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New oscillation criteria for second order linear difference equations with positive and negative coefficients

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ABSTRACT

The oscillation of second order neutral difference equations with positive and negative coefficients of the form

$$\Delta^2(x_n + \lambda a_n x_{n-\tau}) + p_n x_{n-\delta} - q_n x_{n-\sigma} = 0, \quad \lambda = \pm 1$$

is investigated. We obtain many new results using the comparison between both first order and second order difference equations. An example is given to show the strength of the obtained results.

1. Introduction

The oscillatory properties of difference equations of many types have been the subject of extensive investigations reflecting the importance of this topic in applications (see [1–3]). Among the equations which did not undergo the deserved investigation are the equations with positive and negative coefficients. This, probably, is due to the difficulty associated with the presence of negative and positive coefficients in one equation. Some authors studied the oscillation and/or nonoscillation of first order difference equations of the form

$$\Delta(x_n - a_n x_{n-\tau}) + p_n x_{n-\delta} - q_n x_{n-\sigma} = 0; \tag{1}$$

see [1,3–6] and the references cited therein. For second order difference equations of this type, namely,

$$\Delta^2(x_n + \lambda a_n x_{n-\tau}) + p_n x_{n-\delta} - q_n x_{n-\sigma} = 0, \quad \lambda = \pm 1 \tag{2}$$

it seems that the nonoscillation theory is more developed than the oscillation theory (e.g., [7–9]). At least we know from [8] that (2) is nonoscillatory provided that

$$\sum_{i=n_0}^{\infty} n|p_n| < \infty \quad \text{and} \quad \sum_{i=n_0}^{\infty} n|q_n| < \infty, \tag{3}$$

when either $0 \le a_n \le a < 1$ or $-1 < a < a_n \le 0$ for $n \ge n_0$.

In fact, as far as this author knows, the only known work on the oscillation of (2) is [10].

Our main objective here is to obtain new oscillation criteria for (2) when δ, σ and τ are nonnegative integers. We use the comparison with both first order delay and second order ordinary difference equations. In particular, we benefit from the theory developed in [11] for the oscillation and nonoscillation of second order functional difference equations and the advances of the oscillation theory of the second order ordinary difference equation

$$\Delta^2 u_n + h_n u_{n+1} = 0.$$

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Our results improve some results of [10] (when restricted to Eq. (2)) and provide a new technique for investigating equations with positive and negative coefficients.

By a solution of (2), we mean a sequence $\{x_n\}$ of real numbers that satisfies (2). Each solution of (2) is generated recursively given its initial values x_{-i} for all $i = 0, 1, \ldots, M$ where $M = \max\{\tau+2, \delta+2, \sigma+2\}$. A solution of (2) is said to be oscillatory if it is neither eventually positive nor eventually negative. Eq. (2) is called oscillatory if all its solutions are oscillatory. If each solution of (2) is either oscillatory or has a limit zero at infinity, then Eq. (2) is called almost oscillatory. This type of oscillation naturally exists in functional equations with higher delays. One can test this easily for equations with constant coefficients specially when the corresponding characteristic polynomial is of odd order.

2. Main results

Throughout this section, we assume that the series,

$$\sum^{\infty} \sum_{l=r-\delta+\sigma}^{r-1} q_l \tag{4}$$

converges. Also, we assume that $\delta \geq 1+\sigma$, $a_n, p_n, q_n \geq 0$ and $Q_n = p_n - q_{n-\delta+\sigma} \geq 0$ for $n \geq n_0$ where $Q_n \not\equiv 0$ on $\{s, s+1, s+2, \ldots\}$ for any positive integer s.

Lemma 1. *Assume that $\{x_n\}$ is a solution of (2) such that $x_n > 0$ for $n \geq n_1 \geq n_0$. If $\lambda = 1$ or*

$$\limsup_{n\to\infty} a_n + \sum_{r=n_2}^{n-1} \sum_{l=r-\delta+\sigma}^{r-1} q_l \leq 1, \quad \textit{for } \lambda = -1, \tag{5}$$

then the sequence $\{y_n\}$ defined by

$$y_n = x_n + \lambda a_n x_{n-\tau} - \sum_{r=n_2}^{n-1} \sum_{l=r-\delta+\sigma}^{r-1} q_l x_{l-\sigma}, \quad n \geq n_2 > n_1 + \nu,\ \nu = \max\{\tau, 1+\delta\} \tag{6}$$

satisfies $\Delta y_n > 0$ eventually.

