On the $\phi_0$-Stability of Comparison Differential Systems

EDET P. AKPAN

Department of Mathematics,
University of Cross River State, Uyo, Nigeria

AND

OLUSOLA AKINYELE

Department of Mathematics,
University of Ibadan, Ibadan, Nigeria

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1. INTRODUCTION

For many years now, the variety of problems of the qualitative properties of differential equations in the context of Lyapunov second method has been successfully investigated in a unified way using the comparison principle [5]. In this method, the qualitative properties of the system of differential equations are inferred from the corresponding properties of the solutions of the system of comparison equations.

In order to successfully employ this method, it is generally required that the comparison systems possess some special properties. The conditions under which the scalar comparison equations possess positivity and stability behaviour have been investigated using scalar Lyapunov function method by Brauer [1], Siljak and Grujic [9], and others.


However, imbedded in the method of vector Lyapunov function is the requirement of quasimonotone nondecreasing property of the comparison system. But quasimonotonicity of the comparison system is not a necessary condition for the system to be stable. Thus the limitation of the application potential of this general and effective method is due to the fact that
comparison systems may have the desired property like stability without being quasimonotone.

In 1974, Lakshmikantham [4] observed that this difficulty is due to the choice of the cone relative to the comparison system, namely $R^n_+$, the cone of nonnegative elements in $R^n$, and that a possible approach to overcome this limitation is to choose an appropriate cone other than $R^n_+$ to work in a given situation depending on the problem at hand. In 1977, Lakshmikantham and Leela [6] initiated the development of the theory of differential inequalities through cones and the method of cone-valued Lyapunov functions.

It is natural to expect extensions of both the investigations of Ladde [3] and Lakshmikantham and Leela [6] to other types of stability using the method of cone-valued Lyapunov functions. In this paper we investigate the $\phi_0$-stability, uniform $\phi_0$-stability, asymptotic $\phi_0$-stability of comparison differential systems. We also obtain necessary and sufficient conditions for generalized exponential asymptotic $\phi$-stability and uniform asymptotic $\phi_0$-stability. We further obtain various stability results for a very general system of differential equations using the method of cone-valued Lyapunov functions. In the process we extend and generalize various results in [3, 6].

2. Notations and Definitions

Let $R^n$ denote the $n$-dimensional Euclidean space with any convenient norm $\| \cdot \|$, and scalar product $(, )$. $R_+ = [0, \infty)$, $R = (-\infty, \infty)$, $R^n_+ = \{ U \in R^n : U_i \geq 0, i = 1, 2, ..., n \}$. $C[R_+ \times R^n, R^n]$ denotes the space of continuous functions mapping $R_+ \times R^n$ into $R^n$.

**Definition 2.1.** A proper subset $K$ of $R^n$ is called a cone if (i) $\lambda K \subset K$, $\lambda \geq 0$; (ii) $K + K \subset K$; (iii) $K = \bar{K}$; (iv) $K^0 = \phi$; (v) $K \cap (-K) = \{ 0 \}$, where $\bar{K}$ and $K^0$ denote the closure and interior of $K$, respectively, and $\partial K$ denotes the boundary of $K$. The order relation on $R^n$ induced by the cone $K$ is defined as follows: Let $x, y \in K$, then $y \leq_k x$ iff $y - x \in K$ and $x <_{k^0} y$ iff $y - x \in K^0$.

**Definition 2.2.** The set $K^*$ is called the adjoint cone if $K^* = \{ \phi \in R^n : (\phi, x) \geq 0, \text{ for } x \in K \}$ satisfies properties (i)-(v) of Definition 2.1.

$x \in \partial K$ if and only if $(\phi, x) = 0$ for some $\phi \in K_0^*$, $K_0 = K - \{ 0 \}$.

**Definition 2.3.** A function $g : D \to R^n$, $D \subset R^n$ is said to be quasimonotone relative to the cone $K$ if $x, y \in D$ and $y - x \in \partial K$ imply that there exists $\phi_0 \in K_0^*$ such that $(\phi_0, y - x) = 0$ and $(\phi_0, g(y) - g(x)) \geq 0$. 
Consider the differential system

$$x' = f(t, x), \quad x(0) = x_0,$$

(1)

where $f \in C[R_+ \times R^n, R^n]$. Define $S_\rho$ by $S_\rho = \{x \in R^n : \|x\| < \rho, \ \rho > 0\}$. Let $K \subset R^n$ be a cone in $R^n$, $n \leq N$, and $V \in C[R_+ \times S_\rho, K]$. Define for $(t, x) \in R_+ \times S_\rho$, $h > 0$, the function $D^+ V(t, x)$ by $D^+ V(t, x) = \limsup_{h \to 0} (1/h) [V(t+h, x+hf(t, x)) - V(t, x)]$.

Consider the comparison differential system

$$u' = g(t, u), \quad u(t_0) = u_0,$$

(2)

where $g \in C[R_+ \times K, R^n]$, and $K$ is a cone in $R^n$. Let $S(\rho) = \{u \in K : \|u\| < \rho, \ \rho > 0\}, \ v \in C[R_+ \times S(\rho), K]$ and define for $(t, u) \in R_+ \times S(\rho)$, $h > 0$, the function $D^+ v(t, u)$ by $D^+ v(t, u) = \limsup_{h \to 0} (1/h) [v(t+h, u+hg(t, u)) - v(t, u)]$.

**DEFINITION 2.4.** The trivial solution $x = 0$ of (1) is equistable, if for each $\varepsilon > 0$, $t_0 \in R_+$, there exists a positive function $\delta = \delta(t_0, \varepsilon)$ that is continuous in $t_0$ for each $\varepsilon$ such that the inequality $\|x_0\| < \delta$ implies $\|x(t, t_0, x_0)\| < \varepsilon$, $t \geq t_0$.

Other stability notions can be similarly defined. (See [5].)

**DEFINITION 2.5.** The trivial solution $u = 0$ of (2) is $\phi_0$-equistable if given $\varepsilon > 0$, there exists $\delta = \delta(t_0, \varepsilon)$, continuous in $t_0$ for each $\varepsilon$ such that the inequality $(\phi_0, u_0) < \delta$ implies $(\phi_0, r(t)) < \varepsilon$, $t \geq t_0$, where $\phi_0 \in K^*_\phi$. In Definition 2.5, and for the rest of this paper, $r(t)$ denotes the maximal solution of (2) relative to the cone $K \subset R^n$.

Other $\phi_0$-stability concepts can be similarly defined.

**DEFINITION 2.6.** A function $a(\cdot)$ is said to belong to the class $\mathcal{X}$ if $a \in C[0, \rho], R_-$, $a(0) = 0$, and $a(r)$ is strictly increasing in $r$.

