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A nonlinear parabolic problem from combustion theory: attractors and stability

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Abstract

A parabolic (convection-diffusion) problem in a half-line, arising when modeling the temperature profile of an adiabatic solid in radiation-driven combustion, is considered. Both the coefficient in the "convective" term (the velocity of the burning front) and the Neumann datum (the prescribed heat influx into the burning body) are nonlinearly related to the proper value of the solution at the boundary. In addition, the velocity is allowed to vanish below some threshold value. Under the main assumptions of "intensely irradiated boundary" and initial data that behave suitably as $x \to -\infty$, it is proven that there exists a global attractor for the evolution semigroup associated with the problem. Furthermore, the stabilization of all solutions towards the equilibrium solution (a uniformly propagating front) is derived for a class of Neumann data, which are of some interest for applications.

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1. Introduction

Consider the nonlinear initial-boundary value problem in one space dimension

$$u_t = u_{xx} - R\,(u\,(0,t))\;u_x, \qquad x < 0,\; t > 0, \tag{1a}$$

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$$u_x(-\infty, t) = 0, \qquad t > 0, \tag{1b}$$

$$u_x(0, t) = Q(u(0, t)), \qquad t > 0, \tag{1c}$$

$$u(x, 0) = u_0(x), \qquad x \leqslant 0, \tag{1d}$$

where $R(u)$ and $Q(u)$ are regular nonnegative functions throughout the real axis and $u_0(x)$ is a continuous function tending to zero as $x \longrightarrow -\infty$. A number of practically relevant problems are suitable cases for treatment via (1). As a typical situation of interest, we might refer to solid-propellant rocket propulsion theory, when modeling combustion of homogeneous materials under the influence of external irradiation of the "interface" between condensed and gas phases (see the discussion in [9] and references quoted therein). In this context, within the framework of quasiplanarity of all spatial variations and concentrated surface kinetics, the burning solid is assumed to occupy the half-line $x < 0$ and the origin $x = 0$ is attached to the propagating ("flame") front, moving at a rate $-R$ in the lab reference frame. Then, by scaling to nondimensional units, the heat transfer yields the governing equation (1a) for the temperature profile $u(x, t)$ at time t. The rate R depends on the front temperature: a common example is the Arrhenius function $R(u) \propto \exp\left\{-\alpha (u - u_*)^{-1}\right\}$, where α, u_* are constants. A worthwhile feature of the model is that the burning rate R, supported at the interface, is allowed to vanish at (and below) the "ignition temperature" $u = u_*$, hence the model possibly includes nonmoving fronts. Regarding initial and boundary conditions, the prescribed limiting value of u_0 is the ambient temperature, which is taken to be zero without loss of generality; (1b) means no heat exchange with the surrounding medium and, finally, (1c) is the overall heat flux balance at the front. The Neumann datum Q includes: (i) radiant flux, impinging on the burning surface from an external source of thermal nature (a laser) and constant intensity; (ii) chemical heat release, simply proportional to R and coming from exothermic chemical reactions; (iii) gas phase heat release, due to heat feedback from the gas. The latter (say, Q_g) is typically a nonlinear function of R, vanishing as $R \to \infty$ (flame blow-off) and as $R \to 0$ (no burning), and positive for $R > 0$ (gas phase heats up condensed phase during burning). Q_g comes from suitable gas-phase transient submodels: for an illustration, consider the so-called KTSS heat feedback $Q_g(R) \propto R^{-1}\left(1 - \exp\left\{-\beta R^2\right\}\right)$, where β is a constant.

Similar problems, arising as models of a variety of physical processes, have been investigated in the literature. In particular, we would like to mention a series of papers by Ball and Peletier [1,2] and by Frankel and Roytburd [6–8], where problems closely related to (1) are considered. For example, according to the above nomenclature, [8] deals with the gasless case (no heat feedback) in pressure-driven combustion (absence of external radiation) and the energy conservation equation (1a) includes a damping term $-\gamma u$ (volumetric heat loss). On the contrary, in solid-propellant combustion modeling, the heat feedback term is essential. This is because the surface temperature adjustments due to solid–gas coupling are expected to have a large influence on the burning phenomenon [5]. Moreover, in model (1) it is assumed $\gamma = 0$ since the propellant condensed phase is commonly considered adiabatic, possibly suffering heat losses from the burning surface only [4]. As a consequence, the solution procedure of [8] now fails because it relies crucially on the presence of the damping term for proving that the contribution of the solution from initial data decays with time and, ultimately, for establishing the existence of a compact attractor.

Concerning well-posedeness, it is shown in [9] that for exponentially decaying, continuous data u_0 problem (1) has a unique global-in-time classical solution $u(x,t)$; in addition, the same exponential behavior holds for u and its spatial derivative for all positive t. Our aim in this paper is to address some questions of long-time asymptotics. This study is motivated on the basis of experimental as well as numerical evidences [4]. For instance, it is observed that under suitable operating conditions the thermal profile approaches a traveling wave, i.e., a solution which appear to be travelling with constant shape and velocity in the laboratory frame and is time-invariant in the reference frame anchored to the propagating front. In [11] the asymptotic linearized stability of travelling waves and related questions, such as the existence of Hopf bifurcations, are investigated.

The paper is organized as follows. In Section 2, the evolution semigroup associated with the problem is properly defined and some preliminary properties needed in the subsequent asymptotic analysis are established. In Section 3, uniform boundedness of the solution at large time and existence of absorbing sets are proved under the supplementary assumption of "strong irradiation" (namely, a condition on the lower bound of Q); we anticipate that the techniques we use are based on comparison principles. Then the main results are derived. Section 4 is devoted to the existence of a global attractor, i.e., a compact set which attracts all bounded set, and in Section 5 we exhibit a condition on Neumann data yielding existence of a Lyapunov function for the semigroup and approach to equilibrium of the dynamics. The paper concludes with a section of final remarks (Section 6) and a technical section (Appendix A), which includes the proofs of continuity properties of the semigroup.

2. The semigroup

For the mathematical setting of the problem we assume that

H1. $R(u)$ and $Q(u)$ are twice differentiable nonnegative functions such that $R, |R'|, |R''|, Q, |Q'|, |Q''| \leqslant M$ for some constant M

and for later use we set

$$R_{\max} = \sup R(u), \qquad Q_{\min} = \inf Q(u), \qquad Q_{\max} = \sup Q(u).$$

Let $\eta > 0$ be given. We denote by X^0_η the Banach space of real functions $z(x) \in C^0(-\infty, 0]$ such that $\|z\|_{0,\eta} \equiv \sup_{x<0} e^{-\eta x}|z(x)| < \infty$ and by X^1_η the Banach space of real functions $z(x) \in C^1(-\infty, 0]$ such that $\|z\|_{1,\eta} \equiv \|z\|_{0,\eta} + \|z'\|_{0,\eta} < \infty$. Furthermore, let Ω be the quarter-plane $x < 0$, $t > 0$ and Γ be the half-line $x = 0$, $t > 0$. Then a function $u = u(x,t)$ is a (classical global) solution of problem (1) if $u \in C^0\left(\overline{\Omega}\right) \cap C^1(\Omega)$, $u_{xx} \in C^0(\Omega)$, $u_x \in C^0(\Omega \cup \Gamma)$ and (1) hold.

