Regularity of the Surface Density of States

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We prove that the integrated surface density of states of continuous or discrete Anderson-type random Schrödinger operators is a measurable locally integrable function rather than a signed measure or a distribution. This generalizes our recent results on the existence of the integrated surface density of states in the continuous case and those of A. Chahrour in the discrete case. The proof uses the new $L^p$-bound on the spectral shift function recently obtained by Combes, Hislop, and Nakamura. Also we provide a simple proof of their result on the Hölder continuity of the integrated density of bulk states.

Key Words: random Schrödinger operators; surface states; spectral shift function; density of states.

1. INTRODUCTION AND MAIN RESULTS

Recently Combes, Hislop, and Nakamura [8] proved a remarkable inequality for the $L^p$-norm of the spectral shift function. A generalization of this inequality was then found by Hundertmark and Simon [14]. As an application of this inequality Combes, Hislop, and Nakamura prove Hölder continuity of the integrated density of states for a wide class of random Schrödinger operators. In this article, using the $L^p$-bound, we will prove that the integrated surface density of states of continuous or discrete

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Schrödinger operators is a measurable, locally integrable function thus extending results of [5, 9, 19]. Also we will provide a simple proof of the Hölder continuity of the integrated density of (bulk) states for some Anderson type models. It is based on the combination of the $L^p$-bound with the Birman–Solomyak formula for the spectral shift function [2]. Although based on the same ideas as in [8], our proof is simpler for the model we consider.

So we first consider random Schrödinger operators of the form $H_w = H_0 + V_w$ on $L^2(\mathbb{R}^n)$, $n \geq 2$ with $H_0 = -\Delta$ and $V_w$ being the random potential of Anderson type centered near a hypersurface in $\mathbb{R}^n$. More precisely we consider a decomposition $Z_n = Z_{n_1} \oplus Z_{n_2}$ with $n = n_1 + n_2$, $n_2 < n_1$ and introduce random potentials of the form

$$V_w(x) = \sum_{j \in Z_{n_1}} \alpha_0(\omega) f(x-j), \quad (1)$$

where $\alpha_0(\omega)$, $\omega \in \Omega$ is a sequence of random i.i.d. variables with common distribution measure $\kappa$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ is a $\sigma$-algebra on $\Omega = \mathbb{R}^{Z_{n_1}}$, $\mathbb{P}$ the product probability measure on $(\Omega, \mathcal{F})$, $\mathbb{P} = \prod_{j \in Z_{n_1}} \kappa$. Let $E$ denote the expectation with respect to $\mathbb{P}$. For each $\omega = \{\omega_j\}_{j \in Z_{n_1}}$ we define $\alpha_0(\omega) = \alpha(\omega_j)$ with some measurable function $\alpha: \mathbb{R} \rightarrow \mathbb{R}$. The random variables $\{\alpha_0(\omega)\}_{j \in Z_{n_1}}$ form a stationary, metrically transitive random field, i.e., the transformations $T_k: \omega = \{\omega_j\}_{j \in Z_{n_1}} \mapsto \{\omega_{j-k}\}_{j \in Z_{n_1}}$ are measure preserving and ergodic. The single-site potential $f$ is supposed to be compactly supported and in $L^2(\mathbb{R}^n)$. Additionally if $n \geq 4$ the potential $f$ is supposed to belong to $L^r(\mathbb{R}^n)$ with some $r > n/2$.

Throughout this article the constant $r$ will be assumed to be fixed. Instead of the integer lattice in (1) we can alternatively consider an arbitrary lattice as discussed in [19].

Finally we assume that $f$ is sign-definite, i.e., either $f > 0$ or $f < 0$ on sets of positive Lebesgue measure. Below and without loss of generality further we will consider the case $f \geq 0$ only since the case $f \leq 0$ is completely similar. Also supp $\kappa$ is supposed to be bounded, i.e., there is $\alpha_\infty \geq -\infty$ and $\alpha_\infty < \infty$ such that $\alpha_\infty \leq \alpha_0(\omega) \leq \alpha_\infty$ for all $j \in Z_{n_1}$ and all $\omega \in \Omega$. Under these conditions the operator $H_w = H_0 + V_w$ defined in the form sense is self-adjoint on $\mathcal{D}(H_0)$ for all $\omega \in \Omega$. With some obvious modifications the assumptions on $V_w$ can be relaxed by requiring that the expectations of certain quantities are finite.

Let $A$ and $C$ be bounded self-adjoint operators and let $C$ be trace class. The spectral shift function $\zeta(\cdot; A+C, A) \in L^1(\mathbb{R})$ is defined by the trace formula

$$\text{tr}(\phi(A+C)-\phi(A)) = \int_\mathbb{R} \phi'(\lambda) \zeta(\lambda; A+C, A) \, d\lambda \quad (2)$$
which is valid for a sufficiently wide class of continuous functions $\phi$ and $\|\cdot\|_{L^1(\mathbb{R})} \leq \|f\|_{\mathcal{H}}$ where $\|\cdot\|_{\mathcal{H}}$ is the trace norm. For relative trace class perturbations the spectral shift function can be defined by means of the invariance principle (see e.g. [4, 29]). In particular if $A$ and $B$ are self-adjoint possibly unbounded but bounded below with common domain of definition and if $(B+a)^{-p}-(A+a)^{-p}$ is trace class for some $a > 0$ and $p > 1$ then
\begin{equation}
\tilde{\zeta}(\lambda; B, A) = -\tilde{\zeta}(\lambda+a)^{-p}; (B+a)^{-p}, (A+a)^{-p}).
\end{equation}

It vanishes for all $\lambda < \inf\{\text{spec}(B), \text{spec}(A)\}$. A detailed account on the theory of the spectral shift function can be found in the review [4] and in the book [29]. For recent studies we refer to [10, 17] and references therein. The spectral shift function found a number of applications in the theory of random Schrödinger operators [5, 6, 8, 18–20, 22, 27].

