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The Missing Link: Riemann's "Commentatio," Differential Geometry and Tensor Analysis

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Riemann's *Collected Works* contains a seldom mentioned paper on heat conduction, written in Latin and unsuccessfully submitted for a prize to the Académie des Sciences in Paris in 1861. This paper has been presented by many, including Riemann's editors, as a contribution to the development of his geometrical ideas, first outlined in his famous *Habilitationsvortrag* of 1854, *Über die Hypothesen welche der Geometrie zu Grunde liegen.* Through a discussion of the paper, we offer a new perspective on its importance: rather than a development of the mathematics underlying new conceptions of space, it should be seen as a contribution to the development of what later became known as tensor analysis. This interpretation allows a fresh perspective to be brought to the history of a particular field of mathematics. Indeed, both Riemann's geometry and tensor analysis (as developed later) combine in general relativity. However, until then they were developed independently of one another despite both being present in different aspects of Riemann's work. Since the argument proceeds from a detailed consideration of the paper by Riemann, we give the first translation into English of that paper in the Appendix. © 1990 Academic Press, Inc.

Riemanns Gesammelte Werke enthalten einen selten erwähnten Aufsatz über Wärmeleitung. Er ist lateinisch geschrieben und wurde erfolglos 1861 für ein Preisausschreiben der Académie des Sciences in Paris eingereicht. Er wurde von vielen, einschließlich Riemanns Herausgebern, als ein Beitrag zur Entwicklung seiner geometrischen Ideen dargestellt, die zum ersten Mal in seinem berühmten Habilitationsvortrag von 1854 "Über die Hypothesen, welche der Geometrie zu Grunde liegen" vorgestellt wurden. An Hand einer Erörterung des Aufsatzes bieten wir eine neue Sichtweise zu seiner Bedeutung an. Statt als eine Entwicklung der Mathematik, die neuen Raumkonzeptionen zugrunde liegt, sollte er als ein Beitrag zur Entwicklung dessen angesehen werden, was später als Tensoranalysis bekannt wurde. Diese Deutung gestattet, eine neue Sichtweise an die Geschichte eines speziellen Gebiets der Mathematik anzulegen. Tatsächlich kommen Riemanns Geometrie und Tensoranalysis (wie sie später entwickelt wurde) in der allgemeinen Relativitätstheorie zusammen. Bis damals wurden sie jedoch unabhängig voneinander entwickelt, obwohl beide in verschiedenen Aspekten von Riemanns Werk gegenwärtig sind. Da die Argumentation von einer genauen Betrachtung des Riemannschen Aufsatzes ausgeht, geben wir in einem Anhang die erste englische Übersetzung des Aufsatzes. © 1990 Academic Press, Inc.

L'oeuvre de Riemann comprend un mémoire en Latin sur la conduction de la chaleur. Peu cité, ce mémoire fut présenté sans succès à un concours de l'Académie des Sciences en 1861. Les éditeurs de Riemann, entre autres, estiment que ce mémoire aurait contribué au développement des idées géométriques abordées par lui pour la première fois dans son *Habilitationsvortrag* de 1854—*Über die Hypothesen welche der Geometrie zu Grunde liegen*—. Dans le présent article, nous nous proposons de montrer que le mémoire en question mériterait qu'on lui accorde beaucoup plus d'importance qu'on ne l'a fait jusqu'à présent: il conviendrait davantage de l'intégrer dans l'histoire du développement de ce que

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l'on appelera plus tard "analyse tensorielle" que dans l'histoire de l'élaboration mathématique d'une nouvelle conception de l'espace. Nous portons ainsi un regard nouveau sur l'histoire d'un domaine particulier des mathématiques puisque la géométrie de Riemann ainsi que l'analyse tensorielle se retrouveront plus tard toutes les deux dans la théorie de la relativité générale. Pourtant, avant d'en arriver là, ces deux domaines avaient parcouru des chemins bien différents, même s'il est vrai qu'on les recontre l'un et l'autre dans l'oeuvre de Riemann. Etant donné que nous examinons ici le mémoire de Riemann de manière détaillée, nous en donnons en annexe une traduction anglaise complète, la première du genre.

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I. INTRODUCTION

Chapter XXII of G. F. B. Riemann's *Collected Works* is entitled "Commentatio Mathematica, qua respondere tentatur quaestioni ab Illma Academia Parisiensi propositae: . . ." [1876, 391–404]. This chapter is seldom referred to in discussions of Riemann's work, or in histories of mathematics. When it is mentioned, the chapter is often held to constitute a mathematical development of the revolutionary geometrical ideas contained in Riemann's famous *Habilitationsvortrag*, "Über die Hypothesen welche der Geometrie zu Grunde liegen" [1876, 272–287] delivered in 1854 to members of the faculty of Göttingen University including Gauss. "Commentatio Mathematica . . . ," as the full title suggests, is in fact a paper that was submitted for a prize to the Académie des Sciences of Paris in 1861 in answer to a question on heat conduction first proposed by the Académie in 1858. The paper did not win the prize which was withdrawn in 1868.

The purpose of the present paper is to evaluate Riemann's "Commentatio" in terms of its contribution to the earlier ideas of 1854 on metric geometry and to suggest that a reading of the paper as a direct contribution to metric geometry is more the intention of Riemann's editors than of Riemann himself. We propose to examine this question in part through a discussion of the mathematical content of the 1861 paper. This has involved some research into the notes made by Riemann and other papers held in the archives of the University of Göttingen (Niedersächsische Staats- und Universitätsbibliothek Göttingen, abbreviated NSUB), and consultation of the original submission held in the archives of the Académie des Sciences in Paris. The argument we shall develop is that this relatively unknown paper does indeed have a place in the history of mathematics, but not quite the one it is most commonly accorded. We hold that it can justifiably be cited as a significant contribution to tensor analysis and was thus eventually relevant to Einstein's theory of general relativity just as Riemann's conception of space was. However, we hold that Riemann's geometry and tensor analysis developed relatively independently.

The connection between Riemann's work and general relativity is not a new interpretation. However, we shall argue that the significance of the "Commentatio" does not lie in its mathematical development of geometrical ideas. Rather, it

contains an early version of the concept of tensor which developed from, but independently of, Riemann's metric geometry. Tensor analysis and the geometrical ideas came together only later with general relativity. Thus the "Commentatio" is indeed significant in the history of differential geometry but not in the way implied by some who cite it. We shall demonstrate that the mathematical work implied in Riemann's *Habilitationsvortrag* and the calculations in the "Commentatio" are related. However, the latter cannot be interpreted as Riemann consciously providing the mathematical working out of the ideas in the former paper.

The only generally available version of the paper appears in Riemann's *Collected Works*. These were first edited by Heinrich Weber with the cooperation of R. Dedekind in 1876; then a second edition appeared in 1892 which was reprinted in 1902 (with a supplement containing some of Riemann's lectures edited by M. Noether and W. Wirtinger thanks to the efforts of F. Klein). The paper in question was written in Latin since, as Riemann himself wrote in a letter to the Académie, "je ne me suis pas senti assez maître de votre langue pour l'écrire en français" (NSUB, Cod. Ms. Riemann 9). It was apparently received by the Académie on the closing date of 1 July 1861, another fact for which Riemann apologized in his letter, hoping that it was not too late and asking for it to be returned if it was. The paper has never been translated into English (with the exception of M. Spivak's translation of the second half using modern notation [Spivak 1970, 4C1–4C5]).

A related question was first proposed by the Académie in 1855 but rephrased in 1858 (see translation of the question in Appendix) since no entries had been received. The submission date was fixed for 1st July 1861 and the prize advertised as a gold medal valued at 3000 francs [1]. In 1861 the Académie announced that it had received two entries about which some flattering things were said but neither of which deserved the prize. One in particular (we may assume it was Riemann's) was described as too sketchy and the consequences insufficiently developed [Académie des Sciences 1861, 1165; 1865, 286]. Weber tells us in a footnote to the published version that Riemann's health never allowed him to develop his calculations.

The fact that references are made by commentators to the published version of "Commentatio" poses a number of problems. First and most trivial, the published version is available only in Latin and therefore not immediately understandable to some students of the history of mathematics. Second, it contains some minor typographical errors which should be corrected. Third, and most important, a connection is made by the editors to the geometry in Riemann's *Habilitationsvortrag* of 1854, which has caused the "Commentatio" to be interpreted as a development of the calculations missing from the earlier paper. No such connection appears to be made by Riemann himself except for a very brief, unjustified, and apparently irrelevant geometrical example in the essay itself.

From the point of view of the research, the Riemann *Nachlass,* consisting of a number of separate files held at the library in Göttingen, contains some interesting information. File Cod. Ms. Riemann 9 labeled "*Preisaufgabe der Pariser Akademie*" contains material relating to the "Commentatio." We know that Riemann's

notes and papers were first handed over by his wife to Dedekind who "proceeded to arrange them systematically as far as he could" [Neuenschwander 1988, 102]. They were then briefly in the hands of A. Clebsch, Riemann's successor, before coming to Weber in 1874, who started to assemble and edit the *Collected Works*.

It would appear however that during the different stages of editing the *Collected Works*, additions were made by the editors and some sheets were moved from one file to another, probably by Weber. Thus, Cod. Ms. Riemann 9 contains, among other things, sheets of calculations and notes in Riemann's hand, some of which, judging by the style, the paper, and the content, appear to have been made at the time of the 1854 paper. These may have been moved from Cod. Ms. Riemann 16 (his *Habilitationsvortrag*, "Über die Hypothesen welche der Geometrie zu Grunde liegen") to Cod. Ms. Riemann 9 by Weber while preparing his notes to the "Commentatio" which provide a mathematical connection between the two papers. The editorial notes constitute a fuller working out of many of Riemann's calculations appearing in Cod. Ms. Riemann 9.

In the file there are several versions of the editorial notes (in Latin and in German), but not the version (in German) which eventually appeared in print. We know from the preface to the second edition that these notes were extended because "*die Darstellung in der ersten Auflage das Verständnis noch nicht hinlänglich zu fordern schien*" [Riemann 1876, vii]. The file also contains several copies of the "Commentatio," all of which differ from one another in small ways and none of which is exactly the same as the version originally submitted. Letters included in the file indicate that Weber wrote to the Académie requesting the original submission, which was indeed forwarded on condition that he return it. The English version appearing in the Appendix is our translation of the paper that Riemann submitted to the Académie.

The *Comptes Rendus des Séances hebdomadaires de l'Académie des Sciences* chronicle the fate of the diverse entries submitted for the prize. The entries were sent to the Académie anonymously to guarantee fairness. Each entrant's paper was prefaced by an epigraph which could then be used to match the entry to the author's name, and kept in a sealed envelope, should the judges decide to award the prize. Riemann's was one of two entries received by the closing date in 1861 (his was the second). The other entry, received on 8 May 1861 and identified by the epigraph "*le travail c'est la vie,*" was submitted by D. Codazzi who was at the time working at the *liceo* of Pavia before becoming Professor at the University of Pavia. According to D. J. Struik, Codazzi's best known paper was on the applicability of surfaces. Coincidentally this was also an unsuccessful entry in a prize competition of the Académie held two years earlier in 1859 [Struik 1975, 331].