Existence and uniqueness was established in [9]:

Theorem 2.1. *We assume that the hypothesis* H1 *is satisfied. Then, for* u_0 *given in* X_η^0, *there exists a unique solution* u *to* (1). *In addition,* $u(\cdot,t)$ *belongs to* X_η^1 *for all* $t>0$.

It is also helpful to state the following further regularity result, which is important for Section 5. We give the details of the proof in Appendix A.1.

Proposition 2.1. *Let Theorem* 2.1 *hold. Then* $u_t \in C^0(\Omega \cup \Gamma)$ *and, furthermore,* $u_t(\cdot,t)$ *belongs to* X_η^0 *for all* $t>0$.

Theorem 2.1 allows us to define the operators

$$S(t) : u_0 \longmapsto u(\cdot,t), \quad t \geqslant 0$$

from X_η^0 into itself. These operators enjoy the semigroup properties $S(0)=I$ (identity), $S(t+\tau)=S(t)\,S(\tau)$ for $t,\tau \geqslant 0$ (as usual, we write $S(t)\,S(\tau)$ instead of $S(t)\circ S(\tau)$). Furthermore, the following continuity property holds.

Lemma 2.1. *For* $t>0$, $S(t)$ *is a continuous mapping from* X_η^0 *into* X_η^1.

Proof. We only sketch the proof. First note that it suffices to show the assertion for small t. By application of the local existence result [9, Section 3], there exist $\varepsilon>0$ and $\gamma>0$ such that for $0\leqslant t\leqslant\gamma$ and for u_0 given in X_η^0 the solution u to (1) can be written as a sum

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{0} E(x+s(t)-y,t)\,u_0(y)\,dy + 2\int_0^t E(x+s(t)-s(\tau),t-\tau) \\ &\quad\times\left[Q(\theta(\tau)) - \varepsilon\frac{\phi(\tau)}{\sqrt{\tau}}\right]d\tau \\ &= h_1(x,t)+h_2(x,t), \end{aligned} \tag{2}$$

where $E(z,t)=(4\pi t)^{-1/2}\exp\left(-z^2/4t\right)$ is the one-dimensional heat kernel, $s(t) = -\int_0^t R(\theta(\tau))\,d\tau$ and the densities θ,ϕ belong to $C^0[0,\gamma]$. Let u_{0n} be a sequence in X_η^0 converging in X_η^0 to an element u_0 and set $u_n(x,t)=(S(t)\,u_{0n})(x)=h_{1n}(x,t)+h_{2n}(x,t)$ with densities θ_n,ϕ_n and $s_n(t)=-\int_0^t R(\theta_n(\tau))\,d\tau$. We show in Appendix A.2 that $\theta_n\to\theta$ and $\phi_n\to\phi$ uniformly on $0\leqslant t\leqslant\gamma$. Next we define g_n and g by

$$g_n(x,t) = \int_{-\infty}^{0} e^{\eta y}\left|E(x+s_n(t)-y,t)-E(x+s(t)-y,t)\right|dy,$$

$$g(x,t) = \int_{-\infty}^{0} e^{\eta y}\left|E(x+s(t)-y,t)\right|dy.$$

Then we have

$$\|h_{1n}(\cdot,t)-h_1(\cdot,t)\|_{0,\eta}\leqslant\|u_{0n}\|_{0,\eta}\|g_n(\cdot,t)\|_{0,\eta}+\|g(\cdot,t)\|_{0,\eta}\|u_{0n}-u_0\|_{0,\eta}$$

and after straightforward calculations we obtain (C denotes different numerical constants, possibly depending on t)

$$\|g_n(\cdot,t)\|_{0,\eta}\leqslant C\,|s_n(t)-s(t)|\leqslant C\sup_{0\leqslant t\leqslant\gamma}|\theta_n(t)-\theta(t)|\,,\qquad\|g(\cdot,t)\|_{0,\eta}\leqslant C$$

so that $\|h_{1n}(\cdot,t)-h_1(\cdot,t)\|_{0,\eta}\to 0$. Similarly, on replacing E by E_z in the definitions of g_n and g, we see that $\left\|\frac{\partial h_{1n}}{\partial x}(\cdot,t)-\frac{\partial h_1}{\partial x}(\cdot,t)\right\|_{0,\eta}\to 0$. Finally, using the appropriate modifications, the estimation of $\|h_{2n}(\cdot,t)-h_2(\cdot,t)\|_{1,\eta}$ yields

$$\|h_{2n}(\cdot,t)-h_2(\cdot,t)\|_{1,\eta}\leqslant C\left(\sup_{0\leqslant t\leqslant\gamma}|\theta_n(t)-\theta(t)|+\sup_{0\leqslant t\leqslant\gamma}\left|\phi_n(t)-\phi(t)\right|\right).$$

Then we obtain the result. □

Now we establish a lower bound on the solutions of (1) that is essential in the next section. First, since $Q\geqslant 0$, if $u\geqslant 0$ at $t=0$, then by the maximum principle $u\geqslant 0$ for $x\leqslant 0, t>0$, i.e., $S(t)$ is positivity preserving. Actually, this result can be improved. For this purpose we define the function

$$\begin{aligned}\varphi(x,t;\alpha)&=\frac{1}{2}\operatorname{erfc}\left(\frac{|x|}{\sqrt{4t}}+\frac{\alpha}{2}\sqrt{t}\right)+\frac{1}{2}e^{\alpha x}\operatorname{erfc}\left(\frac{|x|}{\sqrt{4t}}-\frac{\alpha}{2}\sqrt{t}\right)\\&=e^{\alpha x}+\frac{1}{\pi}e^{\alpha x/2-\alpha^2t/4}\int_0^\infty\frac{\sin\left(x\sqrt{\lambda}\right)}{\lambda+\alpha^2/4}e^{-\lambda t}\,d\lambda,\end{aligned}\tag{3}$$

where $\alpha>0$ is fixed, and we observe that φ is the solution of the initial-boundary value problem

$$\begin{aligned}\varphi_t&=\varphi_{xx}-\alpha\varphi_x, && x<0,\ t>0,\\\varphi(-\infty,t)&=0, && t>0,\\\varphi(0,t)&=1, && t>0,\\\varphi(x,0)&=0, && x\leqslant 0.\end{aligned}$$

Lemma 2.2. *Let* u_0 *be given in* X^0_η, $u_0 \geqslant 0$. *Then,*

$$S(t)\, u_0 \geqslant \underline{u}$$

for $t \geqslant 0$, *where*

$$\underline{u}(x, t) = Q_{\min} \int_{-\infty}^{x} \varphi(y, t; R_{\max})\, dy \tag{4}$$

Proof. Let $u = S(t)\, u_0$. Since $\varphi \geqslant 0$, the parabolic operator

$$L = \partial^2/\partial x^2 - R(u(0, t))\, \partial/\partial x - \partial/\partial t \tag{5}$$

satisfies $L\left(u - \underline{u}\right) = (R(u(0, t)) - R_{\max})\, \varphi \leqslant 0$ for $x < 0$ and $t > 0$. As $u(x, 0) - \underline{u}(x, 0) = u_0(x) \geqslant 0$, $u(-\infty, t) - \underline{u}(-\infty, t) = 0$ and $u_x(0, t) - \underline{u}_x(0, t) = Q(u(0, t)) - Q_{\min} \geqslant 0$, the maximum principle yields $u(x, t) \geqslant \underline{u}(x, t)$ through the region $x \leqslant 0$, $t \geqslant 0$. □

3. Absorbing sets

We consider as a basic metric space the closed cone of nonnegative elements in X^0_η, denoted by

$$\mathcal{Z}_\eta = \left\{ u_0 \in X^0_\eta : u_0 \geqslant 0 \right\}$$

and we show the existence of absorbing sets in $\mathcal{Z}_\eta$ for the semigroup $S(t)$. The necessary estimates are obtained under stronger assumptions on the functions R and Q that provide uniform bounds on the solutions of (1). Specifically, from now on the following supplementary conditions are assumed.