**DEFINITION 2.7.** (a) A function $v(t, u)$ is said to be positive definite relative to the cone $K$ (or $\phi_0$-positive definite) if there exists $a \in \mathcal{X}$ such that $a[(\phi_0, r(t))] \leq (\phi_0, v(t, u))$, $\phi_0 \in K^*_\phi$.

(b) A function $v(t, u)$ is said to be decrecent relative to the cone $K$ (or $\phi_0$-decrecent) if there exists $b \in \mathcal{X}$, $\phi_0 \in K^*_\phi$, such that $(\phi_0, v(t, u)) \leq b[(\phi_0, r(t))]$. 
3. Stability Theory

In this section, we present results giving sufficient conditions for \( \phi_0 \)-stability, uniform \( \phi_0 \)-stability, and asymptotic \( \phi_0 \)-stability of the trivial solution \( u = 0 \) of (2). We also investigate the corresponding stability concepts of the trivial solution \( x = 0 \) of (1) using the theory of differential inequalities through cones and the method of cone-valued Lyapunov functions.

**Theorem 3.1.** Assume that

(i) \( v \in C[R_+ \times S(\rho), K], v(t, 0) = 0, v(t, u) \) is locally Lipschitzian in \( u \) relative to \( K \), and for each \( (t, u) \in R_+ \times S(\rho), D^+v(t, u) \leq 0 \).

(ii) \( g \in C[R_+ \times K, R^n], g(t, 0) = 0, g(t, u) \) is quasimonotone in \( u \) relative to \( K \).

(iii) For some \( \phi_0 \in K_0^* \) and \( (t, u) \in R_+ \times S(\rho), a[(\phi_0, r(t))] \leq (\phi_0, v(t, u)), a \in \mathcal{X} \).

Then the trivial solution \( u = 0 \) of (2) is \( \phi_0 \)-equistable.

**Proof.** Since \( v(t, 0) = 0 \) and \( v(t, u_0) \) is continuous in \( t_0 \), then given \( a_1(\varepsilon) > 0, t_0 \in R_+ \), there exists \( \delta_1 \), such that \( \|u_0\| < \delta_1 \) implies \( \|v(t, u_0)\| < a_1(\varepsilon), a_1 \in \mathcal{X} \). Now for some \( \phi_0 \in K_0^* \), \( \|\phi_0\| \leq \|\phi_0\| < \|\phi_0\| \delta_1 \) implies \( \|\phi_0\| \|v(t, u_0)\| < \|\phi_0\| a_1(\varepsilon) \). Thus \( \|(\phi_0, u_0)\| < \|\phi_0\| \|u_0\| < \|\phi_0\| \delta_1 \) implies \( |(\phi_0, v(t, u_0))| \leq |\phi_0| \|v(t, u_0)\| < |\phi_0| a_1(\varepsilon) \). It follows that \( (\phi_0, u_0) < \delta \Rightarrow (\phi_0, v(t, u_0)) < a(\varepsilon), \) where \( \|\phi_0\| \delta_1 = \delta, \|\phi_0\| a_1(\varepsilon) = a(\varepsilon), a \in \mathcal{X} \). Let \( u(t) \) be any solution of (2) such that \( (\phi_0, u_0) < \delta \). Then by (i), \( v \) is nonincreasing and so \( v(t, u) \leq v(t_0, u_0), t \geq t_0 \). Thus \( (\phi_0, u_0) < \delta \) implies \( a[(\phi_0, r(t))] \leq (\phi_0, v(t, u)) < (\phi_0, v(t_0, u_0)) < a(\varepsilon) \Rightarrow (\phi_0, r(t)) < \varepsilon, t \geq t_0 \).

**Theorem 3.2.** Let the conditions (i) and (ii) of Theorem 3.1 hold. Assume further that for some \( \phi_0 \in K_0^*, (t, u) \in R_+ \times S(\rho),

\[ a[(\phi_0, r(t))] \leq (\phi_0, v(t, u)) \leq b[(\phi_0, r(t))], \quad a, b \in \mathcal{X}. \]

Then the trivial solution \( u = 0 \) of (2) is uniformly \( \phi_0 \)-stable.

**Proof.** For \( \varepsilon > 0 \), let \( \delta = b^{-1}[a(\varepsilon)] \) independent of \( t_0 \) for \( a, b \in \mathcal{X} \). Let \( u(t) \) be any solution of (2) such that \( (\phi_0, u_0) < \delta \). Then by hypothesis, \( v \) is nonincreasing and so \( (\phi_0, v(t, u)) \leq (\phi_0, v(t_0, u_0)) \). Thus \( a[(\phi_0, r(t))] \leq (\phi_0, v(t, u)) \leq b[(\phi_0, u_0)] < b(\delta) < a(\varepsilon) \).

\( \therefore (\phi_0, u_0) < \delta \) implies \( (\phi_0, v(t)) < \varepsilon. \)
THEOREM 3.3. Let the conditions of Theorem 3.1 hold with $D^+v(t, u) \leq C(t, u, v(t, u))$, $C \in \mathcal{K}$. Then the solution $u = 0$ of (2) is equi-asymptotically $\phi_0$-stable.

Proof. By Theorem 3.1, the trivial solution of (2) is $\phi_0$-equistable. By (3), $v(t, u)$ is monotone decreasing and hence the limit $v^* = \lim_{t \to \infty} v(t, u)$ exists. We claim that $v^* = 0$. Suppose $v^* \neq 0$. Then $C(v^*) \neq 0$, $C \in \mathcal{K}$. Since $C(r)$ is monotone, $C[(\phi_0, v(t, u))] > C[(\phi_0, v^*)]$, and so $D^+((\phi_0, v(t, u))) \leq -C[(\phi_0, v^*)]$. Integrating we obtain $((\phi_0, v(t, u))) \leq -C[(\phi_0, v^*)](t - t_0) + (\phi_0, v(0, u_0))$. Thus as $t \to \infty$ and for some $\phi_0 \in K^*$, we have $(\phi_0, v(t, u)) \to -\infty$. This contradicts the condition $a[(\phi_0, r(t))] \leq (\phi_0, v(t, u))$. It follows that $v^* = 0$. Thus $(\phi_0, v(t, u)) \to 0$ as $t \to \infty$ and so $(\phi_0, r(t)) \to 0$ as $t \to \infty$. Thus given $\varepsilon > 0$, $t_0 \in \mathbb{R}_+$, there exist $\delta = \delta(t_0)$ and $T = T(t_0, \varepsilon)$ such that for $t \geq t_0 + T$, $(\phi_0, u(t)) < \delta$ implies $(\phi_0, r(t)) < \varepsilon$.

THEOREM 3.4. Assume that

(i) $v \in C[\mathbb{R}_+ \times S(p), K]$, $v(t, 0) = 0$, and $v(t, u)$ is locally Lipschitzian in $u$ relative to the cone $K$ for $t \in \mathbb{R}_+$.