Let $L, \ldots, R_n$ be a rectangular box $[a_1, b_1] \times \cdots \times [a_n, b_n]$. We understand the limit
\begin{equation}
V_{\alpha, A}(x) = \sum_{j \in \mathbb{Z}^n} \alpha_j(\omega) f(\cdot - j).
\end{equation}

In [19] we proved that for any $g \in C_0^1(\mathbb{R})$ the limit
\begin{equation}
\lim_{A \to \infty} \frac{1}{\text{meas}_n(A)} \int_{\mathbb{R}} g(\lambda) \tilde{\zeta}(\lambda; H_0 + V_{\alpha, A}, H_0) \, d\lambda =: \mu(g)
\end{equation}
exists almost surely and is non-random. The linear functional $\mu(g)$ is related to the surface density of states $\mu_s(g)$ (see [9]) such that $\mu_s(g) = \mu(g')$ (with $g'$ being the derivative of $g$), where
\begin{equation}
\mu_s(g) = \lim_{A_1, A_2 \to \infty} \frac{1}{\text{meas}_n(A_1 \times A_2)} \text{tr}[\chi_{A_1} \times A_2 (g(H_0 + V_{\alpha, A_1}) - g(H_0))], \quad g \in C_0^3(\mathbb{R}),
\end{equation}

almost surely for arbitrary sequences of boxes $A_1 \subset \mathbb{R}^n, A_2 \subset \mathbb{R}^m$ tending to infinity. Englisch, Kirsch, Schröder, and Simon [9] analyzed the surface states occuring at the boundary between two Anderson-type crystals and proved that the distribution induced by the functional $\mu_s(g)$ (i.e., the surface density of states) is of order (at most) 3. This result applies almost verbatim also to interactions of type (1), so we have
\begin{equation}
\mu_s(g) = \int_{\mathbb{R}} g(\lambda) N'_s(\lambda) \, d\lambda, \quad g \in C_0^3(\mathbb{R}),
\end{equation}
where the distribution $N_s(\lambda)$ of order at most 2 is called the integrated surface density of states.

Using a slightly different approach from ours Chahrour [5] constructed the functional $\mu(g)$ for the case of discrete Schrödinger operators and showed that the integrated surface density of states $N_s(\lambda)$ is a distribution of order (at most) 1. In [6] he proved that for discrete Schrödinger operators with nonrandom periodic potentials $N_s(\lambda)$ is a measurable function.

Further in [19] we proved that the functional $\mu(g)$ induces a signed Borel measure $d\Sigma(\lambda)$ such that for any $g \in C_0(\mathbb{R})$

$$
\mu(g) = \int_{\mathbb{R}} g(\lambda) \, d\Sigma(\lambda).
$$

This result implies that “$N_s(\lambda) \, d\lambda$” is a $\sigma$-finite Borel measure.

We will now extend this result and prove

**Theorem 1.** For continuous Schrödinger operators $H_\omega = H_0 + V_\omega$ with $V_\omega$ being defined by (1) the (signed) surface density of states measure $d\Sigma(\lambda)$ is Lebesgue absolutely continuous. Its Radon–Nikodym derivative belongs to $L^q_{\text{loc}}(\mathbb{R})$ for any $1 \leq q < \infty$.

In other words, Theorem 1 states that the integrated surface density of states $N_s(\lambda)$ is a measurable locally integrable function.

However, it remains unclear whether $N_s(\lambda)$ possesses further regularity properties, e.g. whether it is a function of locally bounded variation such that “$dN_s(\lambda)$” defines a measure. In fact, it is difficult to control the smoothness of $N_s(\lambda)$ since it may oscillate rapidly due to the presence of alternating surface states and surface “holes”.

The results of our article [19] extend almost verbatim (actually with several simplifications) to the case of discrete Schrödinger operators. More precisely we consider discrete Schrödinger operators with random potentials on a hypersurface,

$$
(h_\omega u)(n) = (h_0 u)(n) + \tilde{V}_\omega(n_1) \, \delta(n_2) \, u(n),
$$

$$
(h_0 u)(n) = \sum_{|j|=1}^{n} u(n-j), \quad n = (n_1, n_2) \in \mathbb{Z}^n \oplus \mathbb{Z}^n,
$$

where $\delta(n_2)$ is the Kronecker symbol and $\tilde{V}_\omega(n_1)$ a metrically transitive random field on $\mathbb{Z}^n$.

We will prove the following analogue of Theorem 1.

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Theorem 2. For discrete Schrödinger operators of the form (5) the
(signed) surface density of states measure $d\Xi(\lambda)$ is Lebesgue absolutely
continuous. Its Radon–Nikodym derivative satisfies

$$\left| \frac{d\Xi(\lambda)}{d\lambda} \right| \leq 1$$

for Lebesgue-almost all $\lambda \in \mathbb{R}$.

The proofs of Theorems 1 and 2 will be given in Section 2. There is a
substantial literature on the spectral and scattering theory for the operators
considered here. Continuous Schrödinger operators with interaction (1)
were recently considered in Section 4 of [13]. For the discrete case see e.g.
[15, 16] and the references therein.

Section 3 plays a complementary role. Its aim is to give a simple proof of
the Combes–Hislop–Nakamura result on the Hölder continuity of the
integrated density of (bulk) states for some random Schrödinger operators.
The proof is based on the combination of the Combes–Hislop–Nakamura
$L^p$-bound with the formula of Birman and Solomyak [2]. This combina-
tion is a generalization of Simon’s spectral averaging method which was
used to prove Lipshitz continuity of the integrated density of bulk states
(Wegner’s estimate) for some random Jacobi matrices [27].

2. PROOFS OF THEOREMS 1 AND 2

The two main ingredients of our approach to prove Theorem 1 are
the Banach–Alauglu theorem (see e.g. [24]) and the Combes–Hislop–
Nakamura $L^p$-bound for the spectral shift function [8] (see also its gener-
alization by Hundertmark and Simon in [14]).

It is generally known that the discrete case is much easier to handle than
the continuous case. Indeed in the discrete case (Theorem 2) we actually do
not need the $L^p$-bound for the spectral shift function and will use instead a
well-known bound for finite rank perturbations. We note also that in
the case of finite rank perturbations this bound is implied by the
$L^p$-bound.

In the sequel we will use the following well-known lemma, which is a
direct consequence of the Banach–Alauglu theorem.

Lemma 2.1. Let $1 < q \leq \infty$. Let the sequence of real valued functions
$f_n \in L^q(a, b)$ satisfy

$$\int_a^b |f_n(\lambda)|^q \, d\lambda \leq C \quad (q < \infty) \quad \text{or} \quad \sup_{\lambda \in (a, b)} |f_n(\lambda)| \leq C \quad (q = \infty)$$
uniformly in $n$ for some $C < \infty$. If the sequence $f_n(\lambda) \, d\lambda$ converges weakly to a signed measure $d\mu(\lambda)$ then this measure is absolutely continuous. Its Radon–Nikodym derivative belongs to $L^q(a, b)$.