H2. $R(u) = 0$ for $u \leqslant u_*$, $R(u)$ increasing for $u > u_*$ for some value $u_* > 0$;
H3. $Q_{\min} > u_* R_{\max}$.

By direct computations it follows from (4) and (3)

$$\underline{u}(0, t) = \frac{Q_{\min}}{R_{\max}} - \frac{Q_{\min}}{\pi}\, e^{-t R^2_{\max}/4} \int_0^\infty \frac{\sqrt{\lambda}}{\left(\lambda + R^2_{\max}/4\right)^2}\, e^{-\lambda t}\, d\lambda$$

so that $\underline{u}(0, t)$ is increasing and tends to $Q_{\min}/R_{\max}$ as $t \to \infty$. Using H3, we can choose η such that

$$0 < \eta < \frac{1}{2} R\left(\frac{Q_{\min}}{R_{\max}}\right). \tag{6}$$

Then, by monotonicity there exists $t_1 > 0$ such that

$$\underline{u}\,(0, t_1) = R^{-1}\,(2\eta) \tag{7}$$

and we deduce from Lemma 2.2 that

$$u\,(0, t) \geqslant R^{-1}\,(2\eta) > u_* \tag{8}$$

for all $t \geqslant t_1$, where t_1 is independent of the initial data in $\mathcal{Z}_\eta$. Hence (8) provides a uniform lower bound for $u\,(0, t)$ when t is large.

Remark 3.1. In practical applications condition H2 is always fulfilled. Concerning the physical meaning of H3 and the above discussion, in combustion theory the heat feedback function Q is required to satisfy additionally $Q\,(u) = Q_{\min}$ for $u \leqslant u_*$, $Q\,(u) > Q_{\min}$ for $u > u_*$ and $Q_{\min}$ is the (normalized) external radiant intensity. Thus we have proved that the solid/gas interface of an intensely irradiated material *is always moving* after some characteristic time, and this is of course what we expect physically.

Remark 3.2. The smaller η is chosen the larger the class of admissible initial data becomes, so condition (6) is not really restrictive and, hereafter, it will be tacitly assumed.

Interestingly, the lower bound (8) is useful in constructing a convenient pointwise upper bound for the solution. Define the function

$$\psi\,(x, t) = e^{\eta(x-\eta t)}\mathrm{erf}\left(\frac{|x|}{\sqrt{4t}}\right), \tag{9}$$

which is the solution of the problem

$$\begin{aligned} \psi_t = \psi_{xx} - 2\eta\,\psi_x, &\qquad x < 0,\ t > 0,\\ \psi\,(-\infty, t) = 0, &\qquad t > 0,\\ \psi\,(0, t) = 0, &\qquad t > 0,\\ \psi\,(x, 0) = e^{\eta x}, &\qquad x \leqslant 0. \end{aligned}$$

Lemma 3.1. *Let u_0 be given in $\mathcal{Z}_\eta$, and let $\overline{C}$ be a fixed constant satisfying $\overline{C} \geqslant \|S\,(t_1)\ u_0\|_{0,\eta}$. Then,*

$$S\,(t)\,u_0 \leqslant \overline{u}$$

for $t \geqslant t_1$, *where*

$$\overline{u}(x,t) = \eta\overline{C}\int_{-\infty}^{x}\psi(y,t-t_1)\,dy + Q_{\max}\int_{-\infty}^{x}\varphi(y,t-t_1;2\eta)\,dy.$$

Proof. Let $u = S(t)u_0$. Since $\varphi,\psi\geqslant 0$, the parabolic operator (5) satisfies $L(u-\overline{u}) = (R(u(0,t))-2\eta)\left(\overline{C}\psi + Q_{\max}\varphi\right)\geqslant 0$ for $x<0$ and $t>t_1$. As $u(x,t_1) - \overline{u}(x,t_1) = u(x,t_1) - \overline{C}e^{\eta x}\leqslant 0$, $u(-\infty,t)-\overline{u}(-\infty,t) = 0$ and $u_x(0,t)-\overline{u}_x(0,t) = Q(u(0,t)) - Q_{\max}\leqslant 0$, the maximum principle yields $u(x,t)\leqslant\overline{u}(x,t)$ through the region $x\leqslant 0$, $t\geqslant t_1$. □

Lemma 3.2. *In Lemma* 3.1 *choose further* $\varepsilon > 0$ *and let* t_2 *be such that*

$$\left(2\eta\overline{C} + \left(1+\frac{2}{\pi}\right)Q_{\max}\right)e^{-\eta^2 t_2}\leqslant\varepsilon Q_{\max}.$$

Then

$$\|S(t)u_0\|_{0,\eta}\leqslant\frac{Q_{\max}}{2\eta}(1+\varepsilon)$$

for $t\geqslant t_1+t_2$.

Proof. Let $u = S(t)u_0$. By (9) we see that

$$0\leqslant\int_{-\infty}^{x}\psi(y,t)\,dy\leqslant\frac{1}{\eta}e^{-\eta^2 t}e^{\eta x}. \tag{10}$$

From (3), it follows that

$$\begin{aligned} 0 &\leqslant \int_{-\infty}^{x}\varphi(y,t;2\eta)\,dy\leqslant\frac{1}{2\eta}e^{2\eta x} + \frac{1}{\pi}e^{-\eta^2 t}\int_0^\infty\frac{e^{-\lambda t}}{\lambda+\eta^2}\,d\lambda\int_{-\infty}^{x}e^{\eta y}\sin\left(y\sqrt{\lambda}\right)dy \\ &= \frac{1}{2\eta}e^{2\eta x} + \frac{1}{\pi}e^{-\eta^2 t}e^{\eta x}\int_0^\infty\frac{\eta\sin\left(x\sqrt{\lambda}\right)-\sqrt{\lambda}\cos\left(x\sqrt{\lambda}\right)}{\left(\lambda+\eta^2\right)^2}e^{-\lambda t}\,d\lambda \\ &\leqslant \frac{1}{2\eta}e^{2\eta x} + \frac{1}{\eta\pi}e^{-\eta^2 t}e^{\eta x}\int_0^\infty\frac{1+\sqrt{\rho}}{(\rho+1)^2}\,d\rho \\ &= \frac{1}{2\eta}e^{2\eta x} + \frac{1}{2\eta}\left(1+\frac{2}{\pi}\right)e^{-\eta^2 t}e^{\eta x}. \end{aligned}$$