(ii) For each $t \in \mathbb{R}_+$, $(t, u) \in \mathbb{R}_+ \times S(p)$, and $C \in \mathcal{K}$

$$D^+((\phi_0, v(t, u))) \leq -C[(\phi_0, r(t))] .$$

(iii) $a[(\phi_0, r(t))] \leq (\phi_0, v(t, u)) \leq b[(\phi_0, r(t))]$, $a, b \in \mathcal{K}$.

Then the trivial solution $u = 0$ of (2) is uniformly asymptotically $\phi_0$-stable.

Proof. Let $\varepsilon > 0$ be given. Choose $\delta = \delta(\varepsilon)$ independent of $t_0$. Let $u(t)$ be a solution of (2) such that $(\phi_0, u_0) < \delta$. Then by Theorem 3.2, $u(t)$ is uniformly $\phi_0$-stable. Let $v^* = \{\sup(\phi_0, v(t_0, u_0)) : (\phi_0, u_0) < \delta\}$. Set $T(\varepsilon) = v^*/C[\varepsilon]$, $C \in \mathcal{K}$, then

$$(\phi_0, u(t)) < \delta \Rightarrow (\phi_0, r(t)) < \varepsilon, \quad t \geq t_0 + T(\varepsilon).$$

Suppose (4) is not true, then there would exist at least one $t \geq t_0 + T(\varepsilon)$ such that $(\phi_0, u(t)) < \delta$ implies $(\phi_0, r(t)) \geq \varepsilon$. Since $C \in \mathcal{K}$, from condition (ii), $D^+((\phi_0, v(t, u))) \leq -C(\varepsilon)$. Integrating, we obtain $(\phi_0, v(t, u)) \leq (\phi_0, v(t_0, u_0)) - C(\varepsilon)(t - t_0)$ for $t \geq t_0 + T(\varepsilon)$ and for sufficiently large $t$, this contradicts (iii) in which case (4) is established.

Remark 3.5. The results included in Theorems 3.1–3.4 extend the results contained in Theorems 3 and 4 of Ladde [3], and further generalize.
them to include uniform stability and uniform asymptotic stability of the trivial solution of a very general class of comparison system then in [3]. In particular if \( K = R^*_+ \) and \( \phi_0 = (1, 1, ..., 1) \) and if (2) is replaced by Eq. (8) in [3], then it will be seen that the results of Ladde [3] are included in our results as very special cases.

**Theorem 3.6.** Assume that

(i) \( V \in C[ R_+ \times S_{\rho}, K ] \), \( V(t, x) \) is locally Lipschitzian in \( x \) relative to \( K \), and for \( (t, x) \in R_+ \times S_{\rho} \), \( D^+ V(t, x) \leq_k g(t, V(t, x)) \).

(ii) \( g \in C[ R_+ \times K, R^n] \) and \( g(t, u) \) is quasimonotone in \( u \) relative to \( K \) for each \( t \in R_+ \),

(iii) \( f(t, 0) = 0, g(t, 0) = 0 \), for some \( \phi_0 \in K^{*}_0 \), \( (t, x) \in R_+ \times S_{\rho} \)

\[ b(\|x\|) \leq (\phi_0, V(t, x)) \leq a(t, \|x\|), \quad a, b \in \mathcal{X}. \]

Then the trivial solution \( x = 0 \) of (1) satisfies each one of the stability notions of Definition 2.4 if the trivial solution \( u = 0 \) of (2) satisfies the corresponding one of the stability notions of Definition 2.5.

*Proof.* (a) Let \( 0 < \varepsilon < \rho \) and \( t \in R_+ \). Suppose that the trivial solution \( u = 0 \) of (2) is \( \phi_0 \)-equistable. Then given \( b(\varepsilon) > 0 \), \( t_0 \in R_+ \), there exists \( \delta = \delta(t_0, \varepsilon) > 0 \), such that \( (\phi_0, u_0) < \delta \) implies \( (\phi_0, r(t)) < b(\varepsilon), t \geq t_0 \). Choose \( a(t_0, \|x_0\|) = (\phi_0, u_0) \), then \( (\phi_0, V(t_0, x_0)) \leq a(t_0, \|x_0\|) = (\phi_0, u_0) \Rightarrow V(t_0, x_0) \leq_k u_0 \). Let \( x(t, t_0, x_0) \) be any solution of (1) such that \( V(t_0, x_0) \leq_k u_0 \). Then by Theorem 3.1 in [6], \( V(t, x) \leq_k r(t) \). Now choose \( \delta_1 > 0 \) such that \( a(t_0, \delta_1) = \delta \). Thus the inequalities \( \|x_0\| \leq \delta_1 \) and \( a(t_0, \|x_0\|) < \delta \) hold simultaneously. Thus \( b(\|x\|) \leq (\phi_0, V(t, x)) \leq (\phi_0, r(t)) < b(\varepsilon) \Rightarrow \|x(t, t_0, x_0)\| < \varepsilon \) whenever \( \|x_0\| < \delta_1 \).

(b) In the proof of (a) choose \( \delta = \delta(\varepsilon) \) independent of \( t_0 \) and follow the same argument as in (a) to obtain the result.

(c) Suppose that the trivial solution \( u = 0 \) of (2) is quasi-equiasymptotically \( \phi_0 \)-stable. Then, following the same arguments as in (a) for all \( t \geq t_0 + T(\varepsilon) \) we find that there exists a positive function \( \delta - \delta(t_0, \varepsilon) \) satisfying the inequalities \( \|x_0\| < \delta \) and \( a(t_0, \|x_0\|) < \delta_0 \) simultaneously. It then follows that \( \|x_0\| < \delta_0 \Rightarrow \|x(t, t_0, x_0)\| < \varepsilon, t \geq t_0 + T \). If this were not true, there would exist a divergent sequence \( \{t_k\}, t_k \geq t_0 + T \) and a solution \( x(t, t_0, x_0) \) of (1) such that whenever \( x_0 < \delta \), we have that \( \|x(t, t_0, x_0)\| = \varepsilon \). Using Theorem 3.1 in [6] we are led to a contradiction

\[ b(\varepsilon) \leq (\phi_0, V(t_k, x(t_k, t_0, x_0})) \leq (\phi_0, r(t_k, t_0, u_0)) < b(\varepsilon). \]

(d) Since (a) and (c) are verified together, then \( x = 0 \) is equiasymptotically \( \phi_0 \)-stable.
(e) Since (b) holds, choose $\delta_0$ and $T$ in (c) independent of $t_0$ and proceed as in (c) to obtain the result.