Since the prize was not awarded, the question on heat conduction was maintained until 1865 when a third entry was received from a certain Arthur Picart, a teacher at the Lycée Charlemagne, and accompanied by the epigraph "*Fais ce que dois, advienne que pourra.*" This entry was acknowledged in the Académie's proceedings and indeed discussed at some length but was judged too restricted in its analysis to deserve the prize [Académie des Sciences 1866, 476; 1868, 921].

The effort to ensure impartiality on the part of the Académie explains Riemann's Latin epigraph "*Et his principiis via sternitur ad majora*" [2]. This identification code has undoubtedly served to reinforce among those who mention the "Commentatio" the belief that Riemann himself was conscious of the broader significance of this paper.

In the following sections we shall start by discussing some of the most significant ways in which the "Commentatio" has been mentioned by various authors. The citations are not numerous, nor are they all in accord. Nevertheless it is these citations that led us to take an interest in the "Commentatio." In order to decide on the status of the "Commentatio" we then provide a comprehensive exposition of the contents of the paper. To provide this discussion we make use of our translation into English of the "Commentatio," which appears as an Appendix to this paper.

II. RIEMANN'S INTERPRETERS

Although discussions of Riemann's "Commentatio" are not numerous, references can be found in a variety of different sources including mathematical papers, textbooks on geometry, and accounts of the history of mathematics. In this section we consider a sample of the more significant citations and proceed chronologically. Broadly speaking it will be noted that those commentators writing after the publication of the theory of general relativity tend to describe the "Commentatio" as the development of Riemann's geometrical ideas of 1854. We start by mentioning two mathematicians, E. B. Christoffel and R. Lipschitz, who, some 50 years before general relativity, were independently considering some ideas similar to those on which Riemann was working and who referenced his 1854 paper. Their work is considered to be formative in the development of tensor analysis.

At the time when Riemann, Christoffel, and Lipschitz were working, the term *tensor* had not yet been invented, let alone the concept of tensor *analysis*. The term *tensor* was first used by the crystallographer W. Voigt in 1899 who ordered the elastic, thermal, electric, and magnetic properties of crystals in magnitudes of three types: scalar, vector, and tensor [Goldberg 1976, 62]. The theory of tensor analysis originated in the 1900 paper of Ricci and Levi-Civita, which we discuss below. They called it the theory of absolute differential calculus; it was Einstein, who using this theory in general relativity, adopted the name tensor analysis.

Although the term had not yet been invented, an early form of tensors certainly appears in the work of all three of the above mathematicians. At this point it might be appropriate to describe exactly what mathematics they were considering. Essentially they were engaged in research into the transformation of quadratic differential forms under a change of coordinate system and sought quantities that were form invariant, or covariant, under the transformations. Nowadays tensors are defined by a specific behavior under a coordinate transformation and form invariance is an essential feature of tensor equations.

Riemann provided the motivation for addressing the question of the transformation of one quadratic form into another in his *Habilitationsvortrag*. The line

element in his general geometries is an example of a quadratic differential form and the equivalence of line elements under coordinate transformations is important in his classification of geometries. The papers of both Christoffel and Lipschitz considering quadratic differential forms were published in 1869, that is, before the "Commentatio." They both attributed the stimulus for their work to the *Habilitationsvortrag* (which had been published only 2 years earlier in 1867). We shall now briefly describe their particular contribution and indicate how it was related to the later theory of tensors.

Christoffel is held to have developed the fundamental properties of quadratic differential forms in the first of two papers published in 1869 [1869a, b]. In these papers he laid the foundations of the theory of tensors in the sense that he wrote down equations which describe the behavior of certain expressions under coordinate transformations, and these equations are what essentially constitute the definition of a tensor. He also moved toward the idea of tensor analysis since he defined a new derivative which, unlike the usual derivative, behaves like a tensor quantity under transformations. It is in the first of these two papers that Christoffel introduced his now well-known three-index symbols and also what is now recognizable as the curvature tensor. Both of these quantities appeared in a different form in the earlier "Commentatio" as respectively $p_{\iota,\iota',\iota''}$ and $(\iota\iota', \iota''\iota''')$. However, Christoffel was unlikely to have read the "Commentatio" and his work was developed independently of Riemann's.

It was in the very next paper of the same volume of Crelle's journal in which Christoffel's first paper appeared that Lipschitz [1869] also considered the equivalence of quadratic forms, although his initial brief was the broader one of arbitrary p-forms. He looked for expressions involving the coefficients appearing in the differential form which remain form invariant under coordinate transformations. In the course of this investigation he also introduced three-index symbols similar to those of Christoffel. A particular invariant Ψ that he obtained is recognizable as equivalent to that which appears in the second part of the "Commentatio" as expression (II) and which involves the $(\iota\iota', \iota''\iota''')$ mentioned above. In the special case of two dimensions Lipschitz related his three-index symbols to the coefficients appearing in Gauss' line element for a surface, and the quantity Ψ to the Gaussian curvature of the surface. It is interesting to note that, in spite of referencing Riemann's 1854 paper, Lipschitz's only geometrical references draw explicitly on Gauss' work on surfaces and not on the more general geometry of Riemann. However, as with Christoffel's paper, Lipschitz's investigations were stimulated by Riemann's *Habilitationsvortrag*.

In 1900, G. Ricci and T. Levi-Civita co-authored a paper "*Méthodes de calcul differentiel absolu et leurs applications*" [Ricci and Levi-Civita 1900] in which they refer to the 1869 papers of both Christoffel and Lipschitz. In this paper which is now recognized as a major contribution to the development of absolute differential calculus, Ricci and Levi-Civita introduced the full covariant and contravariant notation which is used nowadays in tensors. Riemann's "Commentatio" is also referenced in this paper specifically because aspects of his earlier treatment of

quadratic differential forms were developed into what Ricci and Levi-Civita called a "*système covariant de* RIEMANN" [Ricci and Levi-Civita 1900, 497]. The quantity that they took up and developed in full tensorial form appeared in the "Commentatio" as $(\iota\iota', \iota''\iota''')$.

The fact that D. M. Y. Sommerville's *Bibliography of Non-Euclidean Geometry* [1911, 36] contains an entry on the "Commentatio" indicates that it might have been known fairly widely in 1911. It is more likely that the entry was due to the "Commentatio" being mentioned in earlier mathematical papers, for example, in that by Ricci and Levi-Civita mentioned above.

Klein, who was at Göttingen after Riemann's death, was instrumental in gathering together Riemann's lecture notes. He expressed great admiration for Riemann's geometrical ideas. However, in Volume I of his work on 19th-century mathematics, he did not link the "Commentatio" to Riemann's geometrical work. Rather, he concentrated on its contribution to relativity theory and wrote: ". . . *'Über eine Frage der Wärmeleitung' (Werke S.391), in welcher der ganze Apparat der quadratischen Differentialformen entwickelt wird, der jetzt in der Relativitätstheorie gebraucht wird*" [Klein 1926, 253] [3]. In Volume II he described the "Commentatio" as supplementing the geometrical ideas introduced in 1854, but then wondered how Riemann could have buried such important work in a prize essay [Klein 1927, 165].

D. J. Struik, the Dutch-born mathematician, historian of mathematics, and Marxist, has quite a lot to say about Riemann in the context of a history of differential geometry [Struik 1933]. Indeed, just as revolutionary new ideas were emerging in electricity (Faraday and Maxwell), geology (Lyell), biology (Darwin), and sociology (Marx), so, according to Struik, Riemann was elaborating a "field theory of space" which was to culminate in Einstein's relativity theory. He is referring here to Riemann's *Habilitationsvortrag* of 1854. He continues: "Riemann wrote but few formulas in his address. He had a chance to work out some of his ideas mathematically in a paper of 1861 on the distribution of electricity on cylinders" [Struik 1933, 175–176]. Struik's footnote referencing the paper on the distribution of electricity actually refers the reader to the "Commentatio" in Riemann's *Collected Works*. This is simply a misreference on Struik's part. The work on electricity on cylinders appears elsewhere in the *Collected Works* as Chapter XXVI, and was not a paper but a series of notes assembled by his editors.

More importantly, Struik is correct in his summary of the mathematical techniques appearing in the "Commentatio" and which we discuss below: the transformation of a quadratic form into a sum of squares using notation that is now part of tensor analysis. He claims there is a connection between the "Commentatio" and the *Habilitationsvortrag* but does not justify that connection. Elsewhere in the same paper, Struik describes the contributions made by Christoffel, Lipschitz, Ricci, and Levi-Civita to the theory of tensor analysis. Riemann's "Commentatio" is not mentioned in this respect, nor is it connected to any of these papers. Thus Struik does not adduce evidence for the geometrical significance of the "Commentatio," nor does he manage to relate its contents to the development of

tensor analysis, as we do in this paper. (Struik is apparently working on a history of tensor analysis at the moment [Rowe 1989, 14].)

One author who does make the connection between tensor analysis and the "Commentatio" is E. T. Bell. In his popular version of history, the mathematical machinery necessary for the formulation of relativity theory was tensor analysis which was "originated essentially by Riemann before 1860, in a posthumously published memoir on heat conduction prepared in competition for a prize offered by the French Academy of Sciences" [Bell 1952, 210]. Although this mention appears in a chapter which describes metric spaces, Bell does not attempt to link the "Commentatio" and tensor analysis as an episode in the development of differential geometry.

A connection between the "Commentatio" and the 1854 paper appears in M. Kline's history of mathematics: "In his *Pariserarbeit* [sic] (1861) Riemann returned to the question of when a given Riemannian space whose metric is

$$ds^2 = \sum_{i,j=1}^{n} g_{ij} ds_i ds_j$$

might be a space of constant curvature or even a Euclidean space" [1972, 894]. First, even if one believes that Riemann did take up his geometrical ideas again in the "Commentatio," then it is incorrect that he considered the conditions necessary for a space to have constant curvature. As we discuss in the next section, the mathematical techniques in the "Commentatio" are similar to those that might be used in considering when a space has zero curvature or is Euclidean.

Second, as is clear from the "Commentatio" itself (see Appendix), Riemann does not in fact consider "Riemannian space" at all. Furthermore Kline continues: "In two key papers Christoffel's major concern was to reconsider and amplify the theme already treated somewhat sketchily by Riemann in his 1861 paper." Although Kline does not state explicitly that Christoffel knew of Riemann's 1861 paper (which he could not have), the impression created is that he did since he is supposed to have "advanced the ideas in both of Riemann's papers" [1972, 896]. Indeed, in the details of his discussion of the "Commentatio," Kline appears to have confused it with the first of Christoffel's 1869 papers.

Kline's account is not the only one which suggests that Christoffel's work in this 1869 paper was a development of the ideas presented in Riemann's "Commentatio." Indeed, G. Temple, after a discussion of the "Commentatio," writes: "These investigations were naturally restricted to two-dimensional surfaces but they led Christoffel (1869) to consider the geometry of an n-dimensional manifold . . ." [1981, 84]. Similarly, R. Torretti in describing the contribution of Riemann in the "Commentatio" to the definition of the curvature tensor (modern nomenclature) writes: "His work was completed by Christoffel (1869)" [1978, 101–102].