Then

$$0 \leqslant e^{-\eta x} u(x,t) \leqslant e^{-\eta x} \overline{u}(x,t) \leqslant \overline{C}\, e^{-\eta^2 (t-t_1)} + \frac{Q_{\max}}{2\eta} \left(e^{\eta x} + \left(1 + \frac{2}{\pi}\right) e^{-\eta^2 (t-t_1)} \right)$$
$$\leqslant \frac{Q_{\max}}{2\eta} + \frac{\varepsilon Q_{\max}}{2\eta}$$

and the result follows. □

We now infer easily that any ball of $\mathcal{Z}_\eta$ centered at 0 of radius $\rho > \rho_0$ is absorbing in $\mathcal{Z}_\eta$ for $S(t)$, where

$$\rho_0 = \frac{Q_{\max}}{2\eta}$$

and if u_0 belongs to the ball $\mathcal{B}_\sigma$ of $\mathcal{Z}_\eta$ centered at 0 of radius σ, then $S(t)\, u_0$ enters the absorbing set at a time $t \leqslant t_0$ and remains in it for $t \geqslant t_0$, $t_0 = t_0(\rho, \sigma)$. Indeed, considering the estimates in more detail, we obtain as a consequence of Lemma 4.1 in [9] that there exists a constant $K = K(t) > 0$ depending on t but independent of u_0 such that

$$\|S(t)\, u_0\|_{0,\eta} \leqslant K(t) \left(1 + \|u_0\|_{0,\eta}\right).$$

For an arbitrary fixed $\sigma > 0$, let u_0 in $\mathcal{Z}_\eta$ be such that $\|u_0\|_{0,\eta} \leqslant \sigma$ and choose the constant $\overline{C}$ of Lemma 3.1 as $\overline{C} = (1+\sigma)\, K(t_1)$. Then Lemma 3.2 applies and the claim follows with $t_0 = t_1 + t_2$.

4. Attractors

To address the existence of the attractors when the existence of the absorbing sets is known, further properties on the evolution operator are needed. A general condition (see, for instance, Temam [10, Chapter 1]) is to prove a decomposition of the evolution in the form

$$S(t) = S_1(t) + S_2(t)$$

that is, $S(t)$ is the perturbation of an operator $S_1(t)$, which is uniformly compact for t large, by an operator $S_2(t)$, which converges to 0 as $t \to \infty$. We start by introducing $S_2(t)$. For this purpose consider the linear problem

$$U_t = U_{xx} - R(u(0,t))\, U_x, \qquad x < 0,\ t > 0,$$

$$U_x(-\infty,t)=0, \qquad t>0,$$
$$U_x(0,t)=0, \qquad t>0,$$
$$U(x,0)=u_0(x), \qquad x\leqslant 0,$$

where $u=S(t)\,u_0$.

Lemma 4.1. *For u_0 given in $\mathcal{Z}_\eta$, there exists a unique classical solution U to the above problem and the mapping $u_0 \longmapsto U(\cdot,t)$ is continuous from X^0_η into X^1_η for $t>0$. Furthermore, $U(\cdot,t)\geqslant 0$, and there is a positive constant K such that*

$$\|U(\cdot,t)\|_{0,\eta}\leqslant K\,\|u_0\|_{0,\eta}\,e^{-\eta^2 t} \tag{11}$$

for $t\geqslant t_1$, t_1 given by (7).

Proof. The techniques of Theorem 2.1 and Lemma 2.1 apply directly, to yield a unique classical solution $U(\cdot,t)\in X^1_\eta$ and its continuous dependence upon initial data. Next, the bounds

$$0\leqslant U(x,t)\leqslant \eta\overline{C}\int_{-\infty}^{x}\psi(y,t-t_1)\,dy,$$

where $\overline{C}\geqslant \|U(\cdot,t_1)\|_{0,\eta}$, $t\geqslant t_1$, are derived in exactly the same way as was Lemma 3.1, so that the estimate (11) follows from (10) and $\|U(\cdot,t)\|_{0,\eta}\leqslant K(t)\,\|u_0\|_{0,\eta}$. $\square$

Then the operators

$$S_2(t): u_0 \longmapsto U(\cdot,t), \quad t\geqslant 0$$

($U(\cdot,t)$ being given by Lemma 4.1) are continuous from $\mathcal{Z}_\eta$ into itself and for every ball $\mathcal{B}_\sigma$

$$\sup_{u_0\in\mathcal{B}_\sigma}\|S_2(t)\,u_0\|_{0,\eta}\to 0 \qquad \text{as } t\to\infty$$

i.e., we obtain the following proposition.

Proposition 4.1. *The operators $S_2(t)$ are uniformly contracting in $\mathcal{Z}_\eta$.*

Next for $t>0$ we define $S_1(t)\in C^0\left(X^0_\eta, X^1_\eta\right)$ by

$$S_1(t)=S(t)-S_2(t).$$

Lemma 4.2. *Let $u_0 \in \mathcal{Z}_\eta$, and let $V = S_1(t)\,u_0$. Then, for all $t > 0$, we have $V(\cdot, t) \geqslant 0$. In addition, there exists a constant $C > 0$ and independent of u_0 such that*

$$0 \leqslant V_x(x, t) \leqslant Ce^{2\eta x} \tag{12}$$

for all $x \leqslant 0$ and $t \geqslant t_1$.

Proof. Let $u = S(t)\,u_0$, $U = S_2(t)\,u_0$, and L be given by (5). Then $V = S_1(t)\,u_0 = u - U$ satisfies $L(V) = 0$, $V(x, 0) = 0$, $V(-\infty, t) = 0$, $V_x(0, t) = Q(u(0, t)) \geqslant Q_{\min} > 0$, hence $V(x, t) \geqslant 0$ through the region $x \leqslant 0$, $t \geqslant 0$. To establish estimates (12) we write $w = V_x$ and by differentiation we see that

$$\begin{aligned} Lw = -w_t + w_{xx} - R(u(0, t))\, w_x = 0, &\qquad x < 0,\ t > 0,\\ w(-\infty, t) = 0, &\qquad t > 0,\\ w(0, t) = Q(u(0, t)), &\qquad t > 0,\\ w(x, 0) = 0, &\qquad x \leqslant 0. \end{aligned} \tag{13}$$

Then $w \geqslant 0$ by the maximum principle. The remaining bound in (12) follows from the claim $w(x, t_1) \leqslant C^* e^{2\eta x}$ and a comparison argument. Indeed, let $C = \max\{C^*, Q_{\max}\}$: then we deduce that $L\left(w - Ce^{2\eta x}\right) = 2C\,(R(u(0, t)) - 2\eta)\, e^{2\eta x} \geqslant 0$ for $x < 0$ and $t > t_1$, $w(x, t_1) - Ce^{2\eta x} \leqslant (C^* - C)\, e^{2\eta x} \leqslant 0$, $w(0, t) - C = Q(u(0, t)) - C \leqslant 0$, and so $w(x, t) \leqslant Ce^{2\eta x}$ for all $x \leqslant 0$, $t \geqslant t_1$. Finally, we check the claim. The solution of (13) has the representation

$$w(x, t) = \int_0^t E_z(x + s(t) - s(\tau), t - \tau)\ f(\tau)\ d\tau, \tag{14}$$

where the density f solves the integral equation

$$Q(u(0, t)) = \frac{1}{2} f(t) + \int_0^t E_z(s(t) - s(\tau), t - \tau)\ f(\tau)\ d\tau.$$