**Theorem 3.7.** Let conditions (i) and (ii) of Theorem 3.6 hold. Assume further that for $c > 0$, $d > 0$, $(\phi_0, u_0) \leq \|x_0\|^d$ and $C \|x\|^d \leq (\phi_0, V(t, x))$. If the trivial solution $u = 0$ of (2) is exponentially asymptotically $\phi_0$-stable, then the trivial solution $x = 0$ of (1) is exponentially asymptotically stable.

**Proof.** Let $x(t, t_0, x_0)$ be any solution of (1) such that $V(t_0, x_0) \leq k u_0$. Then by Theorem 3.1 in [6] we have that $V(t, x) \leq k r(t)$. Thus $C \|x\|^d \leq (\phi_0, V(t, x)) \leq (\phi_0, r(t))$. Since the trivial solution $u = 0$ of (2) is exponentially asymptotically $\phi_0$-stable, then there exist $\sigma > 0$, $\alpha > 0$ both real numbers such that

$$(\phi_0, r(t)) \leq \sigma(\phi_0, u_0) \exp[-\alpha(t-t_0)], \ t \geq t_0,$$

and

$${C \|x\|^d \leq \sigma(\phi_0, u_0) \exp[-\alpha(t-t_0)].}$$

This implies that

$$\|x\| \leq M \|x_0\| \exp[-\beta(t-t_0)], \ t \geq t_0, \sigma/c = M, \alpha/d = \beta.$$

**Remark 3.8.** Theorems 3.6 and 3.7 contain results which further the results in Theorem 3.3 in [6] to include other stability notions and extend the results in Theorem 2.6.13 in [7].

## 4. Converse Theorems

In this section we give necessary conditions under which cone-valued Lyapunov functions can be constructed. Furthermore, we give necessary and sufficient conditions under which we have uniform asymptotic $\phi_0$-stability and the generalized exponential asymptotic $\phi_0$-stability of the trivial solution of the comparison differential system (2).

**Theorem 4.1.** Assume that

(i) $f \in C[R_+ \times S_\rho, R^n]$, $f(t, 0) = 0$, and $f(t, x)$ satisfies $\|f(t, x) - f(t, y)\| \leq L_1(t) \|x - y\|$, $(t, x), (t, y) \in R_+ \times S_\rho$; and $L_1 \in C[R_+, R_+]$, $\int_{t}^{t+\theta} L(s) \, ds < N/\theta$, $N$ is a constant.

(ii) The solution $x(t, 0, x_0)$ of (1) satisfies the estimate

$$\beta_1(\|x_0\|) \leq \|x(t, 0, x_0)\| \leq \beta_2(\|x_0\|), \ t \geq 0, \beta_1, \beta_2 \in \mathcal{K}.$$

(5)
(iii) \( g \in C[R_+ \times K, R^n], \) \( g(t, 0) = 0, \) and \( g(t, u) \) satisfies \( \|g(t, u) - g(t, v)\| \leq L_2(t) \|u - v\|, u, v \in K, \) \( L_2 \in C[R_+, R_+]. \)

(iv) The solution \( u(t, 0, u_0) \) of (2) verifies the estimate

\[
\gamma_2 \leq \|u(t, 0, u_0)\| \leq \gamma_1 \leq \gamma_2 \leq \gamma_1 [\phi_0, u_0]
\]

for some \( \phi_0 \in K^*, t \geq 0, \gamma_1, \gamma_2 \in \mathcal{K}. \)

Then there exists a cone-valued function \( V \) with the properties

(v) \( V \in C[R_+ \times S_\rho, K], V(t, x) \) is locally Lipschitzian in \( x \) for a continuous function \( \beta(t) > 0. \)

(vi) \( D^+ V(t, x) \leq \beta(t, V(t, x)). \)

(vii) \( b(\|x\|) \leq (\phi_0, V(t, x)) \leq a(\|x\|), a, b \in \mathcal{K}. \)

**Proof.** In view of hypotheses (i) and (iii) the existence and uniqueness of solutions of (1) and (2) as well as their continuous dependence on the initial values are assured. Let \( x(t, 0, x_0), u(t, 0, u_0) \) be the solutions of (1) and (2) through \( (0, x_0) \) and \( (0, u_0) \) satisfying (5) and (6), respectively. Let \( \sigma_\nu : S_\rho \rightarrow K \) be a function defined on \( S_\rho \) with values in the cone \( K \subset R^n, \) defined by

\[
\sigma_\nu(x) = \sup_{\delta > 0} G(\|x(t + \delta, t, x)\|) \left( \frac{1 + \alpha \delta}{1 + \delta} \right) w,
\]

where \( x \in S_\rho, G(r) \) is the function defined in Theorem 3.6.9 in [5], \( \alpha > 1, \) \( w \in K \) is any fixed point in \( K. \)

For \( \delta = 0 \) we have from (7) that \( G(\|x\|)w \leq \eta \sigma_\nu(x); \) and \( (\phi_0, G(\|x\|)w) \leq (\phi_0, \sigma_\nu(x)) \) for some \( \phi_0 \in K^*. \) For \( \beta_3 \in \mathcal{K}, \eta > 0, \) \( G(\|x\|)(\phi_0, w) = \eta G(\|x\|) \leq (\phi_0, \sigma_\nu(x)) \Rightarrow \beta_3(\|x\|) \leq (\phi_0, \sigma_\nu(x)). \) From estimate (5), the uniform asymptotic stability of \( x = 0 \) of (1) is implied. This then implies that \( x = 0 \) is uniformly stable. Thus by Theorem 5.4.3 in [8],

\[
\|x(t + \delta, t, x)\| < b(\|x\|), b \in \mathcal{K}.
\]

\[
\therefore \quad G(\|x(t + \delta, t, x)\|) < G(b(\|x\|)).
\]

Since \( (1 + \alpha \delta)/(1 + \delta) < \alpha, \) it follows that

\[
(\phi_0, \sigma_\nu(x)) \leq (\phi_0, xG(b(\|x\|))w) = \eta xG(b(\|x\|)) = \beta_4(\|x\|), \quad \beta_4 \in \mathcal{K}, \eta = (\phi_0, w).
\]

\[
\therefore \quad \beta_3(\|x\|) \leq (\phi_0, \sigma_\nu(x)) \leq \beta_4(\|x\|).
\]
We now show that $\sigma_w(x)$ is locally Lipschitzian in $x$. For $x, y \in S_{\nu}$,

$$
\|\sigma_w(x) - \sigma_w(y)\| = \sup_{\delta \geq 0} G(\|x(t + \delta, t, x)\|) \left(\frac{1 + x\delta}{1 + \delta}\right) w 
$$

$$
- \sup_{\delta \geq 0} G(\|y(t + \delta, t, y)\|) \left(\frac{1 + y\delta}{1 + \delta}\right) w 
$$

$$
\leq l(t, w) \| V_0(t, x) - V_0(t, y) \|,
$$

where $V_0(t, x) = \sup_{\delta \geq 0} G(\|x(t + \delta, t, x)\|) \left(\frac{1 + x\delta}{1 + \delta}\right)$ is the function $V(t, x)$ defined in Theorem 3.6.9 in [5], and by its proof $V(t, x)$ is locally Lipschitzian in $x$, in which case $\sigma_w(x)$ is also locally Lipschitzian in $x$. Obviously $\sigma_w(x)$ is also continuous.