We have already mentioned that Spivak has translated the second half of Riemann's "Commentatio" using modern notation which is different from Riemann's original. He has included the translation in his well-known and fre-

quently quoted *Differential Geometry* because he claims that it addresses the question: "When is a Riemann manifold *flat*?" [Spivak 1970, 4D–1]. The question is valid to the extent that the way in which Riemann tackled the problem on heat conduction made it equivalent to one on the flatness of a manifold. This equivalence was not made explicit in the "Commentatio" and Riemann talked of flat manifolds only in his 1854 paper. Thus Spivak is providing his own interpretation of the paper which is reinforced by the change of notation introduced in his translation.

The clearest case of the unwarranted association of the "Commentatio" and the 1854 paper is to be found in H. Freudenthal's entry on Riemann in the *Dictionary of Scientific Biography*. Here, after a brief mention of the "Commentatio," Freudenthal writes: "That treatise is important for the interpretation of Riemann's inaugural address . . . [This inaugural address, his *Habilitationsvortrag*] contains nearly no formulas. A few technical details are found in an earlier mentioned paper (ibid, pp. 391–423) [ie. the "Commentatio"]" [Freudenthal 1975, 454–455].

Two more recent commentators deserve mention. The first of these is K. Reich [1973, 294] who, in a history of differential geometry from Gauss to Riemann, refers to the "Commentatio." She chooses to present only the second half of the paper thus effectively dissociating it from its context as a problem on heat conduction. Furthermore, like Spivak, she translates the expressions into modern notation with the effect of automatically geometrizing them. In this way she turns Riemann's "form" into a "line element" and transforms some of his expressions into the components of a curvature tensor. E. Scholz, in his *Geschichte des Mannigfaltigkeitsbegriffs von Riemann bis Poincaré* [1980], admits that Riemann's "Commentatio" at first appears to bear no relation to differential geometry [1980, 42]. However, as his commentary proceeds, he is drawn increasingly close to the modern interpretation and writes that after Riemann has transformed his expressions through a linear coordinate change, "damit muss er nun die Frage behandeln, ob die Mannigfaltigkeit mit der Metrik $\sum g_{ij} ds_i ds_j$ 'eben' ist, wie er sich in (1854b) [*Habilitationsvortrag*] ausdrückt" [1980, 44]. Thus, like Spivak, he cannot resist the temptation of interpreting the "Commentatio" in terms of addressing the question of when a manifold is flat, despite the fact that Riemann himself makes no allusion to that question.

We have described a number of different and sometimes conflicting interpretations of Riemann's "Commentatio" proposed by various authors. Whatever particular line they develop, all highlight the importance of the paper to mathematical knowledge and some also to an understanding of the historical process of the development of mathematical ideas. However, there appears to be little consensus among them and even less discussion of the work itself. There is no doubt that the "Commentatio" is an important and often neglected work.

As we have seen, one version of its importance is that the "Commentatio" constitutes the mathematical details missing from the 1854 paper. It is hardly surprising that this opinion should have been perpetrated. Indeed, even if the

commentators had consulted only the introductory sections and notes to be found in the *Collected Works,* they would have encountered a version of this opinion. It is likely, therefore, that some of the interpreters took their cue from these sections and notes. Thus, the very first editorial note to the "Commentatio" itself by Weber refers immediately to the 1854 paper [Riemann 1876, 405]. The short biography by Dedekind at the end of the book mentions the "Commentatio" "*für welche er durch seine Untersuchungen über die Hypothesen der Geometrie schon früher die Grundlagen gewonnen hatte*" [Riemann 1876, 555]. The introduction in English by H. Lewy naturally enough takes up this perspective and states that Riemann "there answered the question which conditions must be fulfilled so that a given quadratic metric can be considered equivalent to a Euclidean metric, and introduced the so-called Riemann curvature tensor whose identical vanishing is shown to be necessary and sufficient" [Riemann 1876, no page number].

We claim that in order to evaluate the different claims described above, a clear presentation and discussion of the paper as it was written by Riemann himself is necessary. This is indeed the purpose of the following section. We found it necessary to produce a translation into English retaining Riemann's notation and observing as far as possible the mathematical conventions of the time. Since it forms an invaluable part of our presentation of Riemann's ideas, we present this translation in the Appendix.

III. RIEMANN'S "COMMENTATIO"

The paper which Riemann submitted as a response to the question posed by the Académie des Sciences is divided into two parts. The two parts are apparently quite distinct and correspond to his division of the original problem. In the first part, he set out to "first solve a more general question" than the one posed by the Académie. He was then able to use this more general solution in the second part to address the specific question posed.

It is the apparent independence of the two parts which may have led Spivak [1970] to translate only the second part. The second part is that which with hindsight one can see explicitly contains the forerunners of certain mathematical expressions that are now essential to differential geometry. Nevertheless we shall argue that it is only by studying the first part that the connection between this paper and his 1854 *Habilitationsvortrag* is revealed. Indeed, only when the paper is considered as a whole does it become clear that it does not, as Struik or Kline, for example, suggest, contain the mathematical explication of the 1854 *Habilitationsvortrag*.

To answer the question posed by the Académie, the thermal properties of a body must be determined so that:

(a) the temperature of the body is dependent on time and two spatial variables only, that is, the body has a certain spatial symmetry with respect to heat conduction;

(b) the set of isothermal curves, that is curves joining points in the body with the same temperature, does not vary.

Determining the thermal properties is equivalent to determining the temperature function u, which appears in Eq. (I), and thus the isothermal curves. In writing the equation of heat conduction in the form (I), Riemann was considering a broader class of bodies than that which the Académie described. The inclusion of the conductivity coefficients $a_{\iota,\iota'}$ under the differentiation operation assumes that they, like the specific heat h, may be dependent on x_1, x_2, x_3, that is dependent on position. Thus, Riemann did not initially set out to consider a homogeneous body. By including the terms involving $a_{\iota,\iota'}$, $\iota \neq \iota'$, he also assumed that the body is anisotropic, that is when a point in the body is heated, heat does not necessarily spread out from it equally in all directions. We shall see that it was important for Riemann's line of argument that the body is given this degree of generality.

The equation (I) of heat conduction is expressed, both implicitly and explicitly, in terms of spatial coordinates x_1, x_2, x_3. The first step in Riemann's solution was to make a transformation to a new coordinate system s_1, s_2, s_3. Thus the quantities in Eq. (I) are transformed so that they become functions of s_1, s_2, s_3; the coefficients $a_{\iota,\iota'}$ transform to $b_{\iota,\iota'}$ and the specific heat h to k. The purpose of such a coordinate transformation is to produce a simplified version of the equation by a judicious choice of coordinate system. In the original equation (I) it was assumed that the temperature u is a function of x_1, x_2, x_3 as well as of time t. The question posed by the Académie related to bodies with a certain symmetry so that, given a particular choice of coordinate system which reflects this symmetry, the temperature can be minimally expressed as a function of time and only two, not three, spatial variables. Thus Riemann introduced the coordinate transformation to generate a situation where u is a function of, say, s_1, s_2, and t only.

We consider in some detail Section 3 of the first part of the "Commentatio," since it is through this section that a formal link can be established between the answer to the problem on heat conduction and Riemann's previous paper on the foundations of geometry. In describing the link as formal we do not mean to imply that it is trivial. What we mean is that Riemann did not analyze the problem on heat conduction from a geometrical point of view and did not use geometry to enhance the understanding of the physical problem. He only gave passing reference to the analogous geometrical analysis in an appended illustration which did not throw any light on the mechanism of heat conduction. As several papers in his collected works and his close collaboration with W. Weber indicate, Riemann should be seen not only as a pure mathematician but also as a physicist. It might be useful to consider Riemann's "Commentatio" from this perspective.

However, there does exist a formal connection between Riemann's geometry and the "Commentatio." Indeed from his description in Section 3 of how he set about answering the problem on heat conduction, we see that the mathematical exercise of transforming one quadratic form into another to which the problem reduced is the same as that which arose in his classification of geometries. We have discussed elsewhere the mathematical details of Riemann's *Habilitationsvortrag* [Farwell and Knee 1990] and only refer to it briefly here. The mathematical details were by no means explicit, but implicitly it suggested that different geometries are equivalent if their line elements may be transformed into one

another by a coordinate transformation. The line elements are quadratic forms, the coefficients $g_{\iota,\iota'}$ of which are now known as components of the metric tensor, a tensor of the second rank. Although this helps us to understand why some interpreters of the "Commentatio" related the geometrical question to the heat conduction question, there is no explicit evidence for that connection.

The relationship that Riemann derived in Section 3 between the two sets of conductivity coefficients $a_{\iota,\iota'}$ and $b_{\iota,\iota'}$ is not the relationship satisfied by the components in different coordinate systems of a tensor of rank two, because of the presence of the term

$$\sum \pm \frac{\partial s_1}{\partial x_1} \frac{\partial s_2}{\partial x_2} \frac{\partial s_3}{\partial x_3},$$

which we recognize as the Jacobian of the transformation from s_1, s_2, s_3 to x_1, x_2, x_3. However, Riemann effectively demonstrated that the cofactors $\alpha_{\iota,\iota'}$ and $\beta_{\iota,\iota'}$ of $a_{\iota,\iota'}$ and $b_{\iota,\iota'}$, respectively, do satisfy the correct relationship. From this relationship he then derived the key equation

$$\sum_{\iota,\iota'} \alpha_{\iota,\iota'}\, dx_\iota\, dx_{\iota'} = \sum_{\iota,\iota'} \beta_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}.$$

As a consequence of deriving this equation, Riemann argued that the transformation of Eq. (I) under a change of coordinate system is equivalent to the transformation of $\Sigma\, \alpha_{\iota,\iota'}\, dx_\iota\, dx_{\iota'}$ into $\Sigma\, \beta_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}$ and vice versa.

Thus Riemann suggested a strategy for solving the first part of the problem: in the context of a more general problem, that is for a nonhomogeneous body, to consider all cases where u is a function of s_1, s_2, and t only; then in these cases to find the coefficients $b_{\iota,\iota'}$ and specific heat k and hence the cofactors $\beta_{\iota,\iota'}$. This part of the solution was presented by Riemann in Sections 4 and 5; we do not dwell on its finer details and only present an outline of the general method he employed to obtain the coefficients $b_{\iota,\iota'}$. Riemann took the equation satisfied by the temperature u in the special case when it does not depend on s_3 and denoted this by $F = 0$. The conductivity coefficients $b_{\iota,\iota'}$ in this equation of heat conduction thus generated a set of m independent equations

$$F_1 = 0,\ F_2 = 0,\ \ldots,\ F_m = 0.$$

The left hand side F of the original equation must then be expressible as a linear combination of the m expressions F_i, $i = 1, 2, \ldots, m$. Having determined the F_i, the unknown coefficients $b_{\iota,\iota'}$ may then be identified. So, for the cases $m = 1$, 2, 3, 4 only, Riemann considered how the form of F_i may in principle be determined.