Proof. Notice that,

$$\Delta y_n = \Delta(x_n + \lambda a_n x_{n-\tau}) - \sum_{l=n-\delta+\sigma}^{n-1} q_l x_{l-\sigma}$$

and

$$\begin{aligned} \Delta^2 y_n &= -(p_n - q_{n-\delta+\sigma}) x_{n-\delta} \\ &= -Q_n x_{n-\delta} \leq 0, \quad n \geq n_2 + 1. \end{aligned} \tag{7}$$

Thus Δy_n is either eventually positive or eventually negative. If $n_3 \geq n_2 + 1$ such that $\Delta y_{n_3} < 0$, then (7) implies that $\Delta y_n < \Delta y_{n_3}$ for $n \geq n_3$. Summing from n_3 to ∞, it follows that $\lim_{n\to\infty} y_n = -\infty$ which is possible only if $\limsup_{n\to\infty} x_n = \infty$. Using the same reasoning as in [11, p.118], a positive integer sequence $\{\bar{n}_k\}$, $\bar{n}_k \to \infty$ as $k \to \infty$ exists such that

$$x_{\bar{n}_k} = \max\{x_n : n_2 + \delta \leq n \leq \bar{n}_k\},$$

hence

$$\begin{aligned} y_{\bar{n}_k} &= x_{\bar{n}_k} + \lambda a_{\bar{n}_k} x_{\bar{n}_k-\tau} - \sum_{r=n_2}^{\bar{n}_k-1} \sum_{l=r-\delta+\sigma}^{r-1} q_l x_{l-\sigma} \\ &\geq x_{\bar{n}_k} + \lambda a_{\bar{n}_k} x_{\bar{n}_k-\tau} - x_{\bar{n}_k} \sum_{r=n_2}^{\bar{n}_k-1} \sum_{l=r-\delta+\sigma}^{r-1} q_l \\ &> \begin{cases} \left(1 - a_{\bar{n}_k} - \sum_{r=n_2}^{\bar{n}_k-1} \sum_{l=r-\delta+\sigma}^{r-1} q_l\right) x_{\bar{n}_k}, & \text{if } \lambda = -1 \\ \left(1 - \sum_{r=n_2}^{\bar{n}_k-1} \sum_{l=r-\delta+\sigma}^{r-1} q_l\right) x_{\bar{n}_k} + a_{\bar{n}_k} x_{\bar{n}_k-\tau}, & \text{if } \lambda = 1. \end{cases} \end{aligned}$$

So, if we choose the integer n_2 such that $\sum_{r=n_2}^{\infty}\sum_{l=r-\delta+\sigma}^{r-1} q_l \le 1$ for $\lambda = 1$ and $a_{\bar{n}_k} + \sum_{r=n_2}^{\bar{n}_k-1}\sum_{l=r-\delta+\sigma}^{r-1} q_l < 1$ for $\lambda = -1$, it follows that

$$y_{\bar{n}_k} \ge 0, \quad \text{for all } k, \tag{8}$$

which is a contradiction. Thus Δy_n must be eventually positive. □

Remark 1. Let $x_n > 0$ for $n \ge n_1 \ge n_0$ and $\bar{n} \ge n_1 + \delta$. Define z_n and u_n by

$$z_n = x_n + \lambda a_n x_{n-\tau}, \qquad u_n = z_n - \sum_{r=\bar{n}}^{n-1}\sum_{l=r-\delta+\sigma}^{r-1} q_l x_{l-\sigma}.$$

Then

$$\Delta u_n = \Delta z_n - \sum_{l=n-\delta+\sigma}^{n-1} q_l x_{l-\sigma} = \Delta y_n, \quad \text{for large } n.$$

Thus $\Delta u_n > 0$ as long as $\Delta y_n > 0$. Also, $\Delta z_n \ge \Delta y_n$ which implies that $\Delta z_n > 0$ provided that $\Delta y_n > 0$. Therefore, n_2 can be chosen as large as needed without violating the positivity of Δy_n. So, without further mentioning, we always choose n_2 sufficiently large to suit some eventually satisfied properties of x_n which may be assumed in the proofs of the next results.