For reader’s convenience we recall the proof. By the Banach–Alaoglu theorem we can find a subsequence $f_{n(i)}$ of $f_n$ which converges in the weak-* topology, i.e., there exists $f_\infty \in L^q(a, b) \subset L^1(a, b)$ such that

$$ \int f_{n(i)}(\lambda) \, g(\lambda) \, d\lambda \to \int f_\infty(\lambda) \, g(\lambda) \, d\lambda, \quad g \in L^{q/(q-1)}(a, b). $$

Thus the measure $d\mu(\lambda) = f_\infty(\lambda) \, d\lambda$ is absolutely continuous.

We start with the proof of Theorem 2 which is much easier than the one for Theorem 1. For an arbitrary rectangular box $A \subset \mathbb{R}^n$ with integer-valued vertices we define

$$ V_{n, A}(n) = \left\{ \begin{array}{ll} \overline{F}_\alpha(n_1) \delta(n_2), & n_1 \in A, \\ 0, & \text{otherwise}. \end{array} \right. $$

Adopting the results of our article [19] to the case of discrete Schrödinger operators we have

**Proposition 2.1.** For any $g \in C^1_0(\mathbb{R})$ the limit

$$ \lim_{A \to \infty} \frac{1}{\operatorname{meas}_n(A)} \int_A g(\lambda) \, \xi(\lambda; h_0 + V_{n, A}, h_0) \, d\lambda $$

exists and defines a linear functional $\mu(g)$ on $C^1_0(\mathbb{R})$. This functional extends to all $g \in C_0(\mathbb{R})$ with the representation

$$ \mu(g) = \int g(\lambda) \, d\Xi(\lambda) $$

and with $\Xi$ being a signed Borel measure.

The main idea behind the proof of Proposition 2.1 is to consider the random fields $V^+(n)$ and $V^-(n)$ such that $V^+(n) = \max \{V_n(n), 0\}$ and $V^-(n) = \min \{V_n(n), 0\}$. It is straightforward to see that $V^+(n)$ and $V^-(n)$ are stationary, $\mathbb{Z}^n$-metrically transitive random fields. By the chain rule for the spectral shift function we have

$$ \xi(\lambda; h_0 + V_{n, A}, h_0) = \xi(\lambda; h_0 + V^+_{n, A} + V^-_{n, A}, h_0 + V^+_{n, A}) + \xi(\lambda; h_0 + V^+_{n, A}, h_0). $$

(2.1)
The first term on the r.h.s. of (2.1) is non-positive and the second is non-negative. The next step is to prove the almost sure existence of the limits

$$\lim_{\lambda \to \infty} \frac{1}{\text{meas}_n(A)} \int_{\mathbb{R}} g(\lambda) \, \xi(\lambda; h_0 + V_{\omega, A}, h_0) \, d\lambda =: \mu^+(g)$$

and

$$\lim_{\lambda \to \infty} \frac{1}{\text{meas}_n(A)} \int_{\mathbb{R}} g(\lambda) \, \xi(\lambda; h_0 + V_{\omega, A} + V_{\omega, A}^+, h_0 + V_{\omega, A}) \, d\lambda =: \mu^-(g)$$

for all \( g \in C^0_{\text{loc}}(\mathbb{R}) \). But this follows from arguments used in [19] or [5]. The functionals \( \mu^\pm(g) \) are sign-definite. By the Riesz representation theorem they define Borel measures \( \mathcal{E}^\pm(\cdot) \). Moreover we have \( \mu(g) = \mu^+(g) + \mu^-(g) \) and therefore \( \mathcal{E}(\cdot) = \mathcal{E}^+(\cdot) + \mathcal{E}^-(\cdot) \), where \( \mathcal{E}(\cdot) \) is a positive and \( \mathcal{E}^-(\cdot) \) a negative Borel measure.

Now we note that \( V_{\omega, A} \) is a finite rank perturbation,

$$\text{Rank } V_{\omega, A} \leq \text{meas}_n(A).$$

Therefore we have

$$\frac{1}{\text{meas}_n(A)} |g(\lambda; h_0 + V_{\omega, A}, h_0)| \leq 1.$$ 

Applying Lemma 2.1 with \( q = \infty \) and using the fact that \( C^0_{\text{loc}} \)-functions are dense in \( C_0 \) from Proposition 2.1 we immediately obtain that the measure \( d\mathcal{E} \) is absolutely continuous. This completes the proof of Theorem 2.

We turn to the proof of Theorem 1. We have the following analogue of Proposition 2.1 (see [19, Section 5.2]):

**Proposition 2.2.** For any \( g \in C^0_{\text{loc}}(\mathbb{R}) \) the limit

$$\lim_{\lambda \to \infty} \frac{1}{\text{meas}_n(A)} \int_{\mathbb{R}} g(\lambda) \, \xi(\lambda; H_0 + V_{\omega, A}, H_0) \, d\lambda$$

exists and defines a linear functional \( \mu(g) \) on \( C^0_{\text{loc}}(\mathbb{R}) \). This functional extends to all \( g \in C_0(\mathbb{R}) \) and admits the representation

$$\mu(g) = \int g(\lambda) \, d\mathcal{E}(\lambda)$$

with \( \mathcal{E} \) being a signed Borel measure.
Remark. Proposition 2.2 was proved in [19] under the assumption \( \text{supp}f \subseteq [-1/2, 1/2]^d \). With rather obvious modifications the proof carries over to the case of single-site potentials with an arbitrary compact support considered in the present work. In particular, one needs to apply Theorem 2.1 of [19] to estimate the effect of overlapping potentials.

Let \( s_j(T) \) denote the singular values of a compact operator \( T \). For any \( 0 < p < \infty \) define the functional \( T \mapsto |T|_p \) by

\[
|T|_p = \sum_j s_j(T)^
u.
\]

As is well known, for \( p \geq 1 \) this functional defines a norm. The set of compact operators \( T \) with finite \( |T|_p \) we denote by \( \mathcal{F}_p \). In particular, \( \mathcal{F}_1 \) is the space of all trace class operators and \( \mathcal{F}_2 \) is the space of all Hilbert–Schmidt operators. If \( T_i \in \mathcal{F}_p, \ 0 < p_i < \infty, \ i = 1, 2 \) then \( T_1T_2 \in \mathcal{F}_p \) with \( p^{-1} = p_1^{-1} + p_2^{-1} \) and

\[
|T_1T_2|_p \leq |T_1|_{p_1} |T_2|_{p_2}. \tag{2.2}
\]

The proof of this inequality can be found in [3, Corollary 11.11] (actually there is a misprint there).