Before we describe the second part, let us summarize the main details of the first part. In order to answer the question put by the Académie it must first be supposed that a set of conductivity coefficients $a_{\iota,\iota'}$ is given for the body in question. These coefficients $a_{\iota,\iota'}$ are defined relative to a coordinate system x_1, x_2, x_3

and are constant, since the body is homogeneous. However, in this coordinate system the temperature function u may depend on all three spatial coordinates as well as time. Since the body possesses a certain symmetry we know it is possible to choose a coordinate system in which u is expressible in terms of only two spatial coordinates and time. In order to express u in this way, Riemann transformed to a new coordinate system, but in so doing relaxed the condition of homogeneity. Methods for obtaining the conductivity coefficients $b_{\iota,\iota'}$ and hence the cofactors $\beta_{\iota,\iota'}$ in this coordinate system were presented.

Having obtained the cofactors $\beta_{\iota,\iota'}$ and hence the expression $\sum \beta_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}$ in the coordinate system s_1, s_2, s_3, Riemann announced at the end of the first part that "it now remains for us to consider when the expression $\sum \beta_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}$ can be transformed into the given form $\sum \alpha_{\iota,\iota'}\, dx_\iota\, dx_{\iota'}$" in which the coefficients $\alpha_{\iota,\iota'}$ are constant. This takes us to the contents of the second part of the paper. The subheading to the second part is "Concerning the transformation of the expression $\sum b_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}$ into the given form $\sum a_{\iota,\iota'}\, dx_\iota\, dx_{\iota'}$." The notation he employed in this subheading and indeed throughout the second part was different to that employed in the first part, in particular in his announcement at the end of the first part of what is yet to be done. As a free standing section, the second part is self-consistent, but the notation of the first part is not carried forward into it. To enable us to present this commentary on the paper as a whole, we make the following changes to Riemann's notation in the second part: $b_{\iota,\iota'}$ and $\beta_{\iota,\iota'}$ are interchanged and $a_{\iota,\iota'}$ is replaced by $\alpha_{\iota,\iota'}$. The notation appearing in the translation is Riemann's original.

It is assumed then that the conductivity coefficients $a_{\iota,\iota'}$ and hence the cofactors $\alpha_{\iota,\iota'}$ are known and the expression $\sum \beta_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}$ has been determined. Riemann then asked what is the condition that must be satisfied by the $\beta_{\iota,\iota'}$ in order that the expression $\sum \beta_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}$ can be transformed into the given form $\sum \alpha_{\iota,\iota'}\, dx_\iota\, dx_{\iota'}$ in which the coefficients $\alpha_{\iota,\iota'}$ are constant. It is always possible to transform the given $\sum \alpha_{\iota,\iota'}\, dx_\iota\, dx_{\iota'}$ into the form $\sum dx_\iota^2$. Thus Riemann's question was equivalent to asking when the expression $\sum \beta_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}$ can be transformed into $\sum dx_\iota^2$. In answering this question, Riemann thus identified the cases which are *apparently* nonhomogeneous; "apparently" because the conductivity coefficients $b_{\iota,\iota'}$ and hence $\beta_{\iota,\iota'}$ are dependent on the coordinates s_1, s_2, s_3, but the dependence is only a manifestation of the coordinate system. Thus the problem has been reduced to exactly the same mathematical exercise as that implicit in Riemann's *Habilitationsvortrag,* in which the classification of geometries in part amounts to identifying those geometries which are *apparently* flat, that is, geometries with line elements which have coordinate-dependent coefficients, but the dependence is only a manifestation of the coordinate system.

The conditions to be satisfied by the quantities $\beta_{\iota,\iota'}$ are given in Eq. (I) of the second part. The three-index quantity $p_{\iota,\iota',\iota''}$ appearing in this equation is related to what is now known as a Christoffel symbol [Christoffel 1869a] and the four-index quantity $(\iota\iota', \iota''\iota''')$ has evolved into a component of the Riemann–Christoffel curvature tensor. Even though with hindsight it is possible to make obvious connec-

tions between quantities that appear in condition (I) and key quantities in the theory of differential geometry, we emphasize that condition (I) was here derived in the context of the problem on heat conduction. A condition analogous to Eq. (I) could be derived in a geometrical context and thus be interpreted as the condition that must be satisfied by the metric associated with a space in order that the space be flat or have zero curvature. Riemann did not in this paper explicitly interpret the condition in that way. He did, however, illustrate the investigations of this part "by a geometrical example, which, although unusual, will be a useful addition." In its setting in the problem on heat conduction, the geometrical illustration is not useful in itself and gives no indication of how the geometry serves any purpose in understanding the mechanism of heat conduction.

Thus in analyzing the contents of Riemann's "Commentatio" we have shown what might have led some authors to make the connection between it and Riemann's metric geometry. More importantly, this analysis indicates that this connection was not explicitly made by Riemann himself.

IV. A NEW PERSPECTIVE

In this paper we have presented a commentary on Riemann's "Commentatio Mathematica. . . ." Our investigation leads us to conclude the following:

(a) the "Commentatio" does not explicitly contain any geometrical analysis and is not a conscious mathematical elaboration of the geometrical concepts described in the *Habilitationsvortrag* of 1854;

(b) the next stage in the development of Riemann's geometry of n-dimensional manifolds following its conception in the 1854 *Habilitationsvortrag* was tensorial and not geometrical. We develop these two points here.

First, we consider the contents of Riemann's "Commentatio." In the accounts of the "Commentatio" in the mathematical literature, we have identified two themes, although in some descriptions, the two are blurred.

For example, Struik and Kline suggest that it contains the mathematical implications of Riemann's *Habilitationsvortrag* and as such is the next stage in the development of metric geometry after that 1854 paper. Thus they both claim that the "Commentatio" is part of the development of differential geometry. On the other hand, Klein sees the "Commentatio" clearly as making a contribution to the development of tensor analysis. Other authors, such as Torretti and Spivak, tend to conflate these two views either explicitly or implicitly.

This conflation stems from interpreting the paper from a modern perspective, since tensor analysis is today very much a part of the theory of differential geometry. However, it is our view that, in order to come to an understanding of the "Commentatio," we must distinguish the development of metric geometry from that of tensor analysis. Only then can a version of the link between them be offered which does not impose the classifications of the present on 19th-century mathematics. We shall return to this point.

In the Appendix to this paper we present the first full translation of the "Commentatio" into English. Spivak's translation uses modern notation; for example, the quantities $b_{\iota,\iota'}$ and $\beta_{\iota,\iota'}$ used by Riemann were translated by Spivak into g_{ij} and γ_{ij} [Spivak 1970, 4C–2]. In the differential geometry of today, the symbols g_{ij} are used to denote the components of the metric tensor. In this modern notation the quantity $\sum b_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}$ appears as $\sum g_{ij}\, dy^i\, dy^j$, which usually denotes the invariant line element ds^2, of fundamental importance in Riemann's geometry of n-dimensional manifolds [Farwell & Knee 1990]. Thus Spivak's choice of notation in his translation is itself an interpretation which he has given to the "Commentatio." This interpretation suggests to a modern reader that in the "Commentatio" Riemann used tensor notation to describe the mathematics of metric geometry. But this modern perspective is one which had already associated these two aspects of mathematics for at least 50 years by the time Spivak was writing.

If, like Spivak and Kline, we only focus on the second part of the "Commentatio," then we may be led to describe it as the mathematical development of Riemann's geometrical ideas presented for the first time in the much less explicitly mathematical *Habilitationsvortrag*. We have already commented in Section III that the problem on heat conduction was reduced by Riemann to a mathematical question on the equivalence of two forms: "when the expression $\sum \beta_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}$ can be transformed into the required form $\sum \alpha_{\iota,\iota'}\, dx_\iota\, dx_{\iota'}$." This same question undoubtedly also arises in metric geometry when the equivalence of two geometries is considered. It is not surprising therefore that some commentators will have imputed to Riemann a continuation of his geometrical work as a result of the geometrical example contained in this paper. The only evidence for this point of view is the juxtaposition of that geometrical example and the analysis of the problem of heat conduction in terms of quadratic differential forms.

Riemann did not consider the equivalence of forms in relation to geometry, but rather in the context of heat conduction. Riemann's mathematical derivations in the second part of the "Commentatio" contain no reference to heat conduction, but equally they contain no reference to geometry. The one allusion to geometry is an illustration, which is not linked to heat conduction and does not obviously therefore serve as a "useful addition."

It is interesting to note that the mathematical details associated with Riemann's derivation of the condition for the equivalence of the forms $\sum \beta_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}$ and $\sum dx_\iota^2$ are much fuller than elsewhere in the "Commentatio." Unlike in the first part, there are no steps missing in the derivation of the condition appearing in the paper. Indeed, the complete calculations in draft form exist in Riemann's hand in Cod. Ms. Riemann 9. We have already pointed out in Section III that the notation used by Riemann in the first and second parts is different. This, together with the more expansive approach in the second part, suggests that perhaps this second part had been worked out before the first, maybe even before Riemann considered the problem on heat conduction.

There can be no doubt that Riemann responded to the question posed by the Académie in the way that he did because of his prior interest in manifolds and

particularly in the equivalence of different metrics associated with manifolds. Weber's editorial note certainly makes the connection between the condition for a given manifold to be flat and the condition for equivalence of forms which Riemann derived in the second part. It is even possible that the second part was worked out by Riemann with this connection to geometry in mind. However, there is no evidence for this and Riemann himself never makes explicit the connection.

For the reasons outlined above, we cannot concur with the hypothesis that the "Commentatio" is the explicit development of the concepts in the *Habilitationsvortrag*. We are able to reach this conclusion only by considering the "Commentatio" as a whole and thus by putting the second part in the context of the problem on heat conduction. It is the first part of the paper which provides Riemann's motivation for considering the transformation of forms presented in the second part.

It is natural to ask whether Riemann actually answered the question, posed by the Académie. After all, the members of the Académie who considered the submissions did not deem the problem solved, even though Riemann himself felt he had answered the question. In a copy appearing in Cod. Ms. Riemann 9 of a letter to the Académie, he writes in a confident tone and offers to polish up the paper before, in his opinion, its inevitable publication. It is highly probable that members of the Académie were not anticipating such an abstract solution to their problem. It is clear that Riemann presented a strategy for the solution of their problem in a general sense rather than its specific solution. For this reason perhaps the members of the Académie called to referee Riemann's submission felt that he had not worked through the problem in sufficient detail.

Finally we address the question of whether the "Commentatio" is significant in the development of differential geometry. Those authors who claim it is significant in this respect do so because they consider it to contain the mathematical details that were missing in Riemann's *Habilitationsvortrag*, in which the concept of metric geometry arose. Since our analysis of the "Commenatio" does not confirm this, it may seem that our conclusion suggests that it is not important in the history of differential geometry. Our conclusion is quite the opposite; the "Commentatio" certainly does play a significant role in what is now known as differential geometry not because it contributes directly to the development of geometry but because it is important in the development of tensor analysis.