Define a cone-valued function $V(t, x)$ by

$$
V(t, x) = u(t, 0, \sigma_w(x(0, t, x))),
$$

where $u(t, 0, u_0)$ are solutions of (2) passing through $(0, u_0)$, and $\sigma_w(x)$ is as defined in (7). By hypotheses (i) and (iii) and the choice of $\sigma_w(x), u, x,$ and $\sigma_w(x)$ are continuous so that $V(t, x)$, defined in (9), is continuous in all its arguments. From conditions (i), (iii), and Corollary 2.7.1 in [5] we obtain

$$
\| V(t, x) - V(t, y) \| = \| u(t, 0, \sigma_w(x(0, t, x))) - u(t, 0, \sigma_w(y(0, t, y))) \| 
$$

$$
\leq \| \sigma_w(x(0, t, x)) - \sigma_w(y(0, t, y)) \| \exp \int_0^t L_2(s) ds 
$$

$$
\leq l(t, w) \| x_0 - y_0 \| \exp \int_0^t L_2(s) ds 
$$

$$
\leq l(t, w) \| x - y \| \exp \int_0^t L_1(s) ds \exp \int_0^t L_2(s) ds 
$$

$$
= \beta(t) \| x - y \|,
$$

where $\beta(t) = l(t, w) \exp[\int_0^t (L_1(s) + L_2(s)) ds] > 0$, $(t, x, y) \in R_+ \times S_{\nu}$, and so (v) is proved. Next, for $h > 0$ sufficiently small,

$$
V(t + h, x + hf(t, x)) - V(t, x) 
$$

$$
\leq \beta(t) \| x + hf(t, x) - x(t + h, t, x) \| 
$$

$$
+ \beta(t) \| V(t + h, x(t + h, t, x)) - V(t, x) \|.
$$
Divide both sides by \( h \) and take \( \text{lim sup} \) as \( h \to 0^+ \) to obtain

\[
D^+ V(t, x) \leq \frac{1}{h} \text{Lim Sup}_{h \to 0^+} \left[ V(t + h, x(t + h, t, x)) - V(t, x) \right] = \text{Lim Sup}_{h \to 0^+} \left[ u(t + h, 0, \sigma_w(x(0, t, w))) - u(t, 0, \sigma_w(x(0, t, x))) \right] = u'(t, 0, \sigma_w(x, t, x)) = g(t, V(t, x)).
\]

Now from (5) we have

\[
\beta_2^{-1}(\|x\|) \leq \|x(t, t, x)\| \leq \beta_1^{-1}(\|x\|), \quad \beta_1, \beta_2 \in \mathcal{U}. \quad (10)
\]

Since \((\phi_0, V(t, x)) = (\phi_0, u(t, 0, \sigma_w(x(0, t, x))))\), (6), (8), (10) imply

\[
(\phi_0, V(t, x)) \geq \gamma_1[(\phi_0, \sigma_w(x(0, t, x)))] \geq \gamma_1[\beta_3[\beta_2^{-1}(x)]] = b(x), \quad b \in \mathcal{U}.
\]

Also (6), (8), (10) imply \((\phi_0, V(t, x)) \leq a(\|x\|), a \in \mathcal{U}\). This completes the proof of Theorem 4.1.

When \( \phi_0 \) and \( K \) in Theorem 4.1 are replaced by \( \phi_0 = (1, 1, \ldots, 1) \) and \( K = R^n \), respectively, we have the following:

**Theorem 4.2.** Assume that

(i) \( f \in C[R_+ \times S^\nu, R^n] \), \( f(t, 0) = 0 \), and \( f(t, x) \) satisfies \( \|f(t, x) - f(t, y)\| \leq L_1(t) \|x - y\| \), \( (t, x), (t, y) \in R_+ \times S^\nu \); and \( L_1 \in C[R_+, R_+] \), \( \int_0^t L(u) \, du \leq N|\theta|, N \) is a constant.

(ii) The solution \( x(t, 0, x_0) \) of (i) satisfies the estimate

\[
\beta_1(\|x_0\|) \leq \|x(t, 0, x_0)\| \leq \beta_2(\|x_0\|), \quad t \geq 0, \beta_1, \beta_2 \in \mathcal{U}.
\]

(iii) \( g \in C[R_+ \times R^n, R^n] \), \( g(t, 0) = 0 \), and \( g(t, u) \) satisfies \( \|g(t, u) - g(t, v)\| \leq L_2(t) \|u - v\|, u, v \in R^n \), \( L_2 \in C[R_+, R_+] \).

(iv) The solution \( u(t, 0, u_0) \) of (2) verifies the estimate

\[
\gamma_1 \left( \sum_{i=1}^n u_{0i} \right) \leq \sum_{i=1}^n u(t, 0, u_0) \leq \gamma_2 \left( \sum_{i=1}^n u_{0i} \right), \quad \gamma_1, \gamma_2 \in \mathcal{U}.
\]

Then there exists a vector function \( V \) with the properties
(v) \( V \in C[\mathbb{R}_+ \times S_{p}, \mathbb{R}^n] \) and \( V(t, x) \) is locally Lipschitzian in \( x \) for a continuous function \( \beta(t) > 0 \).

(vi) \( D^+ V(t, x) \leq g(t, V(t, x)) \).

(vii) \( b(\|x\|) \leq \sum_{i=1}^{r} V_i(t, x) \leq a(\|x\|), \ a, b \in \mathcal{K}. \)

Remark 4.3. Even in this special case of \( \phi_0 = (1, 1, \ldots, 1) \) and \( K = \mathbb{R}^n \), the result in Theorem 4.2 is new, and considerably improves Theorem 4.5.1 of [5]. Theorem 4.2 replaces the strong requirements of continuous partial derivatives of the functions \( f(t, x) \) and \( g(t, u) \) with a milder Lipschitz continuity of those functions, when the conditional requirements \( x_0 \in \mathcal{M}_{(n-k)}, \) and \( u_{0i} = 0 \) for \( i = 1, 2, \ldots, k \) are dropped. Note also that Theorem 4.2 still holds when condition (iv) is replaced by conditions (iv) and (v) of Theorem 4.5.1 of [5].