The proof of Theorem 1.1 heavily relies on the following lemma which is due to Combes, Hislop and Nakamura [8]. A generalization of this result can be found in [14].

**Lemma 2.2.** Let \( A \) be a bounded self-adjoint operator on a separable Hilbert space. Let the trace class operator \( C \) be in \( \mathcal{F}_{1/p} \) for some \( 1 < p < \infty \). Then

\[
\| \zeta(\cdot; A+C, A) \|_{L^p} \leq |C|^{1/p}.
\]

In the case \( p = 1 \) this bound provides the well-known \( L^1 \)-bound for the spectral shift function, \( \| \zeta(\cdot; A+C, A) \|_{L^1} \leq |C|_1 \). The case \( p = \infty \) is relevant in the case of finite rank perturbations, \( \| \zeta(\cdot; A+C, A) \|_{L^\infty} \leq \text{Rank} \ C \).

To proceed, we recall the definition of the Birman–Solomyak spaces \( L^p(L^q) \), \( 1 \leq p, q \leq \infty \). They are the sets of all measurable functions satisfying \( \| \phi \|_{p(L^q)} < \infty \) with

\[
\| \phi \|_{p(L^q)} = \left( \sum_{j \in \mathbb{Z}^d} \left[ \int_{A_0} |\phi(x+j)|^q \, dx \right]^{p/q} \right)^{1/p}
\]

and \( A_0 \) being the unit cube centered at the origin.
We will write 
\[ R_{\omega, A}(-c) = (H_0 + V_{\omega, A} + c)^{-1} \]
and 
\[ R_0(-c) = (H_0 + c)^{-1} \]
for the resolvents of the operators \( H_0 + V_{\omega, A} \) and \( H_0 \) respectively. With the above assumptions on \( f \) we prove

**Proposition 2.3.** Let \( k \) be an integer such that \( k > (v - 1)/2 \) if \( v \geq 4 \) and \( k \geq 1 \) if \( v \leq 3 \). Let \( c \) be a sufficiently large positive number. Then for any \( p > v/(2k + 1) \) such that \( p \leq 4 \) if \( v \leq 3 \) and \( p < 2r \) if \( v \geq 4 \) the difference \( R_{\omega, A}(-c)^k - R_0(-c)^k \) satisfies the inequality

\[
| R_{\omega, A}(-c)^k - R_0(-c)^k |_p \leq C \text{ meas}_n(A)^{1/p} \tag{2.3}
\]
with \( C \) being a constant independent of \( \Lambda \) and \( \omega \).

**Remarks.**

1. The number \( p \) can always be chosen to satisfy \( p < 1 \).

2. A result of this type was already proved by Combes, Hislop, and Nakamura in [8] (Proposition 5.1). The new ingredient in Proposition 2.3 is the volume dependence in the bound (2.3).

3. The fact that \( R_{\omega, A}(-c)^k - R_0(-c)^k \in \mathcal{F}_k \) for \( k > (v - 1)/2, \quad v \geq 4 \), and sufficiently large \( c \) follows from Theorem XI.12 of Reed–Simon [25].

For any measurable function \( W \) we define \( W^{1/2} \) by \( W^{1/2} = \text{sign} \cdot |W|^{1/2} \).

For the proof of Proposition 2.3 and again with the assumptions on \( f \) we need the following

**Lemma 2.3.** Let \( k \geq 1/2 \) and \( p > v/2k, \quad p \geq 1 \). Moreover, let \( p \leq 4 \) if \( v \leq 3 \) and \( p < 2r \) if \( v \geq 4 \). Then there is a constant \( C > 0 \) depending on \( \kappa, f, k, \) and \( v \) only such that

\[
| R_0(-c)^k | V_{\omega, A}^{1/2} |_p \leq C \text{ meas}_n(A)^{1/p} \tag{2.4}
\]

and

\[
| V_{\omega, A}^{1/2} R_0(-c)^k |_p \leq C \text{ meas}_n(A)^{1/p}.\]

**Remark.** Note that both inequalities

\[
v/2k < p < 2r \quad \text{if} \quad v \geq 4,
\]
\[
v/2k < p \leq 4 \quad \text{if} \quad v \leq 3.
\]
can always be satisfied. Indeed, for any \( k \geq 1/2 \) and \( v \leq 3 \) the inequality \( v/2k < 4 \) holds. Since \( 2r > v \) for \( v \geq 4 \) and any \( k \geq 1/2 \) we have \( v/2k < 2r \).

**Proof.** We consider the operator \( R_0(-c)^k | V_{\omega, A}^{1/2} \). The operator \( V_{\omega, A}^{1/2} R_0(-c)^k \) may be discussed similarly. Define the function

\[
g(x) = (x^2 + c)^{-k}, \quad x \in \mathbb{R}^+. \tag{2.4}
\]

It is easy to verify that \( g \in L^q(\mathbb{R}^+) \) and \( g \in l^q(L^2) \) for any \( q > v/2k \).
Suppose first that \( n \leq 3 \). From the assumption \( f \in L^2(\mathbb{R}^n) \) and the support property of \( f \) it follows that \( f \in L^q(\mathbb{R}^n) \) and thus \( f \in L^r(\mathbb{R}^n) \) for any \( 1 \leq q < r \). If \( n \geq 4 \) then from the assumption that \( f \in L^2(\mathbb{R}^n) \) for some \( r > n/2 \) and since \( f \) has compact support it follows that \( f \in L^r(\mathbb{R}^n) \) for any \( 1 \leq q < r \).