The first use of tensor notation was in Christoffel's paper of 1869, although the term tensor was not used at the time. The paper addresses the question of the equivalence of quadratic differential forms [Ehlers 1981] which was precisely the problem considered by Riemann in the second part of the "Commentatio." Christoffel's paper is considered to be a significant contribution to the development of tensor analysis. We have already remarked that some of the quantities appearing in Christoffel's paper also appeared in the "Commentatio" as $p_{\iota,\iota',\iota''}$ and $(\iota\iota', \iota''\iota''')$. Christoffel paid much more attention to the position of indices in his quantities

and his four index quantities do satisfy the modern definition of a tensor. Riemann's $(\iota\iota', \iota''\iota''')$ is not truly tensorial, but the exercise of transforming from one differential form to another considered by Riemann in the second part of the "Commentatio" is now common in tensor analysis.

Transformation from one coordinate system to another is an essential part of the definition of a tensor. A tensor quantity is expressed in terms of some coordinate system. What distinguishes a tensor quantity from any other is its behavior under a transformation from one coordinate system to another. Even though the coordinates appearing in the expression of the tensor change under the coordinate transformation, the overall form of the tensor expression remains unchanged. Thus, a tensor is form invariant under coordinate transformations.

Riemann's metric geometry requires a coordinate free description: the properties of a manifold must be independent of the coordinates used to describe it. In a similar way, mathematical models of physics must be coordinate independent. Coordinate systems are used to set up the model but they are features of the model and not of the physics. Thus equations using coordinates and describing physical processes should not change form under coordinate transformations. The equation of heat conduction is a mathematical description of a physical process; thus it is not surprising that Riemann exploited coordinate transformations in his solution of the problem on heat conduction. The coordinate transformations change the mathematical description to one where he is able to use, for example, in this case, only two spatial variables, but they do not change the physical phenomenon being described. The technique of transforming one quadratic differential form into another is peculiar neither to heat conduction nor to geometry. There are other areas of mathematical physics in which it would be appropriate to consider the same technique and which could have been used as illustrations in the "Commentatio."

Christoffel acknowledged Riemann's *Habilitationsvortrag* in his paper of 1869 and it is clear that the trigger for this paper was Riemann's metric geometry. However, none of the contents of Christoffel's paper suggests that he saw a connection between the work presented in it and geometry. In Riemann's "Commentatio," and independently in Christoffel's paper, quantities which are now known as the components of the Riemann–Christoffel curvature tensor appear. This tensor is today most important in differential geometry. However, the forerunner of this tensor was not first defined in the context of geometry and the line that can be traced between Riemann's "Commentatio" and Christoffel's paper and the modern curvature tensor does not pass through geometrical concepts only.

Tensor analysis developed independently of geometry for decades. Indeed there was no significant development in metric geometry *per se* until it linked with tensor analysis. What provided the impetus for the linking between tensor analysis and metric geometry was the theory of general relativity. Tensors may have continued to develop independently of geometry, and metric geometry may have

remained dormant, but for the theory of relativity. We do not take up this argument here but develop it elsewhere in a forthcoming paper, "The Recovery of Certainty."

Our analysis of Riemann's "Commentatio" and its obvious connection to tensors suggest that it did indeed make a significant contribution to the development of differential geometry, but not directly as some would suggest. Tensors provide the missing link between the "Commentatio" and modern differential geometry.

APPENDIX

A mathematical treatise in which an attempt is made to answer the question proposed by the most illustrious Academy of Paris:

> To determine the caloric state of any solid homogeneous body such that a system of isothermal curves, at a given instant, remains isothermal after a given time, in such a way that the temperature at a point can be expressed as a function of time and two other independent variables.
>
> *Et his principiis via sternitur ad majora.*

1

We shall consider the question proposed by the Academy in such a way that we shall first solve a more general question:

the properties of a body which determine the conduction of heat and the distribution of heat within it such that there exists a system of lines which remain isothermal.

Then:

from the general solution of this problem we shall distinguish those cases in which the properties vary from those in which the properties remain constant, that is where the body is homogeneous.

First Part

2

In order to address the first question, we must consider the conduction of heat in any body. If u denotes the temperature at a given time t at a point (x_1, x_2, x_3), the general equation satisfied by u is of the following form:

$$\begin{aligned} &\partial\left[\left(a_{1,1}\frac{\partial u}{\partial x_1} + a_{1,2}\frac{\partial u}{\partial x_2} + a_{1,3}\frac{\partial u}{\partial x_3}\right)\Big/\partial x_1\right] \\ &\qquad + \partial\left[\left(a_{2,1}\frac{\partial u}{\partial x_1} + a_{2,2}\frac{\partial u}{\partial x_2} + a_{2,3}\frac{\partial u}{\partial x_3}\right)\Big/\partial x_2\right] \\ &\qquad\qquad + \partial\left[\left(a_{3,1}\frac{\partial u}{\partial x_1} + a_{3,2}\frac{\partial u}{\partial x_2} + a_{3,3}\frac{\partial u}{\partial x_3}\right)\Big/\partial x_3\right] = h\frac{\partial u}{\partial t}. \end{aligned} \tag{I}$$

In Eq. (I), the quantities a represent the conductivity coefficients, h the specific heat for the whole volume, that is, the product of the specific heat density and some given functions of x involving x_1, x_2, x_3, for example. We confine our

investigation to the case where the conductivity in opposite directions is the same so that the relation between the quantities a is

$$a_{\iota,\iota'} = a_{\iota',\iota}.$$

Moreover, since heat moves from a warmer to a cooler area, the quadratic form

$$\begin{pmatrix} a_{1,1}, & a_{2,2}, & a_{3,3} \\ a_{2,3}, & a_{3,1}, & a_{1,2} \end{pmatrix}$$

is positive.

3

Now in Eq. (I) in place of the rectangular coordinates x_1, x_2, x_3 let us introduce any three new independent variables s_1, s_2, s_3.

The new form of Eq. (I) can easily be deduced because it is a necessary and sufficient condition that, if δu is a variation, however infinitely small of u, then the variation of the integral

$$\delta \int\int\int \sum_{\iota,\iota'} a_{\iota,\iota'} \frac{\partial u}{\partial x_\iota} \frac{\partial u}{\partial x_{\iota'}} \, dx_1 \, dx_2 \, dx_3 + \int\int\int 2h \frac{\partial u}{\partial t} \delta u \, dx_1 \, dx_2 \, dx_3 \tag{A}$$

over the volume of the body depends only on the value of δu on the surface. When the new variables are introduced, expression (A) becomes

$$\delta \int\int\int \sum_{\iota,\iota'} b_{\iota,\iota'} \frac{\partial u}{\partial s_\iota} \frac{\partial u}{\partial s_{\iota'}} \, ds_1 \, ds_2 \, ds_3 + \int\int\int 2k \frac{\partial u}{\partial t} \delta u \, ds_1 \, ds_2 \, ds_3, \tag{B}$$

where, for the sake of brevity,

$$\sum_{\iota,\iota'} a_{\iota,\iota'} \frac{\partial s_\mu}{\partial x_\iota} \frac{\partial s_\nu}{\partial x_{\iota'}} \Big/ \sum \frac{\partial s_1}{\partial x_1} \frac{\partial s_2}{\partial x_2} \frac{\partial s_3}{\partial x_3} = b_{\mu,\nu}, \qquad h \Big/ \sum \frac{\partial s_1}{\partial x_1} \frac{\partial s_2}{\partial x_2} \frac{\partial s_3}{\partial x_3} = k.$$

But introducing the forms

$$\begin{pmatrix} a_{1,1}, & a_{2,2}, & a_{3,3} \\ a_{2,3}, & a_{3,1}, & a_{1,2} \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} b_{1,1}, & b_{2,2}, & b_{3,3} \\ b_{2,3}, & b_{3,1}, & b_{1,2} \end{pmatrix} \tag{2}$$

the determinants of which we call A and B, and the adjoint forms

$$\begin{pmatrix} \alpha_{1,1}, & \alpha_{2,2}, & \alpha_{3,3} \\ \alpha_{2,3}, & \alpha_{3,1}, & \alpha_{1,2} \end{pmatrix} \tag{3}$$

$$\begin{pmatrix} \beta_{1,1}, & \beta_{2,2}, & \beta_{3,3} \\ \beta_{2,3}, & \beta_{3,1}, & \beta_{1,2} \end{pmatrix}, \tag{4}$$

it will be found that

$$A = B \sum \pm \frac{\partial s_1}{\partial x_1} \frac{\partial s_2}{\partial x_2} \frac{\partial s_3}{\partial x_3}$$

and

$$\beta_{\mu,\nu} = \sum_{\iota,\iota'} \alpha_{\iota,\iota'} \frac{\partial x_\iota}{\partial s_\mu} \frac{\partial x_{\iota'}}{\partial s_\nu}$$

and therefore

$$\sum_{\iota,\iota'} \alpha_{\iota,\iota'} \, dx_\iota \, dx_{\iota'} = \sum_{\iota,\iota'} \beta_{\iota,\iota'} \, ds_\iota \, ds_{\iota'}$$

and

$$h/A = k/B.$$

whence it can easily be seen that the transformation of Eq. (I) can be reduced to the transformation of the expression $\sum_{\iota,\iota'} \alpha_{\iota,\iota'} \, dx_\iota \, dx_{\iota'}$.

Thus we are able to solve our general problem, first by determining the functions $b_{\iota,\iota'}$ and k of s_1, s_2, s_3 such that u is independent of any one of these three variables, and second by forming the expression $\sum \beta_{\iota,\iota'} \, ds_\iota \, ds_{\iota'}$. Given values for the quantities $a_{\iota,\iota'}$ and h, then in order to determine whether u can be a function of time and only two other variables and in which cases this is so, it is necessary to find when $\sum \beta_{\iota,\iota'} \, ds_\iota \, ds_{\iota'}$ can be transformed into a given form $\sum \alpha_{\iota,\iota'} \, dx_\iota \, dx_{\iota'}$ [4]; and we shall see below that this question can be examined using a method similar to that employed by Gauss in his theory of curves on surfaces.

First therefore, let us determine the functions $b_{\iota,\iota'}$ and k of s_1, s_2, s_3 such that u is independent of any one of these three variables. To simplify the notation, let us denote the variables s_1, s_2, s_3 by α, β, γ and the form (2) by

$$\begin{pmatrix} a, & b, & c \\ a', & b', & c' \end{pmatrix}.$$

Then if u does not depend on γ, the differential equation satisfied by u will be of the form

$$a \frac{\partial^2 u}{\partial \alpha^2} + 2c' \frac{\partial^2 u}{\partial \alpha \partial \beta} + b \frac{\partial^2 u}{\partial \beta^2} + e \frac{\partial u}{\partial \alpha} + f \frac{\partial u}{\partial \beta} - k \frac{\partial u}{\partial t} = F = 0, \qquad \text{(II)}$$

where

$$\frac{\partial a}{\partial \alpha} + \frac{\partial c'}{\partial \beta} + \frac{\partial b'}{\partial \gamma} = e, \qquad \frac{\partial b}{\partial \beta} + \frac{\partial c'}{\partial \alpha} + \frac{\partial a'}{\partial \gamma} = f.$$

By giving γ different values in Eq. (II), different equations for the six differential quotients of u will be obtained with coefficients that are independent of γ. But if the m equations

$$F_1 = 0, F_2 = 0, \ldots, F_m = 0$$

are independent, that is, all the others can be written in terms of these, then for any value of γ, the equation $F = 0$ follows from these m equations, whence F must be of the form

$$c_1F_1 + c_2F_2 + \cdots + c_mF_m,$$

in which only the quantities c depend on γ.