Theorem 4.4. Assume that

(i) \( g \in C[\mathbb{R}^n \times K, \mathbb{R}^n], \ g(t, 0) = 0, \ g(t, u) \) is quasimonotone in \( u \) relative to \( K \) for each \( t \in \mathbb{R}_+ \), and for \( (t, u), (t, v) \in \mathbb{R}_+ \times K \) \( \|g(t, u) - g(t, v)\| \leq L(t) \|u - v\|, L \in C[\mathbb{R}_+, \mathbb{R}_+]. \)

(ii) \( (\phi_0, r(t)) \leq \beta(\phi_0, u(t)), \ \beta \in \mathcal{K}, \ u(t) \) is a solution of (2).

Then the trivial solution \( u = 0 \) of (2) is uniformly asymptotically \( \phi_0 \)-stable if and only if there exists a function \( v \) with the following properties.

(iii) \( v \in C[\mathbb{R}_+ \times S_{p}, K], \ v(t, 0) = 0, \ v(t, u) \) is locally Lipschitzian in \( u \) relative to \( K \) for each \( t \in \mathbb{R}_+ \).

(iv) For some \( \phi_0 \in K^* \),

\[ a[(\phi_0, r(t))] \leq (\phi_0, v(t, u)) \leq b[(\phi_0, r(t))], \quad a, b \in \mathcal{K}. \]

(v) For \( (t, u) \in \mathbb{R}_+ \times S_{p}, \) and for \( p \in \mathcal{K}, \ p(t) \to \infty \) as \( t \to \infty \), \( D^+ v(t, u) \leq -p'(t) v(t, u), \) where \( p'(t) \) exists.

Proof. The sufficiency is straightforward, since the proof follows the same type of arguments used in Theorem 3.4.

Necessity. Condition (i) assures the existence and uniqueness of solutions of (2) as well as their continuous dependence on initial data. Let \( u = u(t, 0, u_0) \) so that \( u_0 = u(0, t, u) \). Define a cone-valued function \( v(t, u) \) by

\[ v(t, u) = \exp(-p(t)) C[(\phi_0, r(t))] u(t, 0, \sigma_w(u(0, t, u))), \]

where \( C[(\phi_0, r(t))] = (1/D)[1 - \exp(-D(\phi_0, r(t)))], \ D > 0, \ p \in \mathcal{K}, \ p'(t) \) exists and \( p(t) \to \infty \) as \( t \to \infty \), \( \sigma_w(x) \) is the function defined in (7); and \( u(t, u_0) \) is any solution of (2). When \( u = 0 \), then the right hand side of (11) vanishes so that \( v(t, 0) = 0. \)
Using (i) and Corollary 2.7.1 in [S] and for \( u_1, u_2 \in S(\rho) \),
\[
\|v(t, u_1) - v(t, u_2)\| \\
= \|\exp(-p(t)) C[(\phi_0, r(t))] u(t, 0, \sigma_u(u_1(0, t, u_1))) \\
- \exp(-p(t)) C[(\phi_0, r(t))] u(t, 0, \sigma_u(u_2(0, t, u_2)))\| \\
\leq |N(t)| \|\sigma_u(u_1(0, t, r_1)) - \sigma_u(u_2(0, t, u_2))\| \\
\leq l(t, w) |N(t)| \|u_1 - u_2\| \exp \int_0^t L(s) \, ds \\
= \beta(t) \|u_1 - u_2\|,
\]
where \( \beta(t) = l(t, w) |\exp(-p(t)) C[(\phi_0, r(t))]| \exp \int_0^t L(s) \, ds \geq 0 \). Now
\[
\|v(t + \delta, u^*) - v(t, u)\| \\
\leq \|v(t + \delta, u^*) - v(t + \delta, u)\| \\
+ \|v(t + \delta, u) - v(t + \delta, u(t + \delta, t, u))\| \\
+ \|v(t + \delta, u(t + \delta, t, u)) - v(t, u)\|.
\]
Since \( v(t, u) \) is locally Lipschitzian in \( u \) and \( u \) is continuous in \( \delta \), then the
first two terms in the right hand side of the inequality are small whenever
\( \|u - u^*\| \) and \( \delta \) are small. Using (11), the third term tends to zero as \( \delta \)
tends to zero. Therefore \( v(t, u) \) is continuous in all its arguments. Since
\( u = 0 \) is uniformly asymptotically \( \phi_0 \)-stable, then given \( \varepsilon > 0 \), there exist
two numbers \( \delta = \delta(\varepsilon) \) and \( T = T(\varepsilon) \), independent of \( t_0 \) such that
\( (\phi_0, u_0) < \delta \Rightarrow (\phi_0, r(t)) < \varepsilon \), for \( t \geq T(\varepsilon) \). And so
\[
(\phi_0, v(t, u)) = \exp(-p(t)) C[(\phi_0, r(t))]((\phi_0, u(t, 0, \sigma_u(u(0, t, u)))) \\
\leq \varepsilon C[(\phi_0, r(t))] \\
= b[(\phi_0, r(t))], \quad b \in \mathcal{X} \\
(\phi_0, v(t, u)) = \exp(-p(t)) C[(\phi_0, r(t))]((\phi_0, u(t, 0, \sigma_u(u(0, t, u)))) \\
\geq qC[(\phi_0, r(t))] \beta^{-1}[(\phi_0, r(t))] \quad \text{by condition (ii)} \\
- a[(\phi_0, r(t))], \quad a \in \mathcal{X} \text{ since } C, \beta^{-1} \in \mathcal{X},
\]
where \( q = \inf_{t \in [0, \varepsilon]} \{\exp(-p(t)), \delta\}, \delta > 0 \).
\[
\therefore \quad a[(\phi_0, r(t))] \leq (\phi_0, v(t, u)) \leq b[(\phi_0, r(t))], \quad a, b \in \mathcal{X}.
\]
\[
v(t + h, u + hg(t, u)) - v(t, u) \leq_k \beta(t) \|u + hg(t, u) - u(t + h, t, u)\| \\
+ v(t + h, u(t + h, t, u)) - v(t, u).
\]
Dividing both sides by \( h > 0 \) and taking \( \lim \sup \) as \( h \to 0^+ \), and using (11) and uniqueness of solutions of (2) we obtain

\[
D^+ v(t, u) \leq_k \limsup_{h \to 0^+} \frac{1}{h} [\exp(-p(t + h)) - \exp(-p(t)) C[(\phi_0, r(t))] u(t, 0, \sigma_w(u(0, t, u)))]
\]