We have

$$|E_z(s(t) - s(\tau), t - \tau)| = \frac{1}{2}\left|\frac{s(t) - s(\tau)}{t - \tau}\right|\ E(s(t) - s(\tau), t - \tau) \leqslant \frac{R_{\max}}{2\sqrt{4\pi(t - \tau)}}$$

hence

$$|f(t)| \leqslant 2Q_{\max} + \left(\sup_{0 \leqslant \tau \leqslant t} |f(\tau)| \right) \frac{R_{\max}}{\sqrt{\pi}} \sqrt{t}$$

so that

$$\sup_{0 \leqslant \tau \leqslant t^*} |f(\tau)| \leqslant 4Q_{\max},$$

where $R_{\max}\sqrt{t^*/\pi} \leqslant 1/2$ and $mt^* = t_1$ for some positive integer m. By induction we obtain

$$\sup_{0 \leqslant \tau \leqslant mt^*} |f(\tau)| \leqslant 4^m Q_{\max} = K.$$

Therefore, for $0 \leqslant t \leqslant t_1$, it follows from (14)

$$w(x,t) \leqslant \frac{K}{2} \int_0^t \left| \frac{x + s(t) - s(\tau)}{t - \tau} \right| E(x + s(t) - s(\tau), t - \tau)\, d\tau \leqslant I_1(x,t) + I_2(x,t)$$

and

$$I_1(x,t) = \frac{K}{2} |x| \int_0^t \frac{E(x + s(t) - s(\tau), t - \tau)}{t - \tau}\, d\tau \leqslant \frac{K}{2} e^{4\eta^2 t} e^{2\eta x},$$

$$I_2(x,t) = \frac{K}{2} R_{\max} \int_0^t E(x + s(t) - s(\tau), t - \tau)\, d\tau \leqslant \frac{K R_{\max} \sqrt{t}}{2\sqrt{\pi}} e^{4\eta^2 t} e^{2\eta x}$$

hence the desired result $w(x, t_1) \leqslant C^* e^{2\eta x}$. $\square$

The above Lemma shows that the set $\bigcup_{t \geqslant t_1} S_1(t)\mathcal{B}_\sigma$ is uniformly bounded in $X^1_{2\eta} \cap \{u \geqslant 0\}$. But $X^1_{2\eta}$ is embedded compactly in X^0_η, then for every ball $\mathcal{B}_\sigma$ there exists t_0 such that $\bigcup_{t \geqslant t_0} S_1(t)\mathcal{B}_\sigma$ is relatively compact in $\mathcal{Z}_\eta$. We summarize this in the following statement.

Proposition 4.2. *The operators $S_1(t)$ are uniformly compact in $\mathcal{Z}_\eta$ for t large.*

Now we are in a position to state the main result of this section.

Theorem 4.1. *The semigroup* $\{S(t) : t \geqslant 0\}$ *possesses a compact attractor* $\mathcal{A}$ *in* $\mathcal{Z}_\eta$ *which is global* (*i.e., it attracts the bounded sets in* $\mathcal{Z}_\eta$); *in addition,* $\mathcal{A}$ *is the maximal bounded attractor in* $\mathcal{Z}_\eta$.

Proof. We know that $\mathcal{B}_\rho = \left\{u_0 \in \mathcal{Z}_\eta : \|u_0\|_{0,\eta} \leqslant \rho\right\}$, $\rho > Q_{\max}/2\eta$, is absorbing in $\mathcal{Z}_\eta$ for $S(t) = S_1(t) + S_2(t)$ and Propositions 4.1 and 4.2 hold. We can then apply Theorem I.1.1 in [10] to the evolution semigroup $S(t)$ in $\mathcal{Z}_\eta$ because of the invariance of the cone $\mathcal{Z}_\eta$ under the action of both $S_1(t)$ and $S_2(t)$. The theorem is proved, $\mathcal{A}$ being the ω-limit set of $\mathcal{B}_\rho$. □

5. Approach to equilibrium

The maximal attractor $\mathcal{A}$ of Theorem 4.1 contains, in particular, the "permanent regimes", i.e., the ω-limit sets $\omega(u_0)$ corresponding to all initial data u_0. Then the question of approach to equilibrium arises; precisely, given any u_0, we ask whether the set $\omega(u_0)$ consists of a single stationary solution v and the trajectory starting from u_0 converges to v as $t \to \infty$. We investigate the behavior at infinity of positive solutions of (1) in a case of special interest for the applications, namely when the set of equilibrium points of the associated semigroup is reduced to one point.

Observe first that the equilibrium solutions of (1) are simply the exponential profiles

$$v(x) = \overline{u} \exp\left(x\overline{R}\right), \tag{15}$$

where $\overline{u} > u_*$ is a root of the trascendental equation

$$Q(u) = uR(u)$$

and $\overline{R} = R(\overline{u})$. A root $\overline{u}$ exists since $Q(u) - uR(u)|_{u=u_*} \geqslant Q_{\min} > 0$ and $Q(u) - uR(u)|_{u=\infty} = -\infty$. A simple condition for uniqueness is that $Q(u)/R(u)$ is a monotone decreasing function, as indeed is verified in practical examples [11]. Note also that $v \in \mathcal{Z}_{\overline{R}} \subset \mathcal{Z}_\eta$, since

$$\overline{R} = R\left(\frac{Q(\overline{u})}{R(\overline{u})}\right) > R\left(\frac{Q_{\min}}{R_{\max}}\right) > 2\eta.$$

Now define the map

$$\Phi(u) = \frac{1}{2}\int_{-\infty}^{0}\left(\overline{R}^2 u(x)^2 + u'(x)^2\right)dx - \int_0^{u(0)} Q(u)\, du \tag{16}$$

for $u \in X_\eta^1$. This map is continuous and plays the role of Lyapunov function for the semigroup $\{S(t) : t \geqslant 0\}$.

Lemma 5.1. *Let u_0 be given in $\mathcal{Z}_\eta$, and let $0 < s < t$. Then*

$$\Phi\left(S(t)\,u_0\right) - \Phi\left(S(s)\,u_0\right) = -\int_s^t d\tau \int_{-\infty}^0 \left(u_{xx} - \overline{R}u_x\right)^2 dx - \frac{1}{2}\int_s^t \Psi\left(u\left(0,\tau\right)\right) d\tau,$$

where $u = S(t)\,u_0$, $\Psi(u)$ being defined by

$$\Psi(u) = \left\{Q(u) - \overline{R}u\right\}\left\{\left(\overline{R} - R(u)\right)Q(u) + \overline{R}\left(Q(u) - uR(u)\right)\right\} \tag{17}$$

for $u \in \mathbb{R}$.

