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Cycles in 2-connected graphs

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Abstract

Let $\mathscr{G}_n$ be a class of graphs on n vertices. For an integer c, let $ex(\mathscr{G}_n, c)$ be the smallest integer such that if G is a graph in $\mathscr{G}_n$ with more than $ex(\mathscr{G}_n, c)$ edges, then G contains a cycle of length more than c. A classical result of Erdös and Gallai is that if $\mathscr{G}_n$ is the class of all simple graphs on n vertices, then $ex(\mathscr{G}_n, c) = \frac{c}{2}(n-1)$. The result is best possible when $n-1$ is divisible by $c-1$, in view of the graph consisting of copies of K_c all having exactly one vertex in common. Woodall improved the result by giving best possible bounds for the remaining cases when $n-1$ is not divisible by $c-1$, and conjectured that if $\mathscr{G}_n$ is the class of all 2-connected simple graphs on n vertices, then

$$ex(\mathscr{G}_n, c) = \max\{f(n, 2, c),\ f(n, \lfloor c/2 \rfloor, c)\},$$

where $f(n,t,c) = \binom{c+1-t}{2} + t(n-c-1+t)$, $2 \leqslant t \leqslant c/2$, is the number of edges in the graph obtained from K_{c+1-t} by adding $n-(c+1-t)$ isolated vertices each joined to the same t vertices of K_{c+1-t}. By using a result of Woodall together with an edge-switching technique, we confirm Woodall's conjecture in this paper.

Keywords: Cycles; 2-connected graphs; Extremal graphs

1. Introduction

The graphs considered here are finite, undirected, and simple (no loops or parallel edges). The sets of vertices and edges of a graph G are denoted by $V(G)$ and $E(G)$, respectively.

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A classical result of Erdös and Gallai [2, Theorem 2.7] is that for an integer $c \geqslant 2$, if G is a graph on n vertices with more than $\frac{c}{2}(n-1)$ edges, then G contains a cycle of length more than c. The result is best possible when $n-1$ is divisible by $c-1$, in view of the graph consisting of copies of K_c all having exactly one vertex in common. However, when $n-1$ is not divisible by $c-1$, the bound $\frac{c}{2}(n-1)$ can be decreased. The first improvement was obtained by Woodall [4] for the case when $c \geqslant \frac{n+3}{2}$, and later Woodall [5] completed all the rest cases by proving that if $c \geqslant 2$, and $n = t(c-1)+p+1$ where $t \geqslant 0$ and $0 \leqslant p < c-1$, and G is a graph on n vertices with more than $t\binom{c}{2} + \binom{p+1}{2}$ edges, then G contains a cycle of length more than c. This result is best possible, in view of the graph consisting of t copies of K_c and one copy of K_{p+1}, all having exactly one vertex in common. Caccetta and Vijayan [1] gave an alternative proof of the result, and in addition, characterize the structure of the extremal graphs. We note that all the extremal graphs here are not 2-connected. What is the maximum number of edges a 2-connected graph can have without cycles of length more than c? For $2 \leqslant t \leqslant c/2$, define

$$f(n,t,c) = \binom{c+1-t}{2} + t(n-c-1+t),$$

which is the number of edges in the 2-connected graph obtained from K_{c+1-t} by adding $n-(c+1-t)$ isolated vertices each joined to the same t vertices of K_{c+1-t}. Woodall [5] proposed the following conjecture.

Conjecture 1.1. *If $2 \leqslant c \leqslant n-1$, and G is a 2-connected graph on n vertices with more than*

$$\max\{f(n,2,c),\ f(n,\lfloor c/2 \rfloor,c)\}$$

edges, then G contains a cycle of length more than c.

Toward to a proof of the conjecture, Woodall [5] obtained the following result.

Theorem 1.2 (*Woodall [5]*). *If $2 \leqslant c \leqslant \frac{2n+2}{3}$, and G is a 2-connected graph on n vertices with more than $f(n,\lfloor c/2 \rfloor,c)$ edges, then G contains a cycle of length more than c.*

By using this result and an edge-switching technique, we confirm Conjecture 1.1 by Theorem 3.1 in Section 3. Woodall [5] also conjectured that if, furthermore, G has minimum degree k, then the right bound should be $\max\{f(n,k,c),\ f(n,\lfloor c/2 \rfloor,c)\}$ (this conjecture is still open).