For the case \( 1 \leq p \leq 2 \) using the inequality \((a+b)^q \leq a^q + b^q\), \( a, b \geq 0 \), \( 0 \leq q \leq 1 \) \((a+b)^q = (a+b)^{q-1} = (a+b)^{q-1} = (a+b)^q\) we estimate as follows

\[
\|V_{\omega, A}\|^p_{L^p(\mathbb{R}^n)} \leq \left\| \left( \sum_{j \in \mathbb{Z}^n} |\xi_j(\omega)|^p \right)^{1/p} \right\|_{L^p(\mathbb{R}^n)} = \sum_{j \in \mathbb{Z}^n} \left\| \left( \sum_{k \in \mathbb{Z}^n} |\xi_j(\omega)|^p \right)^{1/p} \right\|_{L^p(\mathbb{R}^n)}
\]

with \( C_1 \) being some constant depending on \( \kappa, f, p, \) and \( \nu \) only. In the case \( 2 \leq p < \infty \) let \( \tilde{A} \subset \mathbb{Z}^n \) be the set of all \( k \in \mathbb{Z}^n \) such that \( V_{\omega, A}(x-k) \neq 0 \) on a subset of \( \Delta_0 \) of positive Lebesgue measure. Since \( f \) is compactly supported there is obviously a constant \( C > 0 \) such that \#\( (\tilde{A}) \leq C \) meas\( \omega, A \). Let \( J(k) \), \( k \in \tilde{A} \) be the set of all \( j \in \mathbb{Z}^n \) such that \( f(x+j-k) \neq 0 \) on a subset of \( \Delta_0 \) of positive Lebesgue measure. Obviously, \#\( (J(k)) \) is uniformly bounded for all \( k \). We denote by \( \chi_{\Delta_0} \) the characteristic functions of the unit cube \( \Delta_0 = [-1/2, 1/2]^n \) and write

\[
V_{\omega, A}(x) = \sum_{k \in \tilde{A}} u_k(x) \quad \text{with} \quad u_k(x) = \sum_{j \in J(k)} \xi_j(\omega) f(x-j) \chi_{\Delta_0}(x-k)
\]

such that \( \text{supp } u_k(\cdot-k) \subseteq \Delta_0 \). We estimate

\[
\|u_k\|^p_{L^p(\mathbb{R}^n)} \leq \left( \sum_{j \in J(k)} |\xi_j(\omega)| \|f\|_{L^p(\mathbb{R}^n)} \right)^{p/2}
\]

\[
= \left( \sum_{j \in J(k)} |\xi_j(\omega)| \right)^{p/2} \|f\|^{p/2}_{L^p(\mathbb{R}^n)}.
\]
Therefore
\[
\|V_{w,\lambda}^{1/2}\|_{L^p(\mathbb{R}^d)} = \left\| \sum_{k \in \Lambda} |\xi_k|^{1/2} \xi_k \right\|_{L^p(\mathbb{R}^d)} \leq \sum_{k \in \Lambda} \left\| \xi_k \right\|_{L^{p/2}(\mathbb{R}^d)}^{1/2} \leq C_2 \text{meas}_d(A) \quad (2.6)
\]
with \(C_2\) depending again on \(\kappa, \phi, p,\) and \(v\) only.

To estimate the norm of \(R_0(\omega) V_{w,\lambda}^{1/2}\) we use the Birman–Solomyak inequality [1] (see also [26])
\[
|w g(-i)\|_q \leq C_q \|w\|_{L^q(\mathbb{R})} \|g\|_{L^q(\mathbb{R})}, \quad 1 \leq q \leq 2
\]
and the Seiler–Simon inequality (see [26])
\[
|w g(-i)\|_q \leq (2\pi)^{-1/2} \|w\|_{L^q(\mathbb{R})} \|g\|_{L^q(\mathbb{R})}, \quad 2 \leq q < \infty.
\]
Setting \(w = |V_{w,\lambda}^{1/2}\) and \(g\) given by (2.4) in these inequalities and then using the estimates (2.5) and (2.6) proves the lemma.

**Proof of Proposition 2.3.** First we consider the case \(v \leq 3\) and \(k = 1\). By the resolvent equation
\[
R_{w,\lambda}(\omega) - R_0(\omega) = -R_0(\omega) V_{w,\lambda}^{1/2} (I + V_{w,\lambda}^{1/2} R_0(\omega) V_{w,\lambda}^{1/2})^{-1} V_{w,\lambda}^{1/2} R_0(\omega).
\]
Since the operator norm of \((I + V_{w,\lambda}^{1/2} R_0(\omega) V_{w,\lambda}^{1/2})^{-1}\) is uniformly bounded, we obtain from (2.2) and Lemma 2.3
\[
|R_{w,\lambda}(\omega) - R_0(\omega)|_p \leq C |R_0(\omega) V_{w,\lambda}^{1/2}|_p \|V_{w,\lambda}^{1/2} R_0(\omega)\|_p \leq C \text{meas}_d(A)^{1/p}.
\]
We turn to the case of arbitrary \(v\). For any \(k \in \mathbb{N}\) we have
\[
R_{w,\lambda}(\omega)^k - R_0(\omega)^k = (-1)^{k-1} \frac{d^{k-1}}{(k-1)!} \frac{\partial}{\partial \omega} \left[ R_{w,\lambda}(\omega) - R_0(\omega) \right]^{k-1}
\]
\[
= \sum_{j+m+n = k-1} c_{jmn} R_0(\omega)^{j+1} |V_{w,\lambda}^{1/2} K_m V_{w,\lambda}^{1/2} R_0(\omega)^{n+1},
\]
with some coefficients $c_m$. The operators $K_m$ are given by

$$K_m = \frac{d^m}{dc_m} (I + V_{\omega,d}^{1/2} R_0(-c) |V_{\omega,d}|^{1/2})^{-1}, \quad m \in \mathbb{N} \cup \{0\}.$$  \hspace{1cm} (2.7)

Applying Lemma 2.3 we obtain

$$R_0(-c)^{i+1} |V_{\omega,d}|^{1/2} \in \mathcal{F}_p, \quad \text{for} \quad p > \frac{v}{2(l+1)},$$

$$V_{\omega,d}^{1/2} R_0(-c)^{p+1} \in \mathcal{F}_p, \quad \text{for} \quad p > \frac{v}{2(n+1)}$$

with

$$|R_0(-c)^{i+1} |V_{\omega,d}|^{1/2}| \leq C \text{ meas}_\alpha(A)^{1/p},$$

$$|V^{1/2}_{\omega,d} R_0(-c)^{p+1}| \leq C \text{ meas}_\alpha(A)^{1/p}.$$  \hspace{1cm} (2.8)

We turn to the discussion of the operators $K_m$. Obviously we have

$$\frac{d}{dc} (I + V_{\omega,d}^{1/2} R_0(-c) |V_{\omega,d}|^{1/2})^{-1}$$

$$= (I + V_{\omega,d}^{1/2} R_0(-c) |V_{\omega,d}|^{1/2})^{-1}$$

$$\times V_{\omega,d}^{1/2} R_0(-c)^2 |V_{\omega,d}|^{1/2} (I + V_{\omega,d}^{1/2} R_0(-c) |V_{\omega,d}|^{1/2})^{-1}.$$  \hspace{1cm} (2.9)