Lemma 2. *Assume that* $\{x_n\}$ *and* $\{y_n\}$ *are as in Lemma 1. If* $\lambda = 1$ *or* $\limsup_{n\to\infty} x_n > 0$,

$$\lambda = -1, \quad \textit{and} \quad \limsup_{n\to\infty} a_n + \sum_{r=n_2}^{n-1}\sum_{l=r-\delta+\sigma}^{r-1} q_l < 1. \tag{9}$$

Then $y_n > 0$ *eventually.*

Proof. If $\lambda = 1$, we choose n_2 so large that $\Delta z_n > 0$ for all $n \ge n_2 - \delta$ and $\sum_{r=n_2}^{n-1}\sum_{l=r-\delta+\sigma}^{r-1} q_l \le 1$. Therefore,

$$\begin{aligned} y_n &\ge z_n - \sum_{r=n_2}^{n-1}\sum_{l=r-\delta+\sigma}^{r-1} q_l z_{l-\sigma} \\ &\ge z_n\left(1 - \sum_{r=n_2}^{n-1}\sum_{l=r-\delta+\sigma}^{r-1} q_l\right) \\ &\ge 0, \quad \text{for } n \ge n_2. \end{aligned}$$

If (9) holds and $\limsup_{n\to\infty} x_n = \infty$, then using similar arguments as those employed in the proof of Lemma 1 we can easily prove the eventual positivity of y_n. When $\limsup_{n\to\infty} x_n = S$, then we choose n_2 so large that for any $\epsilon > 0$, we have

$$x_n < S + \epsilon, \quad n \ge n_2 - \delta$$

and

$$a_n + \sum_{r=n_2}^{n-1}\sum_{l=r-\delta+\sigma}^{r-1} q_l \le \beta < 1, \quad \text{for all } n > n_2.$$

Hence,

$$\begin{aligned} y_n &> x_n - (S+\epsilon)\left(a_n + \sum_{r=n_2}^{n-1}\sum_{l=r-\delta+\sigma}^{r-1} q_l\right) \\ &\ge x_n - (S+\epsilon)\beta, \quad n > n_2. \end{aligned}$$

Let $\{n_k\}$ be such that $n_k \to \infty$ as $k \to \infty$ and $\lim_{k\to\infty} x_{n_k} = S$. Then

$$y_{n_k} > x_{n_k} - (S+\epsilon)\beta \quad \text{and} \quad \lim_{k\to\infty} y_{n_k} \ge S - (S+\epsilon)\beta,$$

which implies that $\lim_{k\to\infty} y_{n_k} \ge (1-\beta)S > 0$. Combining this with the increasing nature of y_n, we easily conclude that $y_n > 0$ eventually. This completes the proof. □

Lemma 3. *Assume that $\{x_n\}$ is a solution of (2) such that $x_n > 0$ for all $n \geq n_1 \geq n_0$ and either $a_n = 0$ for infinitely many values of n or $a_n > 0$ for $n \geq n_1 \geq n_0$, $\tau < \delta$, $\sum_{n-\delta+\tau}^{n-1} \frac{Q_j}{a_{j+\tau-\delta}} > 0$ for $n \geq n_1$, (5) holds and the difference equation*

$$\Delta v_n + \frac{Q_n}{a_{n+\tau-\delta}} v_{n+\tau-\delta} = 0 \tag{10}$$

is oscillatory. Then $x_n > a_n x_{n-\tau}$ eventually.

Proof. Since Lemma 1 holds for $\lambda = -1$, then Remark 1 implies that $\Delta z_n > 0$ eventually and hence z_n is either eventually negative or eventually positive. If $a_n = 0$ for infinitely many values of n, then $z_n = x_n$ for those values and hence z_n is eventually positive. When $a_n > 0$ for $n \geq n_1 \geq n_0$ we assume for the sake of contradiction that $z_n < 0$ eventually. It follows that

$$\Delta y_n < \Delta z_n < -z_n < a_n x_{n-\tau},$$

for sufficiently large n. Substituting into (7), we obtain that Δy_n is an eventually positive solution of the inequality

$$\Delta(\Delta y_n) + \frac{Q_n}{a_{n+\tau-\delta}} \Delta y_{n+\tau-\delta} \leq 0. \tag{11}$$

Thus (10) is nonoscillatory according to [3, Corollary 7.6.1], which is an apparent contradiction. Then z_n is also eventually positive in this case.

The following result can be proved using similar reasoning as above.

Lemma 4. *Assume that (5) holds, $\sum_{n-\delta}^{n-1} Q_j > 0$ eventually, and the difference equation*

$$\Delta v_n + Q_n v_{n-\delta} = 0, \tag{12}$$

is oscillatory. Then $x_n < a_n x_{n-\tau}$ eventually.