Throughout this paper, for $x, y \in V(G)$, xy denotes the edge with ends x and y. If $xy \in E(G)$, we say that y is a *neighbor* of x, or y is *joined* to x. Let H be a subgraph of G, $N_H(x)$ is the set of the neighbors of x which are in H, and $d_H(x) = |N_H(x)|$ is the *degree* of x in H. When no confusion can occur, we shall write $N(x)$ and $d(x)$, instead of $N_G(x)$ and $d_G(x)$. $G-H$ denotes the graph obtained from G by deleting all the vertices of H together with all the edges with at least one end in H, while for $F \subseteq E(G)$, $G \setminus F$ denotes the graph obtained from G by deleting all the edges of F. If $xy \notin E(G)$, $G + xy$ is the graph obtained from G by adding the new edge xy. For subgraphs F and H, $E(F,H)$

denotes the set, and $e(F, H)$ the number, of edges with one end in F and the other end in H. Define $N_H(F) = \cup_{x \in F} N_H(x)$. For simplicity, we write $E(F)$ and $e(F)$ for $E(F, F)$ and $e(F, F)$, respectively. In particular, $e(G) = |E(G)|$. Let $S \subseteq V(G)$. S is a *cut set*, and a *cut vertex* when $|S| = 1$, of G if $G - S$ has more components than G. S is an *independent set* if $E(S) = \emptyset$. A subgraph H is *induced* by S if $V(H) = S$ and $xy \in E(H)$ if and only if $xy \in E(G)$.

Let $C = a_1 a_2 \cdots a_c$ be a cycle. We assume that C has an orientation which is consistent with the increasing order of the indices of a_i, $1 \leqslant i \leqslant c - 1$, and the edge $a_c a_1$ is from a_c to a_1. For $a \in V(C)$, define a^- and a^+ to be the vertices on C immediately before and after a, respectively, according to the orientation of C, and $a^{--} = (a^-)^-$ and $a^{++} = (a^+)^+$. Thus, if $a = a_i$, then $a^- = a_{i-1}$ and $a^+ = a_{i+1}$, where $a_0 = a_c$ and $a_{c+1} = a_1$.

2. Local structure and edge-switching

Definition 2.1. Let C be a cycle in a graph G. We say that C is *locally maximal* if there is no cycle C' in G such that $|E(C')| > |E(C)|$ and $|E(C') \cap E(C, G - C)| \leqslant 2$.

Definition 2.2. Let xy be an edge in a graph G and let $A \subseteq N(y) \setminus (N(x) \cup \{x\})$. The *edge-switching* graph of G *with respect to* A (from y to x), denoted by $G[y \to x; A]$, is the graph obtained from G by deleting all the edges $yz,\ z \in A$ and adding all the edges $xz,\ z \in A$. In notation,

$$G[y \to x; A] = (G \setminus \{yz : z \in A\}) \cup \{xz : z \in A\}.$$

When $A = N(y) \setminus (N(x) \cup \{x\})$, the above definition is identical with the one in [3].

Lemma 2.3. *Let C be a locally maximal cycle in a 2-connected graph G and R a component of $G - C$. Suppose that $x, x' \in N_C(R)$ with $x \neq x'$ and $y \in N_R(x)$.*

(i) *Let $Z = N_R(y) \setminus (N_R(x) \cup \{x\})$. Then C remains a locally maximal cycle in $G[y \to x; Z]$.*

(ii) *If D is a subgraph of R such that $N_{R-D}(D) = \{y\}$ and $N_D(y) \cap N_D(x) = \emptyset$, then, for $A = N_D(y)$, C remains a locally maximal cycle in $G[y \to x; A]$, and furthermore, if $N_C(R - D) = \{x\}$, then C also remains a locally maximal cycle in $G[y \to x; A] + yx'$.*

Proof. (i) Let $Z = \{z_1, z_2, \ldots, z_k\}$ and $F = \{xz_i : 1 \leqslant i \leqslant k\}$, and so

$$G[y \to x; Z] = (G \setminus \{yz_i : 1 \leqslant i \leqslant k\}) \cup F.$$

If C is not a locally maximal cycle in $G[y \to x; Z]$, then there is a cycle C' in $G[y \to x; Z]$ with $|E(C')| > |E(C)|$ and

$$|E(C') \cap E(C, G[y \to x; Z] - C)| \leqslant 2. \tag{2.1}$$

Let $C' = a_1 a_2 \cdots a_p$. Since C is locally maximal in G, and by (2.1), we have that $1 \leqslant |E(C') \cap F| \leqslant 2$.

Case 1. $|E(C') \cap F| = 1$. Suppose that $x = a_t$ and $a_{t+1} \in \{z_1, z_2, \ldots z_k\}$. If $y \notin V(C')$, replacing xa_{t+1} by xya_{t+1}, we obtain that $C'' = a_1 \cdots xya_{t+1} \cdots a_p$. If $y \in V(C')$, say $y = a_s$ and we may assume that $s > t + 1$, by (2.1), it must be that $a_{s-1} \in V(R)$, and so by the construction of $G[y \to x; Z]$, we have that $a_{s-1} \in N(x)$ in G. Then, let $C'' = a_1 \cdots xa_{s-1}a_{s-2} \cdots a_{t+1}ya_{s+1} \cdots a_p$. In either case, C'' is a cycle contradicting the local maximality of C.