Let $m, i \in \mathbb{N}$, $i \leq m$ be given. By $M_{m,i}$ we denote the set of all multiindices $\overrightarrow{m} = (m_1, \ldots, m_i)$ with $m_1, \ldots, m_i \in \mathbb{N}$ satisfying the following conditions

$$m_1, \ldots, m_i \geq 2,$$

$$m_1 + \cdots + m_i = m + i.$$  \hspace{1cm} (2.10)

Applying the formula (2.8) to (2.7) recursively we obtain that for any $m \geq 1$ the operator $K_m$ can be represented in the form

$$K_m = \sum_{i=1}^{m} \sum_{\overrightarrow{m} \in M_{m,i}} c_{m_{\overrightarrow{m}}}(I + V_{\omega,d}^{1/2} R_0(-c) |V_{\omega,d}|^{1/2})^{-1}$$

$$\times \prod_{j=1}^{i} V_{\omega,d}^{1/2} R_0(-c)^{m_j} |V_{\omega,d}|^{1/2} (I + V_{\omega,d}^{1/2} R_0(-c) |V_{\omega,d}|^{1/2})^{-1}$$  \hspace{1cm} (2.9)

with $c_{m_{\overrightarrow{m}}}$ being some real numbers.
We prove now that for any \( m \geq 1 \)

\[
K_m \in \mathcal{F}_p, \quad |K_m|_{p_3} \leq C \text{ meas}_n(A)^{1/p} \quad \text{with} \quad p_3 > \frac{v}{2(m+1)}.
\]  

(2.10)

From Lemma 2.3 and using the inequality (2.2) we obtain

\[
V_{\omega, A}^{1/2} R_0(-c)^m |V_{\omega, A}|^{1/2} \in \mathcal{F}_{\bar{q}_i} \quad \text{with} \quad q_i > \frac{v}{2m_j}
\]

and \( q_j < 4 \) for \( v \leq 3 \) and \( q_j < 2r \) for \( v \geq 4 \). Moreover the inequality

\[
|V_{\omega, A}^{1/2} R_0(-c)^m |V_{\omega, A}|^{1/2}|_{\bar{q}_i} \leq C \text{ meas}_n(A)^{1/m_j}
\]

holds. Thus

\[
\prod_{j=1}^i V_{\omega, A}^{1/2} R_0(-c)^m |V_{\omega, A}|^{1/2} (I + V_{\omega, A}^{1/2} R_0(-c) |V_{\omega, A}|^{1/2})^{-1} \in \mathcal{F}_{\bar{q}_i}
\]

with

\[
\bar{q}_i > \frac{v}{2 \sum_{j=1}^i m_j} = \frac{v}{2(m+i)}
\]

and such that \( \bar{q}_i < 4 \) for \( v \leq 3 \) and \( \bar{q}_i < 2r \) for \( v \geq 4 \). Moreover the estimate

\[
\left| \prod_{j=1}^i V_{\omega, A}^{1/2} R_0(-c)^m |V_{\omega, A}|^{1/2} (I + V_{\omega, A}^{1/2} R_0(-c) |V_{\omega, A}|^{1/2})^{-1} \right|_{\bar{q}_i} \leq C \text{ meas}_n(A)^{1/q_i}
\]

holds. In the equation (2.9) the worst case occurs for \( i = 1 \). Thus the estimate (2.10) is proved.

From Lemma 2.2 it now follows that

\[
R_{\omega, A}(-c)^k - R_0(-c)^k \in \mathcal{F}_p
\]

with \( p \) satisfying

\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} < \frac{2(l+1) + 2(n+1) + 2m + 2}{v} = \frac{2(k+2)}{v}
\]

and the estimate (2.3) holds. Since the inequality \( p > v/(2(k+2)) \) is satisfied with any \( p > v/(2(k+1)) \) this completes the proof of Proposition 2.3.  \( \blacksquare \)
Now we are in the position to complete the proof of Theorem 1. We choose some $\kappa > (n - 1)/2$ if $v \geq 4$ and set $\kappa = 1$ if $v \leq 3$. Fix some $p$ satisfying $1 > p > v/(2k + 1)$. Consider an arbitrary interval $(a, b)$ of the real line. Using the invariance principle for the spectral shift function (3) we estimate

$$
\int_a^b \left| \frac{\xi(\lambda; H_\omega + V_\omega A, H_\omega)}{\text{meas}_\omega(A)} \right|^{1/p} \, d\lambda \\
\leq \left( \text{meas}_\omega(A) \right)^{-1/p} \int_a^b \left| \xi((\lambda + c)^{-k}; R_\omega A(\omega, -c)^k, R_\omega A(\omega, 0)^k) \right|^{1/p} \, d\lambda \\
= \left( \text{meas}_\omega(A) \right)^{-1/p} k^{-1} \int_{(a+b)^{-k}} \left| \xi(t; R_\omega A(\omega, -c)^k, R_\omega A(\omega, -c)^k) \right|^{1/p} t^{-1/k - 1} \, dt \\
\leq \left( \text{meas}_\omega(A) \right)^{-1/p} k^{-1} (b+c)^{k+1} \int_\mathbb{R} \left| \xi(t; R_\omega A(\omega, -c)^k, R_\omega A(\omega, 0)^k) \right|^{1/p} \, dt.
$$

Now applying Lemma 2.2 we get

$$
\int_a^b \left( \frac{\xi(\lambda; H_\omega + V_\omega A, H_\omega)}{\text{meas}_\omega(A)} \right)^{1/p} \, d\lambda \leq C \left( \text{meas}_\omega(A) \right)^{-1/p} \left| R_\omega A(\omega, -c)^k - R_\omega A(\omega, 0)^k \right|_p.
$$

By Proposition 2.3 the r.h.s. of this inequality is bounded uniformly in $A$ and $\omega \in \Omega$. Thus Lemma 2.1 with $q = p^{-1}$ proves the absolute continuity of $d\mathcal{E}$. Choosing $k$ sufficiently large the number $p$ can be made arbitrary small.

3. Hölder continuity of the integrated density of bulk states

Here we give a simple proof of the Hölder continuity of the integrated density of bulk states for some random Schrödinger operators based on the new $L^p$-bound of Combes, Hislop, and Nakamura and on the formula of Birman and Solomyak [2].

To be concrete, we consider the model, where the single-site potential $f$ is supposed to be compactly supported, bounded, and also bounded from below by a positive multiple of the characteristic function $\chi$ of the unit cube centered at the origin,

$$
H_\omega = -\Delta + \sum_{j \in \mathbb{Z}^d} \alpha_j(\omega) f(j - j) \quad \text{on} \quad L^2(\mathbb{R}^d), \quad f \geq c_0 \chi, \quad c_0 > 0.
$$
The distribution $\kappa$ will be supposed to be absolutely continuous, $d\kappa = p(\alpha) \, d\alpha$, and compactly supported, i.e., $\text{supp } p \subseteq [\alpha_-, \alpha_+]$.