For \( \phi_0 \)-equitable if for \( \varepsilon > 0 \), there exists \( \delta = \delta(t_0, \varepsilon) \), continuous in \( t_0 \) for each \( \varepsilon \) such that

\[
(\phi_0, \|u_0\|_{\phi_0}) < \delta \Rightarrow (\phi_0, \|r(t)\|_{\phi_0}) < \varepsilon, \quad t \geq t_0; \quad \phi_0 \in K^*_0.
\]
\[ u = u_0 + \int_{r_0}^{r} g(s, u(s)) \, ds \quad v = v_0 + \int_{r_0}^{r} g(s, v(s)) \, ds \]

\[ \|u - v\|_G \leq_k \|u_0 - v_0\|_G + \int_{r_0}^{r} L(s) \|u - v\|_G \, ds \]

\[ \leq_k \|u_0 - v_0\|_G \exp \left[ \int_{r_0}^{r} L(s) \, ds \right], \quad \text{by Gronwall's inequality.} \]

Similarly

\[ \|u_0 - v_0\|_G \leq_k \|u - v\|_G + \int_{r_0}^{r} L(s) \|u - v\|_G \, ds \]

\[ = \|u - v\|_G - \int_{r_0}^{r} L(s) \|u - v\|_G \, ds \]

\[ \leq_k \|u - v\|_G \exp \left[ - \int_{r_0}^{r} L(s) \, ds \right], \quad \text{since } \exp \left( \int_{r_0}^{r} L(s) \, ds \right) \geq 1. \]

\[ \|u_0 - v_0\|_G \exp \left[ - \int_{r_0}^{r} L(s) \, ds \right] \leq \|u - v\|_G. \]

**Theorem 4.8.** Assume that

(i) \( g \in C[R_+ \times K, R^n] \), \( g(t, 0) = 0 \), and \( g(t, u) \) is quasimonotone in \( u \) relative to \( K \) for each \( t \in R_+ \) and for \( (t, u), (t, v) \in R_+ \times K \) and \( L \in C[R_+, R_+] \), \( \|g(t, u) - g(t, v)\|_G \leq_k L(t) \|u - v\|_G \).

Then the trivial solution \( u = 0 \) of (2) is generalized exponentially asymptotically \( \phi_0 \)-stable if and only if there exists a cone-valued function with the following properties:

(ii) \( v \in C[R_+ \times S^*(\rho), K] \), \( v(t, 0) = 0 \), and \( v(t, u) \) is locally Lipschitzian in \( u \) relative to \( K \) for each \( t \in R_+ \) and for a continuous function \( \beta(t) \geq 0 \).

(iii) \( (\phi_0, \|r(t)\|_G) \leq (\phi_0, v(t, u)) \leq \sigma(t, t_0)(\phi_0, \|r(t)\|_G) \) for some \( \phi_0 \in K_0^* \), \( \sigma \in C[R_+ \times R_+, R_+] \), \( (t, u) \in R_+ \times S^*(\rho) \).

(iv) \( D^+ v(t, u) \leq_k -p'(t) v(t, u), \ p \in \mathcal{X} \) for \( (t, u) \in R_+ \times S^*(\rho) \), \( p'(t) \) exists, \( p(t) \to \infty \) as \( t \to \infty \).

**Proof.** Sufficiency. Integrating (iv) we obtain

\[ v(t, u) \leq_k v(t_0, u_0) \exp(p(t_0) - p(t)) \]

\[ (\phi_0, v(t, u)) \leq (\phi_0, v(t_0, u_0)) \exp(p(t_0) - p(t)) \]

\[ (\phi_0, \|r(t)\|_G) \leq \sigma(t, t_0)(\phi_0, u_0) \exp(p(t_0) - p(t)), \quad \text{from (iii)} \]

\[ (\phi_0, \|r(t)\|_G) \leq M(t)(\phi_0, u_0) \exp(p(t_0) - p(t)). \]
Necessity. Define a cone-valued function as

\[
\varphi(t, u) = \sup_{\delta \geq 0} \left\{ \| u(t + \delta, t, u) \|_C \exp(-p(t + \delta) + p(t)) \right\},
\]

where \( u(t, t_0, u_0) \) are solutions of (2) passing through \((t_0, u_0)\) and by (i) are continuous. Obviously \( \varphi(t, 0) = 0 \). Now for \( (t, u_1), (t, u_2) \in R_+ \times S^*(\rho) \), we have by Lemma 4.7 that

\[
\| v(t, u_1) - v(t, u_2) \|_C \\
= \| \sup_{\delta \geq 0} \left\{ \| u_1(t + \delta, t, u_1) \|_C \exp(-p(t + \delta) + p(t)) \right\} \\
- \sup_{\delta \geq 0} \left\{ \| u_2(t + \delta, t, u_2) \|_C \exp(-p(t + \delta) + p(t)) \right\} \|_C \\
\leq_k \sup_{\delta \geq 0} \left\{ \| u_1(t + \delta, t, u_1) - u_2(t + \delta, t, u_2) \|_C \exp(-p(t + \delta) + p(t)) \right\} \|_C \\
\leq_k \sup_{\delta \geq 0} \left\{ \exp(p(t + \delta) - p(t)) \right\} \| u_1 - u_2 \|_C \exp \int_{t_0}^t L(s) \, ds \\
= \beta(t) \| u_1 - u_2 \|_C,
\]

where

\[
\beta(t) = \sup_{\delta \geq 0} \left\{ \exp(-p(t + \delta) + p(t)) \right\} \exp \int_{t_0}^t L(s) \, ds \geq 0.
\]

Using (12), the continuity of \( \varphi \) follows as in Theorem 4.4. For \( \delta = 0 \), and by the uniqueness of solutions of \( (2) \), we have \( \| r(t) \|_C \leq v(t, u) \) so that \((\phi_0, \| r(t) \|_C) \leq (\phi_0, v(t, u))\).

Since \( u = 0 \) of (2) is generalized exponentially asymptotically \( \phi_0 \)-stable, we have, using Lemma 4.7 that

\[
(\phi_0, \varphi(t, u)) = \sup_{\delta \geq 0} \left\{ \exp(-p(t + \delta) + p(t)) \phi_0, \| u(t + \delta, t, u) \|_C \right\} \\
\leq \sup_{\delta \geq 0} \left\{ \exp(-p(t + \delta) + p(t)) \phi_0, \| r(t + \delta, t, u) \|_C \right\} \\
\leq \sup_{\delta \geq 0} \left\{ \exp(-p(t + \delta) + p(t)) M(t + \delta) \phi_0, \| u_0 \|_C \right\} \\
x \exp(p(t_0) - p(t + \delta)) \\
= \beta(t, t_0)(\phi_0, \| u_0 \|_C) \\
\leq \beta(t, t_0)(\phi_0, \| r(t) \|_C) \exp \int_{t_0}^t L(s) \, ds \\
= \sigma(t, t_0)(\phi_0, \| r(t) \|_C),
\]
where

\[ \sigma(t, t_0) = \sup_{\delta \geq 0} \{ M(t + \delta) \exp(p(t_0) - p(t)) \} \exp \int_{t_0}^{t} L(s) \, ds. \]

The proof of (iv) is similar to that of (v) in Theorem 4.4 with (11) replaced by (12). This completes the proof of Theorem 4.8.