Next, we define $\{h_n^i(q)\}_{n\geq n_0}$ for $i = 1, 2, \ldots$ as follows

$$h_n^1(q) = 1 + a_n + \sum_{l=n-1-\delta+\sigma}^{n-2} q_l,$$

and

$$h_n^{i+1}(q) = 1 + a_n h_{n-\tau}^i(q) + \sum_{l=n-1-\delta+\sigma}^{n-2} h_{l-\delta}^i(q) q_l, \quad \text{for } i = 1, 2, \ldots. \quad \square$$

Theorem 1. *Assume that (9) holds and there exists a positive integer i such that the second order ordinary difference equation*

$$\Delta^2 x_n + Q_n h_{n-\delta}^i(q) x_{n+1} = 0, \tag{13}$$

is oscillatory. Then Eq. (2) is almost oscillatory.

Proof. Let $\{x_n\}$ be a solution of (2) which is not almost oscillatory. Due to the linearity of (2), without loss of generality, we assume that $x_n > 0$ for $n \geq n_1 \geq n_0$. Then Lemmas 1 and 2 imply the existence of an integer $n_3 \geq n_2$ such that

$$y_n > 0 \quad \text{for all } n \geq n_3. \tag{14}$$

Since

$$\begin{aligned} x_n &= y_n + a_n x_{n-\tau} + \sum_{r=n_2}^{n-1} \sum_{l=r-\delta+\sigma}^{r-1} q_l x_{l-\sigma} \\ &> y_n + a_n x_{n-\tau} + \sum_{l=n-1-\delta+\sigma}^{n-2} q_l x_{l-\sigma}, \quad n \geq n_2 + 1. \end{aligned}$$

Then

$$\begin{aligned} x_n &> y_n + a_n y_{n-\tau} + \sum_{l=n-1-\delta+\sigma}^{n-2} q_l y_{l-\sigma} \\ &> y_n + a_n y_{n-\tau} + y_{n-1-\delta} \sum_{l=n-1-\delta+\sigma}^{n-2} q_l \\ &\geq h_n^1(q) y_{n-\nu}, \quad n \geq n_3 + \nu, \end{aligned}$$

where $\nu = \max\{\tau, 1+\delta\}$. Using induction, it is easy to see that

$$x_n > h_n^i(q)y_{n-i\nu}, \quad \text{for all } n > n_3 + i\nu,\ i = 1, 2, \ldots .$$

Since $\Delta^2 y_n + Q_n x_{n-\delta} = 0$, then

$$\Delta^2 y_n + Q_n h_{n-\delta}^i(q) y_{n-\delta-i\nu} \le 0, \quad n \ge n_3 + \delta + i\nu.$$

This implies that (see [11, Lemma 2])

$$\Delta^2 v_n + Q_n h_{n-\delta}^i(q) v_{n-\delta-i\nu} = 0,$$

is nonoscillatory. According to [11, Corollary 8], we obtain that (13) is nonoscillatory which contradicts the assumptions. Then (2) is almost oscillatory as desired. □

We think that more restrictions on the coefficients may help obtaining results on the oscillation rather than almost oscillation. The first one of these results can be obtained by combining Lemmas 3 and 4. This depends on the fact that, when $a_n > 0$ for $n \ge n_1$, the oscillation of (12) implies the oscillation of (10) since $\frac{Q_n}{a_{n+\tau-\delta}} \ge Q_n$, $n \ge n_1$.

Theorem 2. *Assume that (5) holds and either $a_n = 0$ for infinitely many values of n or $\tau < \delta$, $a_n > 0$ and $\sum_{n-\delta}^{n-1} Q_i > 0$ for $n \ge n_1 \ge n_0$. If Eq. (12) is oscillatory, then Eq. (2) is oscillatory.*

Next, the above oscillation criterion is improved when $\tau < \delta$, $\lambda = -1$.

