-\alpha)$ and φ being non-increasing.

By Lemma 4.8 and the above argument, we obtain the desired result:

$$\|Ta\|_{h^{p,\varphi}} \le C. \tag{5.6}$$

Next we turn to Theorem 3.6. □

Proof of Theorem 3.6. Note that C_A is a 1-Calderón-Zygmund operator by Theorem 3.1. Let $b(x) = 1 + iA'(x)$. Then b is a 1-accretive and $b \in \bigwedge_{\mathrm{loc}}^{\alpha}(R^1)$. By the calculus of complex analysis (refer to calculation in [6] or [21, page 407]),

$$\begin{aligned} \widetilde{{}^tC_A}b(x) &= \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} \left\{ \frac{1+iA'(y)}{y-x+i(A(y)-A(x))} - \frac{1+iA'(y)}{y+i(A(y)-A(0))} \chi_{|y|\ge 1}(y) \right\} dy \\ &= \text{constant}, \end{aligned} \tag{5.7}$$

which implies $\widetilde{{}^tC_A}b(x) \in \bigwedge_{\mathrm{loc}}^{\alpha}(R^1)$. Therefore the theorem is proved by Theorem 3.7. □

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