Proof. We consider the auxiliary function

$$\zeta_{a,b}(t) = \frac{1}{2}\int_a^b \left(\overline{R}^2 u(x,t)^2 + u_x(x,t)^2\right) dx,$$

where $a < b < 0$. For $t > 0$, $u = S(t)\,u_0$ is a C^∞ function of $x < 0$ and all spatial derivatives are $O\left(e^{\eta x}\right)$ as $x \to -\infty$. By (1a) the same property holds true for u_{xt}. Therefore we can differentiate $\zeta_{a,b}(t)$ for $t > 0$, then integrate by parts, to obtain

$$\dot{\zeta}_{a,b}(t) = -\int_a^b \left(u_{xx} - \overline{R}^2 u_x\right) u_t\, dx + u_x u_t\big|_{x=a}^{x=b}.$$

Using (1a) we see that

$$\left(u_{xx} - \overline{R}^2 u_x\right) u_t = \left(u_{xx} - \overline{R}u_x\right)^2 + \frac{1}{2}\left\{\left(u_x - Ru\right)\left(\left(\overline{R} - R\right)u_x + \overline{R}\left(u_x - Ru\right)\right)\right\}_x,$$

where $R = R(u(0,t))$ and $Q = Q(u(0,t))$, hence for $0 < s < t$

$$\begin{aligned}\zeta_{a,b}(t) - \zeta_{a,b}(s) = &-\int_s^t d\tau \int_a^b \left(u_{xx} - \overline{R}u_x\right)^2 dx \\ &-\frac{1}{2}\int_s^t \left(u_x - Ru\right)\left(\left(\overline{R} - R\right)u_x + \overline{R}\left(u_x - Ru\right)\right)\Big|_{x=a}^{x=b} d\tau \\ &+\int_s^t u_x u_t\big|_{x=a}^{x=b}\, d\tau.\end{aligned}$$

Taking into account that u_t is continuous up to $x = 0$ (Proposition 2.1), it follows

$$\zeta_{a,b}(t) - \zeta_{a,b}(s) \to -\int_s^t d\tau \int_{-\infty}^0 \left(u_{xx} - \overline{R}u_x\right)^2 dx - \frac{1}{2}\int_s^t \Psi\left(u(0,\tau)\right) d\tau + \int_s^t Q\left(u(0,\tau)\right) u_t(0,\tau)\, d\tau$$

as $a \to -\infty$, $b \to 0$. Since

$$\int_s^t Q\left(u(0,\tau)\right) u_t(0,\tau)\, d\tau = \int_{u(0,s)}^{u(0,t)} Q(u)\, du$$

and

$$\zeta_{a,b}(t) \to \Phi\left(S(t)u_0\right) + \int_0^{u(0,t)} Q(u)\, du$$

the lemma is proved. $\square$

Now suppose that Q fulfills the following condition

$$Q(u)\begin{cases} \geqslant \overline{R}u & \text{if} \quad u < \overline{u}, \\ = \overline{R}u & \text{if} \quad u = \overline{u}, \\ \leqslant \overline{R}u & \text{if} \quad u > \overline{u}, \end{cases} \tag{18}$$

where $v(x) = \overline{u}\exp\left(x\overline{R}\right)$ is an equilibrium solution. A few straightforward consequences of (18) are in order for deducing the main result of this section, Theorem 5.1. First, by monotonicity of R, we see that $Q(u) \geqslant \overline{R}u > Ru$ for $\overline{u} > u$ and $Q(u) \leqslant \overline{R}u < Ru$ for $\overline{u} < u$, so that the equilibrium solution is unique. Furthermore, from (17), we get

$$\Psi(u) \geqslant 0 \quad \text{for all real } u \qquad \text{and} \qquad \Psi(u) = 0 \quad \text{only if} \quad u = \overline{u}. \tag{19}$$

Finally, by Lemma 5.1 and (19), condition $\Phi\left(S(t)u_0\right) = \text{const}$ for all $t > 0$ yields $u_{xx} = \overline{R}u_x$, $u(0,t) = \overline{u}$ for all $t \geqslant 0$, where $u = S(t)u_0$. Therefore $u = \overline{u}\exp\left(x\overline{R}\right) = v$.

Theorem 5.1. *Let Q satisfy condition* (18) *and let u_0 be given in $\mathcal{Z}_\eta$. Then $S(t)u_0 \to v$ in X_η^1 as $t \to \infty$, where v is the unique stationary solution* (15).

Proof. Let $\psi \in \mathcal{Z}_\eta$ denote any point of the ω-limit set $\omega(u_0)$ of u_0 (i.e., there exists $t_n \to \infty$ such that $S(t_n)u_0 \to \psi$ in X_η^0). Then, by Lemma 2.1, we have $S(t + t_n)u_0 =$

$S(t)\,S(t_n)\,u_0 \to S(t)\,\psi$ in X^1_η for all $t > 0$. On the other hand, the function $t \longmapsto \Phi(S(t)\,u_0)$ is nonincreasing by Lemma 5.1 and (19), and bounded from below at infinity by (8), hence $\lim_{t\to\infty} \Phi(S(t)\,u_0) = l$ (say) exists. Consequently, by continuity of Φ, we have $\Phi(S(t)\,\psi) = l$ for all $t > 0$. Thus $\omega(u_0) = \{v\}$ and $S(t)\,u_0 \to v$ in X^1_η. $\square$

Remark 5.1. In combustion applications, where specific parametric submodels for kinetics and transient flame are considered to assess quantitative trends, condition (18) expresses a constraint among relevant physical parameters such as activation energy, latent heat, characteristic times, etc.

6. Concluding remarks

After a brief presentation of a physical model drawn from combustion theory and the governing equations, we have treated the following questions about problem (1): nonlinear stability (the solutions remain bounded as $t \to \infty$), dissipativity (existence of absorbing sets) and description of the long-term dynamics (existence of a global attractor, approach to equilibrium). All results are proved under the key assumption H3, which amounts to saying that the burning surface is strongly irradiated by an external source. Therefore, a mathematical support is given to the general idea that irradiation favors stability, as expected on the basis of experimental observations. The physical explanation is that large radiant flux increases the energy storage of the burning material, thus yielding the gradual conversion of the radiation-sustained combustion wave into an ablation wave which makes burning more and more stable.

However, despite the existence of global solutions, our treatment of stability based on comparison arguments is not immediately applicable if condition H3 is not satisfied. Indeed, in this case it could happen that $\liminf_{t\to\infty} R(u(0,t)) = 0$ and the solution could exhibit some dramatic behaviors, such as deflagration, due to oscillations of increasing amplitude of the boundary value $u(0,t)$, or extinction, in the limiting case $Q_{\min} = 0$ and Neumann data vanishing below the threshold value u_*. Therefore, to fully describe the asymptotic dynamics, we need now a deeper insight into the more involved interplay between burning rate R and heat flux Q, as suggested by the results of the linearized problem [11]. Another (related) question is the extension of the stability analysis to time-dependent Neumann data $Q(u,t)$, assumed to "stabilize" as $t \to \infty$ [2]. We conjecture that the long-term dynamics is described in some sense by the asymptotically dynamical system corresponding to $Q(u) \equiv Q(u,\infty)$. Work in these directions is in progress.