Theorem 3. *Assume that all assumptions of Lemma 3 are satisfied. If there exists a positive integer i such that*

$$\Delta^2 x_n + Q_n h_{n-\delta}^i(0) x_{n+1} = 0, \quad (h_n^i(0) = h_n^i(q)|_{q_n \equiv 0}) \tag{15}$$

is oscillatory, then Eq. (2) is oscillatory.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of Eq. (2). From Lemma 3, we know that $z_n = x_n - a_n x_{n-\tau} > 0$ for all $n \ge \tilde{n}$ (for some $\tilde{n} \ge n_1$). Since $x_n = z_n + a_n x_{n-\tau}$ for $n \ge \tilde{n}$, then

$$x_n > z_n + a_n z_{n-\tau} > (1 + a_n) z_{n-\tau} = h_n^1(0) z_{n-\tau}, \quad n \ge \tilde{n} + \tau.$$

Reusing this inequality, we get

$$x_n > z_n + a_n h_{n-\tau}^1(0) z_{n-2\tau} > h_n^2(0) z_{n-2\tau}, \quad n \ge \tilde{n} + 2\tau.$$

So an induction yields

$$x_n > h_n^i(0) z_{n-i\tau}, \quad n \ge \tilde{n} + i\tau.$$

Substituting into (7), we get

$$\Delta^2 y_n + Q_n h_{n-\delta}^i(0) z_{n-\delta-i\tau} \le 0, \quad n \ge \tilde{n} + i\tau = N_i.$$

Summing from $k \ge N_i$ to $n-1$, then

$$\Delta y_n - \Delta y_k + \sum_{j=k}^{n-1} Q_j h_{j-\delta}^i(0) z_{j-\delta-i\tau} \le 0, \quad n \ge N_i.$$

Since $\Delta y_n > 0$ for $n \ge N_i$, then as $n \to \infty$, we obtain

$$-\Delta y_n + \sum_{j=n}^{\infty} Q_j h_{j-\delta}^i(0) z_{j-\delta-i\tau} \le 0, \quad n \ge N_i.$$

But $\Delta y_n \le \Delta z_n$ for all $n \ge N_i$,

$$-\Delta z_n + \sum_{j=n}^{\infty} Q_j h_{j-\delta}^i(0) z_{j-\delta-i\tau} \le 0, \quad n \ge N_i.$$

Summing from N_i to $n-1$, it follows that

$$z_{N_i} + \sum_{l=N_i}^{n-1} \sum_{j=l}^{\infty} Q_j h_{j-\delta}^i(0) z_{j-\delta-i\tau} \le z_n, \quad n \ge N_i.$$

Now [11, Theorem 3] implies that Eq. (15) is nonoscillatory. This contradiction completes the proof. □

Remark 2. Corollary 1 in [12] relates the oscillation of the special case of Eq. (2); namely,

$$\Delta^2(x_n - ax_{n-\tau}) + p_n y_{n-\sigma} = 0,$$

to the oscillation of the second order ordinary difference equation

$$\Delta^2 x_n + \mu p_n \left(\frac{n-\delta}{n+1}\right) x_{n+1} = 0, \tag{16}$$

where $\mu \in (0, 1)$. Although this result and Theorem 3 are not generally compared, it is easy to see that Eq. (15) is more efficient in application than (16) since $\mu p_n(\frac{n-\delta}{n+1}) < h^i_{n-\delta}(0)$ (according to the celebrated Sturm comparison theorem [11, Theorem 4]).

Theorem 4. *Assume that* $\lambda = 1$ *and* $a_n \in [0, 1]$ *for all* $n \geq n_0$. *If the equation*

$$\Delta^2 x_n + (1 - a_{n-\delta}) Q_n x_{n+1} = 0, \quad n \geq n_0 \tag{17}$$

is oscillatory, then Eq. (2) *is oscillatory.*

Proof. As usual we assume that $\{x_n\}$ is a solution of Eq. (2) such that $x_n > 0$ for $n \geq n_1 \geq n_0$. If $\{y_n\}$ and $\{z_n\}$ are defined as before, it follows from Lemmas 1 and 2 that

$$\begin{aligned}\Delta^2 y_n &= -Q_n x_{n-\delta} = -Q_n(z_{n-\delta} - a_{n-\delta} x_{n-\tau-\delta}) \\ &\leq -Q_n z_{n-\delta} + Q_n a_{n-\delta} z_{n-\tau-\delta} \\ &\leq -Q_n z_{n-\delta} + Q_n a_{n-\delta} z_{n-\delta} \\ &= -Q_n(1 - a_{n-\delta}) z_{n-\delta}, \quad n \geq n_2 + \delta + 1.\end{aligned}$$