The integrated density of states has the following representation (see [23])

$$N(\lambda) = \mathbb{E}\{\text{tr}(\chi \mathcal{E}_{H_0}((\infty, \lambda)))\},$$

where $\mathcal{E}_{H_0}$ denotes the spectral projection corresponding to $H_0$. Lipshitz continuity of $N(\lambda)$ for $f = \chi$ was proved in [21] and for $f$ considered here in [7].

With $I = (\lambda_1, \lambda_2)$ and $A_\alpha = H_\alpha|_{q(\alpha) = 0}$ we consider

$$N(\lambda_2) - N(\lambda_1) = \mathbb{E}\{\text{tr}(\chi \mathcal{E}_{H_\alpha}(I) \chi)\}$$

$$\leq \|p\|_{\alpha} \mathbb{E}\left\{ \int_{\lambda_1}^{\lambda_2} d\alpha \, \text{tr}(\chi \mathcal{E}_{A_{\alpha+\delta f}}(I) \chi) \right\}. \quad (3.1)$$

We need the following simple result:

**Lemma 3.1.** Let $A_1$ and $A_2$ be bounded self-adjoint operators on a separable Hilbert space such that $0 \leq A_1^2 \leq A_2^2$. Let an orthogonal projection $P$ be such that $A_1PA_1 \in \mathcal{F}$ and $A_2PA_2 \in \mathcal{F}$. Then $\text{tr}(A_1PA_1) \leq \text{tr}(A_2PA_2)$.

**Proof.** Consider a monotone sequence of finite-dimensional orthogonal projections $P_n$ strongly approximating $P$ such that $P_n \leq P$ for all $n \in \mathbb{N}$. For any $n$ we obviously have

$$\text{tr}(A_1PA_1) = \text{tr}(P_nA_1^2PA_n) \leq \text{tr}(P_nA_2^2PA_n) = \text{tr}(A_2PA_2). \quad (3.2)$$

Noting that $A_1PA_1 \leq A_2PA_2$ and $A_2PA_2 \to A_2PA_2$ strongly as $n \to \infty$ for $i = 1, 2$ by Theorem 2.16 of [26] we obtain

$$|A_1PA_1 - A_2PA_2|_1 \to 0, \quad i = 1, 2 \quad (3.3)$$

for $n \to \infty$. Thus by (3.2)

$$\text{tr}(A_1PA_1) - \text{tr}(A_2PA_2) \leq -\text{tr}(A_1PA_1 - A_1PA_1) + \text{tr}(A_2PA_2 - A_2PA_2)$$

for any $n \in \mathbb{N}$. Taking the limit $n \to \infty$ and using (3.3) proves the claim. ∎
By Lemma 3.1 we obtain
\[ \text{tr}(\chi E_{A_0 + s_f}(I) \chi) \leq c_0^{-1} \text{tr}(f^{1/2} E_{A_0 + s_f}(I) f^{1/2}) \]

Now we will use the Birman–Solomyak formula [2] (see also [28], [11]). The present formulation is from [28]. Let
\[ L^p_{\text{unif, loc}}(\mathbb{R}^n) := \left\{ \phi \left| \sup_{|x-y| \leq 1} |\phi(y)|^p \, dy < \infty \right. \right\}, \quad p > n/2. \]

**Theorem 3.1.** Let \( A = -\Delta + W \) with \( W \in L^p_{\text{unif, loc}}(\mathbb{R}^n) \) and \( V \in l^1(L^2) \), \( V \geq 0 \). For any compact interval \( I \subset \mathbb{R} \) the following relation is valid
\[
\int_I \xi(\lambda; A + \alpha, V, A + \alpha, V) \, d\lambda = \int_{s^{-}}^{s^{+}} \text{tr}(V^{1/2} E_{A_0 + a}(I) V^{1/2}) \, ds.
\]

Applying this theorem to (3.1) we obtain
\[
N(l_2) - N(l_1) \leq \|p\|_\infty \left\{ \int_{s^{-}}^{s^{+}} \xi(\lambda; A_0 + a + f, A_0 + a - f) \, d\lambda \right\},
\]

From the Hölder inequality it follows that for any \( p > 1 \) and any \( \omega \in \Omega \)
\[
\left\{ \int_{s^{-}}^{s^{+}} |\xi(\lambda; A_0 + a + f, A_0 + a - f)| \, d\lambda \right\}^{1/p} \leq \left\{ \int_{s^{-}}^{s^{+}} \xi(\lambda; A_0 + a + f, A_0 + a - f)^p \, d\lambda \right\}^{1/p} |\lambda_2 - \lambda_1|^{1/p - 1}.
\]

Choose some \( k > (n-1)/2 \) if \( n \geq 4 \) and \( k \geq 1 \) if \( n \leq 3 \) so large that \( k > n/2p \). By the invariance principle for the spectral shift function (3) for any \( \omega \in \Omega \) we have
\[
\int_a^b \xi(\lambda; A_0 + a + f, A_0 + a - f) \, d\lambda
\]
\[= \int_a^b |\xi((\lambda + c)^{-k}; (A_0 + a + f + c)^{-k}, (A_0 + a - f + c)^{-k})|^p \, d\lambda
\]
\[= k^{-1} \int_{(a+c)^{-k}}^{(b+c)^{-k}} |\xi(t; (A_0 + a + f + c)^{-k}, (A_0 + a - f + c)^{-k})|^p t^{-1/k-1} \, dt
\]
\[\leq k^{-1}(b+c)^{k+1} \int_{(a+c)^{-k}}^{(b+c)^{-k}} |\xi(t; (A_0 + a + f + c)^{-k}, (A_0 + a - f + c)^{-k})|^p \, dt. \quad (3.4)
\]
By Proposition 5.1 of [8] we have that
\[(A_w + \alpha_+ f + c)^{-1} - (A_w + \alpha_- f + c)^{-1} \in \mathcal{F}_{1/p}. \tag{3.5}\]
Alternatively we can use our Proposition 2.3. For instance, for \(k = 1\) we have
\[
(A_w + \alpha_+ f + c)^{-1} - (A_w + \alpha_- f + c)^{-1} \\
= [I + (H_0 + \alpha_- f + c)^{-1} V_{\omega_{\{0\}}} - 0]^{-1} \cdot \{ (H_0 + \alpha_+ f + c)^{-1} - (H_0 + \alpha_- f + c)^{-1} \} \\
\times [I - V_{\omega_{\{0\}}} (A_w + \alpha_+ f + c)^{-1}] \\
\]
Since for sufficiently large \(c\) the first and the last factors on the r.h.s. of this equality are bounded uniformly in \(\omega \in \Omega\), the relation (3.5) follows from Proposition 2.3. Since \(\chi \in L^\alpha\) the additional restrictions \(p \leq 4\) if \(\nu \leq 3\) and \(p < 2r\) if \(\nu \geq 4\) can be omitted.