Now let us examine more closely the individual cases $m = 1, 2, 3, 4$ and at the same time simplify the equations independent of γ in terms of which the equation $F = 0$ is expressed.

First case in which $m = 1$. If $m = 1$, then the coefficients in Eq. (II) cannot depend on γ. Now it is always possible to replace γ by a new variable $\int k d\gamma$, in so doing making $k = 1$ and all the coefficients independent of γ. Further, by replacing α and β by suitable new variables, a and b can be made to vanish. This will happen if the expression $bd\alpha^2 - 2c'd\alpha d\beta + ad\beta^2$ [5] (which cannot be the square of a linear differential expression if (2) is a positive form) is reducible to the form $md\alpha' d\beta'$ in which the new variables α' and β' are taken to be independent.

Therefore in this case the differential equation (II) will be reduced to

$$2c' \frac{\partial^2 u}{\partial\alpha\partial\beta} + e\frac{\partial u}{\partial\alpha} + f\frac{\partial u}{\partial\beta} = \frac{\partial u}{\partial t}$$

and in the form (2) a and b will then equal 0, a' and b' will be linear functions of γ, and c' will be independent of γ. In this case the temperature will remain independent of γ provided that the initial temperature is a function of α and β only.

Second case in which $m = 2$. Equation (II) can be written in terms of two equations independent of γ and if $\partial u/\partial t$ is used in one equation, it need not appear in the other. For the sake of brevity, the latter will be written as

$$\Delta u = 0 \tag{1}$$

and the former as

$$\Lambda u = \partial u/\partial t, \tag{2}$$

where Δ and Λ are characteristic expressions involving ∂_α and ∂_β.

It is easy to choose independent variables so that Eq. (1) can be changed into one with Δ given by

$$\begin{aligned} \text{either} &= \partial_\alpha\partial_\beta + e\partial_\alpha + f\partial_\beta \\ \text{or} &= \partial\alpha^2 + e\partial_\alpha + f\partial_\beta \\ \text{or} &= \partial_\alpha, \end{aligned}$$

in which the values $e = 0$ and $f = 0$ are not excluded.

Since from (1) $\partial_t\Delta u = 0$ and from (2) $\Delta\partial_t u = \Delta\Lambda u$, and $\partial_t\Delta u = \Delta\partial_t u$, it follows that

$$\Delta\Lambda u = 0. \tag{3}$$

Now two cases can be distinguished depending on whether Eq. (3) either (case (α)) follows from Eq. (1), that is

$$\Delta\Lambda = \Theta\Delta,$$

where Θ represents a new characteristic expression, or (case (β)) does not follow from Eq. (1) and a new equation independent of Δu is produced.

In order to examine case (α), let us suppose that

$$\Delta = \partial_\alpha\partial_\beta + e\partial_\alpha + f\partial_\beta.$$

Then by using the equation $\Delta u = 0$, $\Delta\Lambda u$ can be reduced to an expression which involves only the derivatives with respect to one or other variable and in which all of the coefficients are equal to zero. Since the term containing $\partial_\alpha\partial_\beta$ can be removed by using the equation $\Delta u = 0$, let us suppose that

$$\Lambda = a\partial\alpha^2 + b\partial\beta^2 + c\partial_\alpha + d\partial_\beta$$

and let us consider the expression

$$\Delta\Lambda - \Lambda\Delta.$$

In this expression, since the coefficients of ∂_α^3, ∂_β^3 must vanish, then $\partial a/\partial\beta = 0$, $\partial b/\partial\alpha = 0$. Hence if the special cases $a = 0$, $b = 0$ are excluded, it is possible to change the independent variables in order to make $a = b = 1$. Then if the coefficients of ∂_α^2, ∂_β^2 are made equal to zero in the reduced expression for $\Delta\Lambda - \Lambda\Delta$, it is found that [6]

$$\frac{\partial c}{\partial\beta} = 2\frac{\partial e}{\partial\alpha}, \frac{\partial d}{\partial\alpha} = 2\frac{\partial f}{\partial\beta}$$

from which it can be deduced that

$$\Delta = \partial_\alpha\partial_\beta + \frac{\partial m}{\partial\beta}\partial_\alpha + \frac{\partial n}{\partial\alpha}\partial_\beta$$

$$\Lambda = \partial\alpha^2 + \partial\beta^2 + 2\frac{\partial m}{\partial\alpha}\partial_\alpha + 2\frac{\partial n}{\partial\beta}\partial_\beta,$$

where m and n denote functions of α and β. These expressions for Δ and Λ used in the two differential Eqs. (1) and (2) should be sufficient to ensure that the coefficients of ∂_α, ∂_β in the reduced expression $\Delta\Lambda$ vanish.

Indeed, using a similar method for the alternative forms for Λ, the simplest expressions for Δ and Λ can be found which satisfy the condition

$$\Delta\Lambda = \Theta\Delta.$$

But we shall not linger on this investigation which is more long-winded than difficult.

This case demonstrates that the temperature always remains independent of γ provided that the initial temperature is any function of α and β satisfying the equation $\Delta u = 0$; from the equations

$$\Delta u = 0$$

$$\Lambda u = \partial u/\partial t$$

it follows that

$$0 = \Theta\Delta u = \Delta\Lambda u = \Delta\partial_t u = \frac{\partial \Delta u}{\partial t}$$

and hence the equation $\Delta u = 0$ is the only one that need be considered provided that u is positive initially and its variation is given by the second equation $\Lambda u = \partial u/\partial t$. Then indeed u satisfies the law of heat conduction, that is the equation $F = 0$.

5

There remains the other special case (β), in which $\Delta\Lambda u = 0$ is independent of $\Delta u = 0$. In order that we might also include the cases $m = 3$ and $m = 4$, let us consider the more general hypothesis that, in addition to the equation $\Delta u = 0$, there exists a linear differential equation $\Theta u = 0$, not containing $\partial u/\partial t$ and independent of $\Delta u = 0$.

If Δ is of the form $\partial_\alpha\partial_\beta + e\partial_\alpha + f\partial_\beta$, then, by using the equation $\Delta u = 0$, the derivative with respect to both variables can be removed from the expression Θ.

Now two cases must be distinguished.

If, from the expression Θ, all the differential quotients with respect to one or other variable, e.g., β are removed together, then a differential equation containing only differential quotients involving α is obtained

$$\sum_\nu a_\nu \frac{\partial^\nu u}{\partial \alpha^\nu} = 0, \tag{1}$$

or a differential equation containing only differential quotients involving t is obtained

$$\sum_\nu a_\nu \frac{\partial^\nu u}{\partial t^\nu} = 0. \tag{2}$$

For in this case, the expressions Λu, $\Lambda^2 u$, $\Lambda^3 u$, . . . , which are equal to the differential quotients of u involving t, can be transformed using the equations $\Delta u = 0$, $\Theta u = 0$ into expressions containing only differential quotients with respect to one or other variable and which are not greater than Θu. Since the number of differential quotients is finite, it is clear that elimination produces an equation of the form (2). The coefficients a_ν in both equations are functions of α, β.

It will be pertinent to observe that both of these equations are always positive even if Δ is not of the form $\partial_\alpha\partial_\beta + e\partial_\alpha + f\partial_\beta$. The special case in which $\Delta = \partial_\alpha^2 + e\partial_\alpha + f\partial_b$ can be dealt with under either heading since, by using the equation $\Delta u = 0$, all derivatives involving β can be removed from both Θu and Λu, and thus an equation of either form is easily obtained. The case in which $f = 0$, like that in which $\Delta = \partial_\alpha$, can be dealt with under the first heading.

Now let us examine the second case more carefully.

The general solution of the equation

$$\sum_\nu a_\nu \frac{\partial^\nu u}{\partial t^\nu} = 0$$

consists of terms of the form $f(t)e^{\lambda t}$, where $f(t)$ is an integral function of t, and λ is independent of t and it is easily seen that each of these functions ought to satisfy Eq. (I). Now we shall demonstrate that λ cannot be a function of x_1, x_2, x_3.

Let kt^n be the highest order term in the function $f(t)$ and let us distinguish two cases.

1. When λ is either real or of the form $\mu + \nu i$ and μ, ν are functions of one real variable α of x_1, x_2, x_3, then by substituting $u = f(t)e^{\lambda t}$ in the left-hand side of Eq. (I), the coefficient of $t^{n+2}e^{\lambda t}$ will be

$$k\left(\frac{\partial\lambda}{\partial\alpha}\right)^2 \sum_{\iota,\iota'} a_{\iota,\iota'} \frac{\partial\alpha}{\partial x_\iota}\frac{\partial\alpha}{\partial x_{\iota'}}.$$

But this quantity cannot disappear unless

$$\frac{\partial\alpha}{\partial x_1} = \frac{\partial\alpha}{\partial x_2} = \frac{\partial\alpha}{\partial x_3} = 0,$$

that is, unless α is a constant, since the form

$$\begin{pmatrix} a_{1,1}, & a_{2,2}, & a_{3,3} \\ a_{2,3}, & a_{3,1}, & a_{1,2} \end{pmatrix}$$

as shown above is positive.

2. When λ is of the form $\mu + \nu i$ and μ, ν are variables independent of x_1, x_2, x_3, then the quantities $\mu + \nu i$ and $\mu - \nu i$ can be used instead of the independent variables α and β and u will consist of the complex conjugate term $\phi(t)e^{\beta t}$ as well as $f(t)e^{\alpha t}$. But if

$$\Delta u = a\frac{\partial^2 u}{\partial\alpha^2} + b\frac{\partial^2 u}{\partial\alpha\partial\beta} + c\frac{\partial^2 u}{\partial\beta^2} + e\frac{\partial u}{\partial\alpha} + f\frac{\partial u}{\partial\beta}$$

then, by substituting $u = f(t)e^{\alpha t}$ in the equation $\Delta u = 0$ and by making the coefficient of $t^{n+2}e^{\alpha t}$ equal to zero, we will obtain $a = 0$ and further $c = 0$ by substituting $u = \Phi(t)e^{\beta t}$. Whence, by using $\Delta u = 0$, the equation $\Lambda u = \partial t$ can be transformed so that it contains only differential quotients involving one or other variable. But by substituting in turn

$$u = f(t)e^{\alpha t}, \qquad u = \phi(t)e^{\beta t}$$

the coefficient of the sum of these differential quotients is found to equal 0, and so the differential quotients in the equation $\Lambda u = \partial u/\partial t$ all ought to disappear as well, as has been proposed, when u according to our assumption is not constant.