Since $y_n \leq z_n$ for all $n \geq \bar{\bar{n}}$ (for some $\bar{\bar{n}} \geq n_2 + \delta + 1$), the above inequality yields

$$\Delta^2 y_n + (1 - a_{n-\delta}) Q_n y_{n-\delta} \leq 0, \quad n \geq \bar{\bar{n}}.$$

As in the previous proofs, the above inequality implies that (17) is not oscillatory. This completes the proof. □

Theorem 5. *Assume that* $\lambda = 1$ *and either,*

$$Q_n \geq a_{n-\delta+\tau}, \tag{18}$$

or

$$Q_n \geq q_{n-\delta+\sigma}, \quad \text{and} \quad p_n \geq a_{n-\delta+\tau}. \tag{19}$$

for all $n \geq n_0$. *If there exists* $\epsilon_0 > 0$ *such that the second order equation*

$$\Delta^2 x_n + \frac{Q_n}{1+\epsilon_0} x_{n+1} = 0, \tag{20}$$

is oscillatory, then Eq. (2) *is oscillatory.*

Proof. Assume that $\{x_n\}$ is a solution of (2) such that $x_n > 0$ for all $n \geq n_1 \geq n_0$. From Eq. (7) and Lemma 1, we see that

$$\sum_{n_1+\delta}^{\infty} Q_i x_{i-\delta} < \infty.$$

If (18) is satisfied, we obtain

$$\sum_{n_1+\delta}^{\infty} a_{i-\delta+\tau} x_{i-\delta} < \infty.$$

This implies that $\lim_{n\to\infty} a_n x_{n-\tau} = 0$. Consequently, $\lim_{n\to\infty} z_n - x_n = 0$ which implies that $\liminf_{n\to\infty} x_n > 0$ and hence,

$$\lim_{n\to\infty} \frac{z_n}{x_n} = 1.$$

So, for any $\epsilon \in (0, \epsilon_0)$, there exists an integer $n_\epsilon \geq n_1 + \delta$ such that

$$x_n > \frac{z_n}{1+\epsilon} \geq \frac{y_n}{1+\epsilon} \quad n \geq n_\epsilon,$$

where, according to Lemma 2, n_ϵ can be chosen so large that $y_n > 0$ for all $n \geq n_\epsilon$. Thus Eq. (7) and the above inequality yield

$$\Delta^2 y_n + \frac{Q_n}{1+\epsilon} y_{n-\delta} < 0, \quad n \geq n_\epsilon + \delta.$$

Since $\epsilon < \epsilon_0$, it follows that y_n satisfies the inequality

$$\Delta^2 y_n + \frac{Q_n}{1+\epsilon_0} y_{n-\delta} < 0, \quad n \geq n_\epsilon + \delta$$

which as in the previous proofs implies that (20) is nonoscillatory. This contradiction proves the theorem when (18) holds. If (19) is satisfied, the proof can be completed similarly. □

3. Applications

Several known first order and second order oscillation results can be used to extract interesting oscillation criteria from our theorems of Section 2. Of course, it is impossible to apply all known results here. So we select the following results from the literature on the equations

$$\Delta x_n + p_n x_{n-k} = 0, \tag{21}$$

and

$$\Delta^2 x_n + c_n x_{n+1} = 0, \tag{22}$$

where $\{p_n\}$ and $\{c_n\}$ are nonnegative real sequences and k is a positive integer.

Eq. (21) is oscillatory if

$$\alpha := \liminf_{n\to\infty} \left(\frac{1}{k} \sum_{i=n-k}^{n-1} p_i \right) > \frac{k^k}{(k+1)^{k+1}}, \quad \text{[3, Theorem 7.5.1]}$$

or

$$\alpha \leq \frac{k^k}{(k+1)^{k+1}} \quad \text{and} \quad \limsup_{n\to\infty} \left(\frac{1}{k} \sum_{i=n-k}^{n-1} p_i \right) > 1 - \frac{\alpha^2}{2(2-\alpha)}. \quad \text{[13]}$$

Eq. (22) is oscillatory if

$$\sum_{i=n_0}^{\infty} c_i = \infty, \quad \text{[14]}$$

or

$$\sum_{i=n_0}^{\infty} c_i < \infty \quad \text{and} \quad \liminf_{n\to\infty} n \sum_{i=n+1}^{\infty} c_i > \frac{1}{4}. \quad \text{[15,16]}$$

More oscillation criteria for (21) can be found in [17,18] while for Eq. (22) the reader is referred to [1,2,15,16,19–22].