Thus from Lemma 2.2 it follows that the l.h.s. of (3.4) is bounded by a constant \(C > 0\) uniformly in \(\omega \in \Omega\). Finally this leads to the estimate
\[
N(\lambda_2) - N(\lambda_1) \leq C_p |\lambda_2 - \lambda_1|^\frac{1}{p} \tag{3.6}
\]
for any \(p > 1\), which proves the Hölder continuity of the integrated density of states.

We sketch now how a combination of the Combes–Hislop–Nakamura \(L^p\)-bounds with the formula of Birman and Solomyak can be used to obtain a Wegner-type estimates in finite volumes. Let \(A \subset \mathbb{R}^n\) be a rectangular box \([a_1 - 1/2, b_1 + 1/2] \times \cdots \times [a_n - 1/2, b_n + 1/2]\) with \(a_k, b_k\) being integers. Denote
\[
H_{\omega}^d = -\mathcal{A}^{(d)} + \sum_{j \in \mathbb{Z}^d} \alpha_j(\omega) f(\cdot - j) \quad \text{on} \quad L^2(\Lambda)
\]
with \(\mathcal{A}^{(d)}\) been the Dirichlet Laplacian. The integrated density of states is defined as (see e.g. [23])
\[
N(\lambda) := \lim_{A \to \infty} \frac{\text{meas}(A) \text{tr} \ E_{H_{\omega}^d}((-\infty, \lambda))}{\text{meas}(A)}
\]
where \(E_{H_{\omega}^d}\) denotes the spectral projection corresponding to \(H_{\omega}^d\). By the metric transitivity of the random field we obtain
\[
\mathbb{E}\{\text{tr} \ E_{H_{\omega}^d}^{\omega}(I)\} = \sum_k \mathbb{E}\{\text{tr} \chi(\cdot - k) \ E_{H_{\omega}^d}^{\omega}(I) \chi(\cdot - k)\} = \text{meas}(A) \mathbb{E}\{\text{tr} \ E_{H_{\omega}^d}^{\omega}(I) \chi\}. \tag{3.7}
\]
By Lemma 3.1 the r.h.s. of (3.8) is bounded by
\[
\text{meas}_r(A) \frac{1}{c_0} \mathbb{E} \{ \text{tr} f^{1/2} \mathbb{E}_{H^{(\omega)}}(I) \ f^{1/2} \}.
\]

With \( A^{(\omega)}_{\omega} = H^{(\omega)} \ |_{B(\omega) = 0} \) we consider
\[
\mathbb{E} \{ \text{tr} f^{1/2} \mathbb{E}_{H^{(\omega)}}(I) \ f^{1/2} \} = \mathbb{E} \left\{ \int_{\mathbb{R}^+} dx \text{tr}(f^{1/2} \mathbb{E}_{H^{(\omega)}}(I) \ f^{1/2}) \right\}
\leq \|p\|_\infty \mathbb{E} \left\{ \int_{\mathbb{R}^+} dx \text{tr}(f^{1/2} \mathbb{E}_{A^{(\omega)}_{\omega} + \alpha f} I f^{1/2}) \right\}.
\]

(3.8)

Applying Theorem 3.1 to the r.h.s. of (3.8) and taking into account (3.7) we obtain
\[
\mathbb{E} \{ \text{tr} \mathbb{E}_{H^{(\omega)}}(I) \} \leq \text{meas}_r(A) \frac{\|p\|_\infty}{c_0} \mathbb{E} \left\{ \int_{\mathbb{R}^+} d\lambda \xi(\lambda; A^{(\omega)}_{\omega} + \alpha_+ f, A^{(\omega)}_{\omega} + \alpha_- f) \right\}
\]

Again from the Hölder inequality it follows that for any \( p > 1 \) and any \( \omega \in \Omega \)
\[
\int_{\mathbb{R}^+} |\xi(\lambda; A^{(\omega)}_{\omega} + \alpha_+ f, A^{(\omega)}_{\omega} + \alpha_- f)| \ d\lambda \leq \left( \int_{\mathbb{R}^+} \xi(\lambda; A^{(\omega)}_{\omega} + \alpha_+ f, A^{(\omega)}_{\omega} + \alpha_- f)^p \ d\lambda \right)^{1/p} |\lambda_2 - \lambda_1|^{1/p}
\]

By the standard arguments (see [25]) from (3.5) it follows that for any \( A \)
\[
(A^{(\omega)}_{\omega} + \alpha_+ f + c)^{-k} - (A^{(\omega)}_{\omega} + \alpha_- f + c)^{-k} \in \mathcal{F}_{1/p}
\]
and moreover for any \( p > 1 \) there is a constant \( \tilde{C}_p \) independent of \( \omega \in \Omega \) and \( A \) such that
\[
|(A^{(\omega)}_{\omega} + \alpha_+ f + c)^{-k} - (A^{(\omega)}_{\omega} + \alpha_- f + c)^{-k}|_{1/p} \leq \tilde{C}_p
\]
uniformly in \( A \).

Thus from Lemma 2.2 it follows that for any \( p > 1 \) the l.h.s. of (3.4) with \( A^{(\omega)}_{\omega} \) instead of \( A_{\omega} \) is bounded by \( \tilde{C}_p = |\lambda_2 - \lambda_1|^{(p - 1)/p} \) with a constant \( \tilde{C}_p > 0 \).
independent of $\omega \in \Omega$ and $A$. This leads to a Wegner-type estimate in finite volumes

$$
E \{ \text{tr} \mathcal{E}_W(\omega)(I) \} \leq \text{meas}_r(A) \frac{c_0}{C_0} \tilde{C}_p |\lambda_2 - \lambda_1|^{\frac{c_1}{p}}
$$

for any $p > 1$. In turn this implies again the Hölder continuity of the integrated density of states (3.6).

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