Therefore, in this second case, the function u consists of a finite number of terms of the form $f(t)e^{\lambda t}$, where λ is constant and $f(t)$ is an integral function of t.

In the first case, since the equation is of the form

$$\sum_{\nu} a_{\nu} \frac{\partial^{\nu} u}{\partial \alpha^{\nu}} = 0, \tag{1}$$

the function u will be of the form

$$u = \sum_{\nu} q_{\nu} p_{\nu}$$

with $p_1, p_2, \ldots$, denoting particular solutions of Eq. (1) and $q_1, q_2, \ldots$ denoting arbitrary constants, that is, functions of β and t only. But if this expression is substituted into the equation

$$\Lambda u = \frac{\partial u}{\partial t},$$

an equation of the form

$$\Sigma PQ = 0$$

will be obtained where the quantities Q are differential coefficients of q and hence functions of β and t only, and the quantities P are functions of α and β only. But we have seen above that if the equation consists of n terms, then there are μ linear equations involving the functions Q and $n - \mu$ equations involving the functions P in which the coefficients are functions of β only, μ denoting any number 0, 1, 2, $\ldots, n$. Therefore expressions for $\partial q/\partial t$ will be obtained in terms of the differential quotients of q involving β and not α.

Now let us examine individual instances of our problem relating to this case.

When $m = 2$ and Δ is of the form $\partial_\alpha \partial_\beta + e\partial_\alpha + f\partial_\beta$, the resultant equation $\Delta\Lambda u = 0$ will have the following form provided that it is independent of differential quotients involving β:

$$\frac{\partial^3 u}{\partial \alpha^3} + r\frac{\partial^2 u}{\partial \alpha^2} + s\frac{\partial u}{\partial \alpha} = 0.$$

Whence u will be of the form

$$ap + bq + c,$$

with a, b, c denoting functions of β and t only, and p and q denoting functions of α and β only. Now the independent variable α can be introduced in place of q, which will give

$$u = ap + b\alpha + c,$$

where p is now the only function of both variables α and β. By substituting this expression into the equations

$$\Delta u = 0, \qquad \Lambda u = \frac{\partial u}{\partial t}$$

the coefficients will easily be determined.

We have yet to consider the case in which one of the equations into which the equation $F = 0$ was split has the form (1) and thus the form

$$r\frac{\partial^2 u}{\partial \alpha^2} + s\frac{\partial u}{\partial \alpha} = 0.$$

Then it will be the case that $u = ap + b$ with a and b denoting functions of β and t only, and p a function of α and β only. If p is replaced by the independent variable α it will follow that

$$u = a\alpha + b, \qquad \frac{\partial^2 u}{\partial \alpha^2} = 0.$$

Therefore if $m = 2$, that is, in the case when the equation $F = 0$ is split into two equations

$$\Delta u = 0$$

$$\Lambda u = \frac{\partial u}{\partial t},$$

we find that there are three possibilities: $\Delta\Lambda = \Theta\Lambda$, or the function u consists of a finite number of terms of the form $f(t)e^{\lambda t}$, where λ is a constant and $f(t)$ an integral function of t, or it takes the form

$$\phi(\beta, t)\chi(\alpha, \beta) + \alpha\phi_1(\beta, t) + \phi_2(\beta, t).$$

If $m = 3$, then the function u either consists of a finite number of terms of the form $f(t)e^{\lambda t}$ or is of the form

$$\phi(\beta, t)\alpha + \phi_1(\beta, t).$$

And so the case $m = 4$ can be solved with no further work.

For if, as well as the equation $\Lambda u = \partial u/\partial t$, three other other equations involving

$$\frac{\partial^2 u}{\partial \alpha^2}, \frac{\partial^2 u}{\partial \alpha \partial \beta}, \frac{\partial^2 u}{\partial \beta^2}, \frac{\partial u}{\partial \alpha}, \frac{\partial u}{\partial \beta}$$

are considered, then an equation of the following form will result,

$$r\frac{\partial u}{\partial \alpha} + s\frac{\partial u}{\partial \beta} = 0,$$

in which case it will be possible to choose the independent variables so that u is a function of only one of them. If the three equations involve

$$\frac{\partial^2 u}{\partial \alpha^2}, \frac{\partial^2 u}{\partial \alpha \partial \beta}, \frac{\partial^2 u}{\partial \beta^2},$$

then Λu, $\Lambda^2 u$, $\Lambda^3 u$ can be expressed in terms of $\partial u/\partial \alpha$, $\partial u/\partial \beta$ and then an equation of the following form will be produced

$$a \frac{\partial^3 u}{\partial t^3} + b \frac{\partial^2 u}{\partial \alpha \partial \beta} + c \frac{\partial u}{\partial t} = 0,$$

whence u will be either

$$pe^{\lambda t} + qe^{\mu t} + r \qquad \text{or} \qquad (p + qt)e^{\lambda t} + r,$$

where we have shown earlier that λ and μ are constants.

Now with p used in place of the independent variable α and substituted in the equation $\Lambda u = \partial u/\partial t$, we find that q cannot be a function of α even when λ and μ are not equal. Therefore p and q can be used as independent variables. Moreover, from the equation $\Lambda u = \partial u/\partial t$, we deduce that r = constant.

Therefore, in this case, u is either a function of t and one other variable, or takes on one of the forms

$$\alpha e^{\lambda t} + \beta e^{\mu t} + \text{const.}, \qquad (\alpha + \beta t)e^{\lambda t} + \text{const.}$$

with the value $\mu = 0$ not excluded.

Having found the various forms for the function u, the equations $F_\nu = 0$ can be very easily found; however, for the sake of brevity we will not write them out in full. Thus in every single case both the form

$$\begin{pmatrix} b_{1,1}, & b_{2,2}, & b_{3,3} \\ b_{2,3}, & b_{3,1}, & b_{1,2} \end{pmatrix}$$

and the adjoint form

$$\begin{pmatrix} \beta_{1,1}, & \beta_{2,2}, & \beta_{3,3} \\ \beta_{2,3}, & \beta_{3,1}, & \beta_{1,2} \end{pmatrix}$$

are determined. If now in the expression $\Sigma\, \beta_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}$, any functions of x_1, x_2, x_3 are substituted in place of the quantities s_1, s_2, s_3, then all the cases in which u is a function of time and two other variables will be clearly obtained. Hence the first question will be solved.

It now remains for us to consider when the expression $\Sigma\, \beta_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}$ can be transformed into the given form $\Sigma\, \alpha_{\iota,\iota'}\, dx_\iota\, dx_{\iota'}$.

Second Part

Concerning the transformation of the expression $\Sigma_1^n\, \iota\, \Sigma_1^n\, \iota'\, b_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}$

into the given form $\sum_{\iota,\iota'} a_{\iota,\iota'}\, dx_\iota\, dx_{\iota'}$:

In the case when the question of the most illustrious Academy is restricted to homogeneous bodies in which the conductivity coefficients are constant, let us

consider the conditions under which the expression $\Sigma\, b_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}$ can be transformed into the form $\Sigma\, a_{\iota,\iota'}\, dx_\iota\, dx_{\iota'}$ with constant coefficients $a_{\iota,\iota'}$ by making the quantities s equal to functions of x. Having done that, let us then briefly consider the transformation to any other form.

Now, the expression $\Sigma\, a_{\iota,\iota'}\, dx_\iota\, dx_{\iota'}$ can always be transformed into $\Sigma\, dx_\iota^2$ provided that it is a positive form in the dx's. So, if $\Sigma\, b_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}$ can be transformed to $\Sigma\, a_{\iota,\iota'}\, dx_\iota\, dx_{\iota'}$, then it can also be transformed to the $\Sigma\, dx_\iota^2$ and vice versa. Therefore let us discover when it can be transformed to $\Sigma\, dx_\iota^2$.

Let the determinant $\Sigma \pm b_{1,1} b_{2,2} \ldots b_{n,n} = B$ and the cofactor $= \beta_{\iota,\iota'}$; then $\Sigma\, \beta_{\iota,\iota'} b_{\iota,\iota'} = B$ and $\Sigma\, \beta_{\iota,\iota'} b_{\iota,\iota''} = 0$, if $\iota \gtrless \iota''$.

If $\Sigma\, b_{\iota,\iota'}\, ds_\iota\, ds_{\iota'} = \Sigma\, dx_\iota^2$ for any values of dx, then by substituting $d + \delta$ for d we can also show that $\Sigma\, b_{\iota,\iota'}\, ds_\iota\, \delta s_{\iota'} = \Sigma\, dx_\iota\, \delta x_\iota$ for any values of dx and δx. Hence, if the quantities ds_ι are expressed in terms of dx_ι and the quantities δx_ι in terms of δs_ι, then it follows that

$$\frac{\partial x_{\nu'}}{\partial s_\nu} = \sum_\iota b_{\nu,\iota} \frac{\partial s_\iota}{\partial x_{\nu'}} \tag{1}$$

and from this

$$\frac{\partial s_\iota}{\partial x_{\nu'}} = \sum_\nu \frac{\beta_{\nu,\iota}}{B} \frac{\partial x_{\nu'}}{\partial s_\nu}. \tag{2}$$

Since

$$\sum_\nu \frac{\partial s_\iota}{\partial x_\nu} \frac{\partial x_\nu}{\partial s_\iota} = 1 \qquad \text{and} \qquad \sum_\nu \frac{\partial s_\iota}{\partial x_\nu} \frac{\partial x_\nu}{\partial s_{\iota'}} = 0 \textit{ if } \iota \gtrless \iota',$$

and using (1) and (2), it can further be established that [7]

$$\sum_\nu \frac{\partial x_\nu}{\partial s_\iota} \frac{\partial x_\nu}{\partial s_{\iota'}} = b_{\iota,\iota'} \tag{3}$$

and

$$\sum_\nu \frac{\partial s_\iota}{\partial x_\nu} \frac{\partial s_{\iota'}}{\partial x_\nu} = \frac{\beta_{\iota,\iota'}}{B}. \tag{4}$$

and by differentiating (3)

$$\sum_\nu \frac{\partial^2 x_\nu}{\partial s_\iota \partial s_{\iota''}} \frac{\partial x_\nu}{\partial s_{\iota'}} + \sum_\nu \frac{\partial^2 x_\nu}{\partial s_{\iota'} \partial s_{\iota''}} \frac{\partial x_\nu}{\partial s_\iota} = \frac{\partial b_{\iota,\iota'}}{\partial s_{\iota''}}.$$