**Remark 4.9.** If in Theorem 4.8, \( \sigma(t, t_0) \) and \( p(t) \) are replaced by \( \beta > 0 \) and \( \pi \), respectively, then the result giving the necessary and sufficient conditions for exponential asymptotic \( \phi_0 \)-stability of (2) is obtained.

**Remark 4.10.** It must be remarked that the full power of the method of cone-valued Lyapunov functions and the \( \phi_0 \)-stability theory as developed in this paper can be seen at work when it is employed in the analysis of the stability of large scale systems. However, the study of the stability of large scale systems using the method of cone-valued Lyapunov functions is currently being investigated and will be reported elsewhere.

5. Example

Consider a simple nonlinear differential system

\[
\begin{align*}
x'_1 &= -14x_1 - x_1^2 \exp(x_1) + 2x_2 - x_2^2 \exp(x_1) \\
x'_2 &= -18x_1 - x_2^2 \exp(x_2) + 2x_2 - x_2^2 \exp(x_2), \quad x_1 \geq 0.
\end{align*}
\]

(13)

We choose a single Lyapunov function \( V(t, x) = |x_1| + |x_2| \) for (13), then it is easy to show that

\[ D^+ V(t, x) \leq 32V(t, x) = g(t, V). \]

Now consider the scalar comparison equation

\[ u' = g(t, u) = 32u, \quad u(t_0) = u_0. \]

(14)

From (14), \( u = u_0 \exp 32t \). Evidently \( V \in C[R_+ \times R_+, R_+] \) and \( D^+ V \leq g(t, V) \), \( g \in C[R_+ \times R_+, R_+] \) and \((x_1^2 + x_2^2)^{1/2} \leq V(t, x) \leq x_1^2 + x_2^2\). This implies that \( a(\|x\|) \leq V(t, x) \leq b(\|x\|) \), where \( a(r) = r \) and \( b(r) = r^2 \Rightarrow a, b \in \mathcal{K}. \) But the trivial solution \( u = 0 \) of (14) is not stable. Then by Theorem 3.3.1 in [5], we cannot deduce the stability properties of (13) from those of (14) even though it is easy to see that (13) is indeed stable.

We now consider a vector Lyapunov function

\[ V(t, x) = (V_1, V_2)^T, \]

(15)
where \( V_1 = |x_1| \) and \( V_2 = |x_2| \). It can easily be checked that \( D^+ V_1 \leq -7 |x_1| + 2 |x_2| \) and \( D^+ V_2 \leq -9 |x_1| + 2 |x_2| \). Therefore

\[
D^+ V \leq \begin{bmatrix}
-7 & 2 \\
-9 & 2
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix} = g(t, V).
\] (16)

Clearly \( a(\|x\|) \leq \sum_{i=1}^{3} V_i(t, x) \leq b(\|x\|) \), where \( a(r) = r \), \( b(r) = r^2 \).

Now consider the comparison system

\[
u' = g(t, u) = Au,
A = \begin{bmatrix}
-7 & 2 \\
-9 & 2
\end{bmatrix}.
\] (17)

The vectorial inequality in (16) and all the other conditions of Theorem 4.4.6 in [5] are satisfied by (15) except that \( g \) in (17) is not quasimonotone nondecreasing in \( u \). Therefore we cannot deduce the stability properties of (13) from those of (17) using the vector Lyapunov function (15).

We now seek to construct a cone \( P \subset \mathbb{R}^2_+ \) relative to which the system (17) is quasimonotone. The eigenvalues of \( A \) in (17) are given by the roots of the equation

\[
\lambda^2 + 9\lambda + 4 = 0, \Rightarrow \lambda_1 = -1, \lambda_2 = -4.
\]

The eigenvectors are \((1, 3)^T\) and \((1, 3/2)^T\) corresponding to \( \lambda_1 = -1 \) and \( \lambda_2 = -4 \), respectively. Choose \( B = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} \), then \( B^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 3 \end{bmatrix} \) and \( B^{-1}AB = \begin{bmatrix} -4 & 0 \\ 0 & 9 \end{bmatrix} \). Thus the off-diagonal elements of \( B^{-1}AB \) are non-negative. Clearly \( B \) is a nonnegative, nonsingular \( 2 \times 2 \) matrix with which the mapping \( u = B\nu \) transforms (17) into

\[
u' = B^{-1}AB\nu.
\] (18)

By Theorem 3.1 in [2], there exists a cone \( P = \{ \sum_{i=1}^{2} u_i b_i; u_i \geq 0, i=1,2 \} \subset \mathbb{R}^2_+ \), generated by the 2 linearly independent column vectors of \( B \) relative to which (17) is quasimonotone.

Now choose

\[
V(t, x) = x(t, 0; \sigma_u(x(0, t, x)))
\] (19)
as a cone-valued Lyapunov function for (13). It is easy to check that the right hand side of (13) satisfies the conditions of Theorem 4.1 and so (19) has the properties

\[
V \in C[\mathbb{R}_+ \times S^*_\rho, P], \quad V(t, x) \text{ is locally Lipschitzian in } x
\]

\[
D^+_{(13)} V(t, x) \leq \rho g(t, V(t, x))
\]

\[
b(\|x\|) \leq (\phi_0, V(t, x)) \leq a(\|x\|), \quad a, b \in \mathcal{K}.
\] (20)
The Lipschitzian property of $g(t, u)$ in (17) assures the uniqueness of the solutions of (17) and so the maximal solution of (17) coincides with any other solution passing through the same point. Thus

$$r(t) = u_0 \exp(tA) \Rightarrow (\phi_0, r(t)) = (\phi_0, u_0) \exp(tA).$$

Now given $\varepsilon > 0$, there exists a $\delta$ such that $(\phi_0, u_0) < \delta$. Choose

$$\delta = \varepsilon \exp(-tA),$$

then we have $(\phi_0, r(t)) < \varepsilon \exp(-tA) \exp(tA) = \varepsilon$. This shows that $u = 0$ of (17) is $\phi_0$-equistable. This, (19), (20), and Theorem 3.6 imply that the trivial solution $x = 0$ of (13) is equistable.

Indeed, since the eigenvalues of $A$ are real and negative then as $t \to \infty$, $u_0 \exp(tA) \to 0$. This implies that the trivial solution $u = 0$ of (17) is asymptotically $\phi_0$-stable. This, (19), (20), and Theorem 3.6 imply that $x = 0$ is asymptotically stable. It is evident therefore that the method of cone-valued Lyapunov functions succeeds in showing that (13) is asymptotically stable whereas the method of scalar and vector Lyapunov functions failed.

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