The following result deals with almost oscillation.

Theorem 6. *Assume that (9) holds. Then Eq. (2) is almost oscillatory if there exists a positive integer i such that either one of the following conditions is satisfied:*

$$\sum_{j=n_0}^{\infty} Q_j h^i_{j-\delta}(q) = \infty, \tag{23}$$

or

$$\sum_{j=n_0}^{\infty} Q_j h^i_{j-\delta}(q) < \infty \quad \textit{and} \quad \liminf_{n\to\infty} n \sum_{j=n+1}^{\infty} Q_j h^i_{j-\delta}(q) > \frac{1}{4}. \tag{24}$$

Theorem 2 implies the following result.

Theorem 7. *Assume that Eq. (5) holds and either $a_n = 0$ for infinitely many values of n or $\tau < \delta$, $a_n > 0$ for $n \geq n_1 \geq n_0$. If either*

$$\alpha := \liminf_{n\to\infty} \left(\frac{1}{\delta} \sum_{i=n-\delta}^{n-1} Q_i \right) > \frac{\delta^\delta}{(\delta+1)^{\delta+1}},$$

or

$$\alpha \leq \frac{\delta^\delta}{(\delta+1)^{\delta+1}} \quad \text{and} \quad \limsup_{n\to\infty} \left(\frac{1}{\delta} \sum_{i=n-\delta}^{n-1} Q_i \right) > 1 - \frac{\alpha^2}{2(2-\alpha)},$$

are satisfied, then Eq. (2) is oscillatory.

As an improvement to the above criteria, we obtain the following result which is an application of Theorem 3.

Theorem 8. *Assume that (5) holds and either $a_n = 0$ for infinitely many values of n or $\tau < \delta$, $a_n > 0$ for $n \geq n_1 \geq n_0$ and either*

$$\alpha := \liminf_{n\to\infty} \left(\frac{1}{\delta-\tau} \sum_{j=n-\delta+\tau}^{n-1} \frac{Q_j}{a_{j+\tau-\delta}} \right) > \frac{(\delta-\tau)^{\delta-\tau}}{(\delta-\tau+1)^{\delta-\tau+1}},$$

or

$$\alpha \leq \frac{(\delta-\tau)^{\delta-\tau}}{(\delta-\tau+1)^{\delta-\tau+1}} \quad \text{and} \quad \limsup_{n\to\infty} \left(\frac{1}{\delta-\tau} \sum_{j=n-\delta+\tau}^{n-1} \frac{Q_j}{a_{j+\tau-\delta}} \right) > 1 - \frac{\alpha^2}{2(2-\alpha)},$$

are satisfied. Assume, further, that either (23) or (24) holds when $q = 0$. Then Eq. (2) is oscillatory.

An application of Theorem 4 leads to the following result.

Theorem 9. *Assume that $\lambda = 1$ and $a_n \in [0, 1]$ for all $n \geq n_0$. If either*

$$\sum_{i=n_0}^{\infty} (1 - a_{j-\delta}) Q_j = \infty, \tag{25}$$

or

$$\sum_{i=n_0}^{\infty} (1 - a_{j-\delta}) Q_j < \infty \quad \text{and} \quad \liminf_{n\to\infty} n \sum_{i=n+1}^{\infty} (1 - a_{j-\delta}) Q_j > \frac{1}{4}, \tag{26}$$

then Eq. (2) is oscillatory.

The following result is an application to Theorem 5.

Theorem 10. *Assume that $\lambda = 1$ and either (18) or (19) holds. If either*

$$\sum_{i=n_0}^{\infty} Q_j = \infty, \tag{27}$$

or

$$\sum_{i=n_0}^{\infty} Q_j < \infty \quad \text{and} \quad \liminf_{n\to\infty} n \sum_{i=n+1}^{\infty} Q_j > \frac{1}{4}, \tag{28}$$

then Eq. (2) is oscillatory.

We refer the reader here to the fact that the second part of (28) implies the existence of a sufficiently small $\epsilon_0 > 0$ such that $\liminf_{n\to\infty} n \sum_{n+1}^{\infty} Q_j > \frac{1}{4}(1 + \epsilon_0)$.