Appendix A

A.1. Proof of Proposition 2.1

Consider the following representation of the solution u to the problem (1)

$$
\begin{aligned}
u\left(x,t\right) = &\int_{-\infty}^{0} E\left(x+s\left(t\right)-z,t\right)\, u_0\left(z\right)\, dz \\
&+\int_0^t E\left(x+s\left(t\right)-s\left(\tau\right),t-\tau\right)\left[Q\left(v\left(\tau\right)\right)-v\left(\tau\right)\, R\left(v\left(\tau\right)\right)\right] d\tau \\
&+\int_0^t E_z\left(x+s\left(t\right)-s\left(\tau\right),t-\tau\right)\, v\left(\tau\right)\, d\tau, \qquad \text{(A.1)}
\end{aligned}
$$

where $s\left(t\right) = -\int_0^t R\left(v\left(\tau\right)\right)\, d\tau$. Then the boundary value $v\left(t\right) = u\left(0,t\right)$ is the solution of the integral equation

$$
\begin{aligned}
v\left(t\right) = &\; 2\int_{-\infty}^{0} E\left(s\left(t\right)-y,t\right)\, u_0\left(y\right)\, dy \\
&+2\int_0^t E\left(s\left(t\right)-s\left(\tau\right),t-\tau\right)\left[Q\left(v\left(\tau\right)\right)-v\left(\tau\right)\, R\left(v\left(\tau\right)\right)\right] d\tau \\
&+2\int_0^t E_z\left(s\left(t\right)-s\left(\tau\right),t-\tau\right)\, v\left(\tau\right)\, d\tau \\
= &\; w\left(t\right)+\left(J_1 v\right)\left(t\right)+\left(J_2 v\right)\left(t\right). \qquad \text{(A.2)}
\end{aligned}
$$

We proceed in four steps.

Step 1: By time translation and relabeling of variables, from now on we may suppose $u_0 \in X_\eta^1$. From the definition of w, the identity

$$
w'\left(t\right) = -2E_z(s(t),t)u_0(0) + 2\int_{-\infty}^{0} E_z(s(t)-y,t)\left[u_0'(y) - R(v(t))u_0(y)\right] dy
$$

and the estimates

$$
\int_{-\infty}^{0} e^{\eta y} E(s(t)-y,t)\, dy \leqslant 1; \qquad \int_{-\infty}^{0} e^{\eta y}\left|E_z(s(t)-y,t)\right|\, dy \leqslant \frac{1}{\sqrt{\pi t}} \qquad \text{(A.3)}
$$

it follows easily

$$
\left|w\left(t\right)\right| \leqslant 2\left\|u_0\right\|_{0,\eta}; \qquad \left|w'\left(t\right)\right| \leqslant \frac{C}{\sqrt{t}}\left\|u_0\right\|_{1,\eta}
$$

for $t > 0$.

Step 2: Since $v\left(t\right)$ is continuous at $t = 0$, we have $\left(J_1 v\right)\left(t\right) \sim \left(1+\left|v\left(0\right)\right|\right) t^{1/2}$, $\left(J_2 v\right)\left(t\right) \sim \left|v\left(0\right)\right| t^{1/2}$ as $t \to 0$. Now assume $v'\left(t\right) \sim t^{-\alpha}$ as $t \to 0$, where $0 < \alpha < 1$. By boundedness of R, Q and their derivatives, by use of the integral

$\int_0^t (t-\tau)^{-1/2}\,\tau^{-\lambda}\,d\tau = Ct^{1/2-\lambda}$ $(0 \leqslant \lambda < 1)$ and by the identities

$$
\begin{aligned}
(J_1 v)'(t) \;=\; & 2E\,(s(t),t)\,[Q\,(v(0)) - v(0)R\,(v(0))] \\
& + 2\int_0^t E(s(t)-s(\tau), t-\tau)\frac{d}{d\tau}\,[Q\,(v(\tau)) - v(\tau)R\,(v(\tau))]\,d\tau \\
& - 2\int_0^t E_z(s(t)-s(\tau), t-\tau)\,[R(v(t)) - R(v(\tau))] \\
& \times [Q\,(v(\tau)) - v(\tau)R\,(v(\tau))]\,d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
(J_2 v)'(t) \;=\; & -E\,(s(t),t)\,\frac{s\,(t)}{t}\,v(0) \\
& - \int_0^t E\,(s(t)-s(\tau), t-\tau)\,\Delta s(t,\tau) v'(\tau)\,d\tau \\
& + \int_0^t E_z\,(s(t)-s(\tau), t-\tau)\,[R(v(t)) - R(v(\tau))]\,\Delta s(t,\tau)\,v(\tau)\,d\tau \\
& - \int_0^t E\,(s(t)-s(\tau), t-\tau)\left[\frac{\partial}{\partial t}\Delta s(t,\tau) + \frac{\partial}{\partial \tau}\Delta s(t,\tau)\right] v(\tau)\,d\tau,
\end{aligned}
$$

where $\Delta s(t,\tau) = \frac{s(t)-s(\tau)}{t-\tau}$, we find

$$
\begin{aligned}
(J_1 v)'(t) \;&\sim\; (1+|v\,(0)|)\left(t^{-1/2} + t^{1/2-\alpha} + t^{1/2}\right), \\
(J_2 v)'(t) \;&\sim\; |v\,(0)|\left(t^{1/2} + t^{-1/2}\right) + (1+|v\,(0)|)\,t^{1/2-\alpha}
\end{aligned}
$$

as $t \to 0$. Such asymptotic behaviors suggest (see [1, Section I]) for the functional setting of eq. (A.2) the Banach space $Y_{\gamma,\alpha}$ consisting of those functions $v \in C^0\left[0,\gamma\right] \cap C^1\left(0,\gamma\right]$ such that the norm

$$
\|v\|_{\gamma,\alpha} = \sup_{0<t<\gamma} |v(t)| + \sup_{0<t<\gamma} \left|t^\alpha v'(t)\right| < \infty
$$

for some $\gamma > 0$ and $1/2 < \alpha < 1$. Indeed, the above heuristic estimates can be made rigorous and we find

$$
\|(J_1 + J_2)\,v\|_{\gamma,\alpha} \leqslant C\left(1 + \|v\|_{\gamma,\alpha}\right)\gamma^{\alpha-1/2}
$$

for all $v \in Y_{\gamma,\alpha}$. Similarly, we show that

$$\|(J_1 + J_2)\, v_1 - (J_1 + J_2)\, v_2\|_{\gamma,\alpha} \leqslant C\, \|v_1 - v_2\|_{\gamma,\alpha}\, \gamma^{\alpha-1/2}$$

for all v_1, v_2 in a given closed ball centered at 0 in $Y_{\gamma,\alpha}$. Since w may be considered as a known term, and $w \in Y_{\gamma,\alpha}$ by Step 1, we conclude that there exists a sufficiently small γ such that the operator J defined by

$$J v = w + (J_1 + J_2)\, v$$

is a contraction which maps the closed ball centered at w in $Y_{\gamma,\alpha}$ into itself. Hence, the time derivative $v'(t) = u_t(0, t)$ is C^1 for $0 < t \leqslant \gamma$, and, by standard continuation arguments, for all $t > 0$.