Now having found expressions for each of

$$\frac{\partial b_{\iota,\iota'}}{\partial s_{\iota''}}, \frac{\partial b_{\iota,\iota''}}{\partial s_{\iota'}}, \frac{\partial b_{\iota',\iota''}}{\partial s_\iota}$$

we obtain

$$2\sum_{\nu}\frac{\partial^2 x_\nu}{\partial s_{\iota'}\partial s_{\iota''}}\frac{\partial x_\nu}{\partial s_\iota}=\frac{\partial b_{\iota,\iota'}}{\partial s_{\iota''}}+\frac{\partial b_{\iota,\iota''}}{\partial s_{\iota'}}-\frac{\partial b_{\iota',\iota''}}{\partial s_\iota}, \tag{5}$$

and if the right-hand side of (5) is denoted by $p_{\iota,\iota',\iota''}$, then we find that

$$2\frac{\partial^2 x_\nu}{\partial s_{\iota'}\partial s_{\iota''}}=\sum_{\iota}\frac{\partial s_\iota}{\partial x_\nu}p_{\iota,\iota',\iota''}, \tag{6}$$

Differentiating the quantities $p_{\iota,\iota',\iota''}$ leads to

$$\frac{\partial p_{\iota,\iota',\iota''}}{\partial s_{\iota'''}}-\frac{\partial p_{\iota,\iota',\iota'''}}{\partial s_{\iota''}}=2\sum_{\nu}\frac{\partial^2 x_\nu}{\partial s_{\iota'}\partial s_{\iota''}}\frac{\partial^2 x_\nu}{\partial s_\iota\partial s_{\iota'''}}-2\sum_{\nu}\frac{\partial^2 x_\nu}{\partial s_{\iota'}\partial s_{\iota'''}}\frac{\partial^2 x_\nu}{\partial s_\iota\partial s_{\iota''}}.$$

Finally, substituting the values found in (6) and (4), we find that

$$\begin{aligned}\frac{\partial^2 b_{\iota,\iota''}}{\partial s_{\iota'}\partial s_{\iota'''}}+\frac{\partial^2 b_{\iota',\iota'''}}{\partial s_\iota\partial s_{\iota''}}-\frac{\partial^2 b_{\iota,\iota'''}}{\partial s_{\iota'}\partial s_{\iota''}}-\frac{\partial^2 b_{\iota',\iota''}}{\partial s_\iota\partial s_{\iota'''}}&\\ +\frac{1}{2}\sum_{\nu,\nu'}(p_{\nu,\iota',\iota'''}p_{\nu',\iota,\iota''}-p_{\nu,\iota,\iota'''}p_{\nu',\iota',\iota''})\beta_{\nu,\nu'}/B&=0.\end{aligned} \tag{I}$$

Hence $\Sigma\, b_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}$ can be transformed into the form $\Sigma\, dx_\iota^2$, provided that the functions b satisfy the equations (I). Let us denote the left-hand side of Eqs. (I) by

$$(\iota\iota',\ \iota''\iota''').$$

In order that the nature of Eqs. (I) may be more closely examined, we consider the expression

$$\delta\delta\,\Sigma\, b_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}-2d\delta\,\Sigma\, b_{\iota,\iota'}\, ds_\iota\,\delta s_{\iota'}+dd\,\Sigma\, b_{\iota,\iota'}\,\delta s_\iota\,\delta s_{\iota'}$$

involving the second-order variations d^2, $d\delta$, δ^2 which satisfy

$$\begin{aligned}\delta'\,\Sigma\, b_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}-\delta\,\Sigma\, b_{\iota,\iota'}\, ds_\iota\,\delta' s_{\iota'}-d\,\Sigma\, b_{\iota,\iota'}\,\delta s_\iota\,\delta' s_{\iota'}&=0\\ \delta'\,\Sigma\, b_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}-2d\,\Sigma\, b_{\iota,\iota'}\, ds_\iota\,\delta' s_{\iota'}&=0\\ \delta'\,\Sigma\, b_{\iota,\iota'}\,\delta s_\iota\,\delta' s_{\iota'}-2\delta\,\Sigma\, b_{\iota,\iota'}\,\delta s_\iota\,\delta' s_{\iota'}&=0,\end{aligned}$$

where δ' denotes any variation. With these conditions the expression above becomes

$$=\Sigma\,(\iota\iota',\ \iota''\iota''')(ds_\iota\,\delta s_{\iota'}-ds_{\iota'}\,\delta s_\iota)(ds_{\iota''}\,\delta s_{\iota'''}-ds_{\iota'''}\,\delta s_{\iota''}). \tag{II}$$

Having obtained the expression in this form, it is self-evident that the form of the expression will remain unchanged even when $\Sigma\, b_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}$ is transformed by changing the independent variables. But if the quantities b are constant, all the coefficients in the expression (II) turn out to equal zero. Hence if $\Sigma\, b_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}$ can be transformed into a similar expression with constant coefficients, expression (II) must likewise disappear.

It is clear that if expression (II) does not vanish, then the expression [8]

$$\frac{\frac{1}{2}\sum (\iota\iota',\iota''\iota''')(ds_\iota\,\delta s_{\iota'} - ds_{\iota'}\,\delta s_\iota)(ds_{\iota''}\,\delta s_{\iota'''} - ds_{\iota'''}\,\delta s_{\iota''})}{\sum b_{\iota,\iota'}\,ds_\iota\,ds_{\iota'}\sum b_{\iota,\iota'}\,\delta s_\iota\,\delta s_{\iota'} - (\sum b_{\iota,\iota'}\,ds_\iota\,\delta s_{\iota'})^2} \tag{III}$$

does not alter even when the independent variables are changed and furthermore remains exactly the same if any independent linear expressions $\alpha ds_\iota + \beta\delta s_\iota$, $\gamma ds_\iota + \delta\delta s_\iota$ are substituted in place of ds_ι and δs_ι. However, the maximum and minimum values of the expression (III) involving ds_ι, δs_ι do not depend on the form of $\sum b_{\iota,\iota'}\,ds_\iota\,ds_{\iota'}$ nor on the values of ds_ι, δs_ι. Hence, the maximum and minimum values can be used to ascertain whether two forms can be transformed into one another.

These investigations can be illustrated by a geometrical example which, although unusual, will be a useful addition.

The expression $\sqrt{(\sum b_{\iota,\iota'}\,ds_\iota\,ds_{\iota'})}$ can be regarded as a line element in a more general space of n dimensions extending beyond the bounds of our intuition. But if in this space all possible shortest lines are drawn from the point $(s_1, s_2, \ldots, s_n)$ with first-order variations of s given by $\alpha ds_1 + \beta\delta s_1 : \alpha ds_2 + \beta\delta s_2 : \ldots : \alpha ds_n + \beta\delta s_n$, where α and β denote any quantities, then these lines will form a surface which can be visualized in the space of our common intuition. Thus expression (III) will then be a measure of the curvature of this surface at the point $(s_1, s_2, \ldots, s_n)$.

In the specific case of $n = 3$, the expression (II) involves second-order terms of the form

$$ds_2\,\delta s_3 - ds_3\,\delta s_2,\ ds_3\,\delta s_1 - ds_1\,\delta s_3,\ ds_1\,\delta s_2 - ds_2\,\delta s_1,$$

from which we obtain six equations which the functions b ought to satisfy in order that the form $\sum b_{\iota,\iota'}\,ds_\iota\,ds_{\iota'}$ can be transformed into one constant coefficients. Also it is not difficult to establish, using well-known methods, that these six conditions are sufficient. However, it must be noted that only three are independent.

In order finally to solve the question posed by the most illustrious Academy, we must substitute into these six equations expressions for functions b, found by the method set out above. By this method, we shall find all the cases in which the temperature u in homogeneous bodies is a function of time and two other variables.

But time does not allow us to present these calculations in full. So we must be content to enumerate particular solutions of the question having set out the methods to be used.

For the sake of brevity, we only look at the simplest case in which temperature u varies according to the rule

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} = aa\,\frac{\partial u}{\partial t} \tag{I}$$

and to which we have shown that the remaining cases can easily be reduced: the case $m = 1$ can be solved giving u as constant on parallel straight lines or on

circular spirals, and if rectangular coordinates z, $r \cos \phi$, $r \sin \phi$ are chosen appropriately, then $\alpha = r$, $\beta = z + K\phi$, where K is constant.

The case $m = 2$ can be solved if $u = f(\alpha) + \phi(\beta)$; case $m = 3$, if $u = \alpha e^{\lambda t} + f(\beta)$, where λ denotes a real constant; then case $m = 4$, as we have seen above, if $u = \alpha e^{\lambda t} + \beta e^{\mu t} + \text{const.}$, or $u = (\alpha + \beta t)e^{\lambda t} + \text{const.}$, or $u = f(\alpha)$.

Now in order to find out more about the form of the function u, we need only note that temperature u, if not of the form $\alpha e^{\lambda t}$, can only be a function of time and one other variable when it is constant on parallel planes, or on coaxial cylinders, or on concentric spheres. If u is of the form $\alpha e^{\lambda t}$, then from the differential equation (I), it follows that

$$\frac{\partial^2 \alpha}{\partial x_1^2} + \frac{\partial^2 \alpha}{\partial x_2^2} + \frac{\partial^2 \alpha}{\partial x_3^2} = \lambda a a \alpha$$

and hence by substituting the values of u in the differential equation (I) in the fourth case, the functions α and β can easily be determined provided we note that, in this case, $\alpha e^{\lambda t}$ and $\beta e^{\mu t}$ can be complex conjugate quantities.

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NOTES

1. To give an idea of the value of this prize, P. Guiral in *La Vie Quotidienne en France à l'Age d'Or du Capitalisme 1852–1879* [1976] tells us that the lowest paid office boy at the time was earning no more than 1000 francs per year, an ordinary administrator 1400 francs per year, and senators could be earning between 25,000 and 30,000 francs per year.

2. Our translation of Riemann's epigraph: And from these beginnings a way is opened to greater things. Scholz [1980, 42 footnote 32)] points out that the phrase is one that Gauss appended in a slightly different form to one of his papers on surfaces. In fact in Gauss' paper written in 1822 it appears as "Ab his via sternitur ad maiora" [Gauss 1873, 189].

3. Our translation: "On Heat Conduction" in which he develops the entire apparatus of quadratic differential forms which are now used in relativity theory.

4. In Riemann's original submission to the Académie this was written using incorrect coordinates as $\sum \alpha_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}$.

5. This expression appears as $bd\alpha^2 + 2c'd\alpha d\beta + ad\beta^2$ in the version of the "Commentatio" in Riemann's *Collected Works*; the plus sign in the second term in the printed version is a typographical error.

6. The paper that Riemann submitted to the Académie has for the second of these two equations

$$\frac{\partial f}{\partial \alpha} = 2\frac{\partial d}{\partial \beta}.$$

The coefficient of ∂_β^2 in $(\Delta\Lambda - \Lambda\Delta)$ is

$$\frac{\partial d}{\partial \alpha} - 2\frac{\partial f}{\partial \beta}.$$

Thus the form of the second equation in the Appendix is the correct one. This is also the form appearing in the version of the paper in the *Collected Works*; Riemann's error was originally corrected by Weber.

7. In Eqs. (3)–(5), the equation following (4) and also that between (6) and (I), the indices on the summation signs and also on the ∂x are missing in Riemann's original version. Riemann also omitted these indices in his rough calculations of this part of the paper which are to be found in Cod. Ms. Riemann 9.

8. The sign attached to this expression is Riemann's; it appears with a minus sign in the *Collected Works*.

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