As an illustrative example, we consider the difference equation

$$\Delta^2 \left(x_n + \frac{\lambda}{n^2} x_{n-\tau} \right) + \frac{2}{n^\beta} x_{n-\delta} - \frac{1}{n^5} x_{n-\sigma} = 0, \tag{29}$$

where $\beta \in R$, τ, δ, σ are positive integers satisfying that $\delta \geq \sigma + 1$, $\delta > \tau$. Since

$$\frac{Q_j}{a_{j+\tau-\delta}} = (j + \tau - \delta)^2 \left[\frac{2}{j^\beta} - \frac{1}{(j - \delta + \sigma)^5} \right].$$

Then

$$\lim_{n\to\infty} \frac{Q_n}{a_{n+\tau-\delta}} \geq 2,$$

for $\beta \leq 2$. Therefore, the constant α of Theorem 8 satisfies that

$$\alpha = 2 > \frac{(\delta-\tau)^{\delta-\tau}}{(\delta-\tau+1)^{\delta-\tau+1}}.$$

Also; since $h_n^i(q) > 1$ then

$$\sum^{\infty} Q_j h_{j-\delta}^i(q) = \infty, \quad \text{if } \beta \leq 1.$$

For $\beta \in (1, 2]$,

$$\liminf_{n\to\infty} n \sum_{j=n+1}^{\infty} Q_j h_{j-\delta}^i(q) > \liminf_{n\to\infty} n \sum_{j=n+1}^{\infty} Q_j.$$

But $\lim_{n\to\infty} n^2 Q_n \geq 2$. Then for all $\epsilon \in (0, 1]$ there exists $n_\epsilon > 0$ such that $Q_n \geq \frac{2-\epsilon}{n^2}$ for $n \geq n_\epsilon$, and hence

$$\sum_{j=n+1}^{\infty} Q_j > \frac{2-\epsilon}{n+1}.$$

Thus

$$\liminf_{n\to\infty} n \sum_{j=n+1}^{\infty} Q_j h_{j-\delta}^i(q) > \liminf_{n\to\infty} \frac{n(2-\epsilon)}{n+1} = 2-\epsilon > \frac{1}{4}.$$

Then all solutions of Eq. (29) oscillate when $\lambda = -1$ and $\beta \leq 2$ according Theorem 8.

For $\lambda = 1$, we drop the condition $\delta > \tau$. If $n_0 > 2+\delta$, then $(1-a_{n-\delta})Q_n > \frac{3}{4}Q_n$ for $n \geq n_0$. Thus (25) holds as long as $\beta \leq 1$. For $\beta \in (1, 2]$, we have

$$\sum_{j=n_0}^{\infty}(1-a_{j-\delta})Q_j < \sum_{j=n_0}^{\infty} Q_j < \sum_{j=n_0}^{\infty} \frac{2}{j^\beta} < \infty.$$

Let $n_\epsilon > n_0$, then similar arguments as before yield,

$$\begin{aligned}\liminf_{n\to\infty} n \sum_{j=n+1}^{\infty} (1-a_{j-\delta})Q_j &> \liminf_{n\to\infty} n \sum_{j=n+1}^{\infty} \frac{3(2-\epsilon)}{4j^2} \\ &= \frac{3}{4}(2-\epsilon) > \frac{1}{4}.\end{aligned}$$

Thus all solutions of Eq. (29) oscillates according to Theorem 9 when $\beta \leq 2$.

It is important to refer to the fact that Eq. (29) is nonoscillatory when $\beta > 2$ according to [8] since

$$\sum_{n=n_0}^{\infty} n|p_n| = \sum_{n=n_0}^{\infty} \frac{2}{n^{\beta-1}} < \infty, \quad \text{and} \quad \sum_{n=n_0}^{\infty} n|q_n| = \sum_{n=n_0}^{\infty} \frac{2}{n^4} < \infty,$$

in this case and $0 < a_n = \frac{1}{n^2} < \frac{1}{(2+\delta)^2} < 1$ for $n \geq n_0$. Thus Eq. (29) oscillates if and only if $\beta \leq 2$.

This example suggests some kind of sharpness of our results when (4) holds. This author believes that there exists $P_n \geq 0$ (related to p_n and q_n) such that (2) is nonoscillatory provided that the second order ordinary difference equation

$$\Delta^2 x_n + P_n x_{n+1} = 0,$$

is nonoscillatory. We also believe that Theorem 4 is true if (17) is replaced by the equation

$$\Delta^2 x_n + \frac{Q_n}{1+a_n} x_{n+1} = 0.$$

Finally, it will be very interesting if one could prove Theorem 5 for $\epsilon_0 = 0$.

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