Step 3: Continuity of $u_t(0, t)$ implies continuity of $u_t(x, t)$ up to the boundary $x = 0$. Without entering into more details, we only observe that differentiation with respect to time of (A.1) yields a three-termed expression

$$u_t(x, t) = I_0(x, t) + I_1(x, t) + I_2(x, t),$$

where

$$\begin{aligned} I_0(x, t) = &-E_z(x + s(t), t)\theta_0(0) \\ &+ \int_{-\infty}^{0} E_z(x + s(t) - y, t) \left[u_0'(y) - R(v(t))\, u_0(y)\right] dy, \end{aligned}$$

$$\begin{aligned} I_1(x, t) = &\ E(x + s(t), t)\, F(v(0)) \\ &- \int_0^t E_z(x + s(t) - s(\tau), t - \tau)\, [R(v(t)) - R(v(\tau))]\, F(v(\tau))\, d\tau \\ &+ \int_0^t E(x + s(t) - s(\tau), t - \tau) \frac{d}{d\tau} F(v(\tau))\, d\tau, \end{aligned}$$

$$\begin{aligned} I_2(x, t) = &-E(x + s(t), t) \frac{x + s(t)}{2t} v(0) \\ &+ \frac{1}{2} \int_0^t E_z(x + s(t) - s(\tau), t - \tau) \\ &\times [R(v(t)) - R(v(\tau))] \left[\frac{x}{t - \tau} + \Delta s(t, \tau)\right] v(\tau)\, d\tau \\ &- \frac{1}{2} \int_0^t E(x + s(t) - s(\tau), t - \tau) \left[\frac{\partial}{\partial t} \Delta s(t, \tau) + \frac{\partial}{\partial \tau} \Delta s(t, \tau)\right] v(\tau)\, d\tau \end{aligned}$$

$$+\int_0^t E_z(x+s(t)-s(\tau),t-\tau)v'(\tau)\,d\tau.$$

Then it is easy to check the following limits as $x \to 0$:

$$I_0(x,t) \to \frac{d}{dt}\int_{-\infty}^0 E(s(t)-y,t)\,u_0(y)\,dy,$$

$$I_1(x,t) \to \frac{1}{2}(J_1 v)'(t),$$

$$I_2(x,t) \to \frac{1}{2}(J_2 v)'(t)+\frac{1}{2}v'(t).$$

It follows from (A.2) that $u_t(x,t) \to v'(t)$ for $t>0$. Finally, the continuity of $v'(t)$ and the uniform convergence of $u_t(x,t)$ on compact time intervals give the two-dimensional continuity of u_t at $x=0$ for $t>0$.

Step 4: $u_t(\cdot,t)$ belongs to X_η^0 for all $t>0$. This property follows from the representation (A.1) by inspection.

A.2. A technical lemma

The densities $\theta(t)$ and $\phi(t)$ in the representation (2) of $u(x,t)$ are determined by imposing the boundary conditions

$$\lim_{x\to 0} u(x,t)=\theta(t), \qquad \lim_{x\to 0} u_x(x,t)=Q(\theta(t))$$

hence, on the basis of the discontinuity property of the single-layer potential [3, Chapter 13] they satisfy the pair of integral equations

$$\begin{aligned}\theta(t) &= \int_{-\infty}^0 E(s(t)-y,t)\,u_0(y)\,dy\\ &\quad +2\int_0^t E(s(t)-s(\tau),t-\tau)\left[Q(\theta(\tau))-\varepsilon\frac{\phi(\tau)}{\sqrt{\tau}}\right]d\tau,\\ \phi(t) &= \frac{\sqrt{t}}{\varepsilon}\int_{-\infty}^0 E_z(s(t)-y,t)\,u_0(y)\,dy\\ &\quad +2\sqrt{t}\int_0^t E_z(s(t)-s(\tau),t-\tau)\left[\frac{1}{\varepsilon}Q(\theta(\tau))-\frac{\phi(\tau)}{\sqrt{\tau}}\right]d\tau.\end{aligned}$$

A local solution of this system may be constructed by a fixed point technique. We write these equations more compactly as a vector equation

$$\mathbf{v}=\mathbf{K}(\mathbf{v},u_0),$$

where $\mathbf{v} = (\theta, \phi)$ and the components of the vector valued operator $\mathbf{K}(\mathbf{v}, u_0) = (K_1(\mathbf{v}, u_0), K_2(\mathbf{v}, u_0))$ are the right sides of the above equations, respectively. It has been established in [9] that, for data u_0 in X^0_η, there exist $\varepsilon > 0$ and $\gamma > 0$ such that $\mathbf{K}$ possesses a unique fixed point in the space $Y = C^0\left[0, \gamma\right] \oplus C^0\left[0, \gamma\right]$, equipped with the supremum norm. In addition, if u_0 belongs to a given bounded set of X^0_η, γ is bounded away from zero and $\mathbf{K}(\mathbf{v}, u_0)$ is a contraction mapping with respect to $\mathbf{v}$, uniformly with respect to u_0.

Now, let u_{0n}, θ_{0n}, ϕ_{0n} and s_n be defined as in the proof of Lemma 2.1. Then $\mathbf{v}_n = \left(\theta_n, \phi_n\right) \in Y$ satisfies $\mathbf{v}_n = \mathbf{K}(\mathbf{v}_n, u_{0n})$, hence

$$\|\mathbf{v}_n - \mathbf{v}\|_Y \leqslant \|\mathbf{K}(\mathbf{v}_n, u_{0n}) - \mathbf{K}(\mathbf{v}, u_{0n})\|_Y + \|\mathbf{K}(\mathbf{v}, u_{0n}) - \mathbf{K}(\mathbf{v}, u_0)\|_Y .$$

We have

$$\|\mathbf{K}(\mathbf{v}_n, u_{0n}) - \mathbf{K}(\mathbf{v}, u_{0n})\|_Y \leqslant \lambda \|\mathbf{v}_n - \mathbf{v}\|_Y ,$$

where $0 < \lambda < 1$, and by the first estimate in (A.3)

$$\begin{aligned} |K_1(\mathbf{v}, u_{0n}) - K_1(\mathbf{v}, u_0)| &= \left| \int_{-\infty}^0 E(s(t) - y, t)\ (u_0(y) - u_{0n}(y))\ dy \right| \\ &\leqslant \|u_{0n} - u_0\|_{0,\eta} \int_{-\infty}^0 e^{\eta y} E(s(t) - y, t)\, dy \\ &\leqslant \|u_{0n} - u_0\|_{0,\eta} . \end{aligned}$$

Likewise, by the second estimate in (A.3), we have

$$\begin{aligned} |K_2(\mathbf{v}, u_{0n}) - K_2(\mathbf{v}, u_0)| &= \frac{\sqrt{t}}{\varepsilon} \left| \int_{-\infty}^0 E_z(s(t) - y, t)\ (u_0(y) - u_{0n}(y))\ dy \right| \\ &\leqslant \frac{\sqrt{t}}{\varepsilon} \|u_{0n} - u_0\|_{0,\eta} \int_{-\infty}^0 e^{\eta y}\, |E_z(s(t) - y, t)|\ dy \\ &\leqslant \frac{1}{\varepsilon\sqrt{\pi}} \|u_{0n} - u_0\|_{0,\eta} . \end{aligned}$$

Combining all of these estimates, we conclude that

$$\|\mathbf{v}_n - \mathbf{v}\|_Y \leqslant \left(1 + \frac{1}{\varepsilon\sqrt{\pi}}\right) (1 - \lambda)^{-1} \|u_{0n} - u_0\|_{0,\eta} \to 0 \qquad \text{as } n \to \infty$$

i.e., $\theta_n \to \theta$ and $\phi_n \to \phi$ uniformly on $0 \leqslant t \leqslant \gamma$.

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