Case 2. $|E(C') \cap F| = 2$. Suppose that $x = a_t$ and so $a_{t-1},\ a_{t+1} \in \{z_1, z_2, \ldots, z_k\}$. If $y \notin V(C')$, let

$$C'' = (C' \setminus \{xa_{t-1}, xa_{t+1}\}) \cup \{ya_{t-1}, ya_{t+1}\}.$$

If $y \in V(C')$, say $y = a_s$ and we may assume that $s > t + 1$, by (2.1), $a_{s-1}, a_{s+1} \in V(R)$, and so $a_{s-1}, a_{s+1} \in N(x)$ in G. Then, let

$$C'' = (C' \setminus \{xa_{t-1}, xa_{t+1}, ya_{s-1}, ya_{s+1}\}) \cup \{xa_{s-1}, xa_{s+1}, ya_{t-1}, ya_{t+1}\}.$$

In either case, C'' is a cycle contradicting the local maximality of C.

(ii) Since $N_D(y) \cap N_D(x) = \emptyset$ and $A = N_D(y)$, using $N_{R-D}(D) = \{y\}$, the same proof as in (i) (with Z replaced by A) yields that C is a locally maximal cycle in $G[y \to x; A]$. (In fact, in this case, if C' is a cycle with $|E(C') \cap E(C, G[y \to x; A] - C)| \leqslant 2$ and $E(C') \cap \{xz : z \in A\} \neq \emptyset$, then $y \notin V(C')$.) Furthermore, if $N_C(R - D) = \{x\}$, let $G^* = G[y \to x; A] + yx'$. If C is not a locally maximal cycle in G^*, then there is a cycle C^* in G^* with $|E(C^*)| > |E(C)|$ and

$$|E(C^*) \cap E(C, G^* - C)| \leqslant 2$$

and moreover, $yx' \in E(C^*)$, which implies that $V(C^*) \cap V(D) = \emptyset$. Furthermore, since $N_C(R - D) = \{x\}$, we have that $x' \in N_C(D)$. Thus, we may obtain a cycle C' from C^* by replacing yx' with a path from y to x' with all internal vertices in D. Then, as seen in (i), C' can be transformed into a cycle contradicting the local maximality of C. This completes the proof of Lemma 2.3. □

Lemma 2.4. *Let C be a locally maximal cycle in a 2-connected graph G and R a component of $G - C$. One of the following two statements holds.*

(i) *$N_R(x) = V(R)$ for every $x \in N_C(R)$.*
(ii) *There is $y \in N_R(x)$ for some $x \in N_C(R)$ and a nonempty set $A \subseteq N_R(y) \setminus (N_R(x) \cup \{x\})$ such that*

$$G' = \begin{cases} G[y \to x; A] & \text{if } G[y \to x; A] \text{ is 2-connected,} \\ G[y \to x; A] + yx' & \text{otherwise,} \end{cases}$$

is 2-connected, where $x' \in N_C(R) \setminus \{x\}$, and moreover, C remains a locally maximal cycle in G'.

Proof. Suppose that (i) does not hold. Then $N_R(x) \neq V(R)$ for some $x \in N_C(R)$, which implies that there is $y \in N_R(x)$ such that

$$Z_y = N_R(y) \setminus (N_R(x) \cup \{x\}) \neq \emptyset.$$

If $G[y \to x; Z_y]$ is 2-connected, then by Lemma 2.3(i), C remains a locally maximal cycle in $G[y \to x; Z_y]$, and (ii) holds with $A = Z_y$ and $G' = G[y \to x; Z_y]$. Suppose thus that this is not the case. Then, x is the unique cut vertex of $G[y \to x; Z_y]$. Let R_y be the smallest component in $G[y \to x; Z_y] - x$ with $V(R_y) \subseteq V(R)$. R_y is defined for each y with $Z_y \neq \emptyset$. (That is, $G[y \to x; Z_y]$ is not 2-connected for each y with $Z_y \neq \emptyset$.) For simplicity, we may assume that y has been chosen such that $|V(R_y)|$ is as small as possible. Then, either $R_y = \{y\}$ or y is a cut vertex of R. We claim that x is joined to every vertex of R_y in G. If this is not true, then there is a $w \in V(R_y)$ with $Z_w \neq \emptyset$. Then, $G[w \to x; Z_w] - x$ has a component that is a proper subset of R_y, which implies that $|V(R_w)| < |V(R_y)|$, contradicting the choice of y. This proves the claim. By the claim, we have that $y \in R_y$. Let $R_1, R_2, \ldots, R_t$ be the components of $G[y \to x; Z_y] - x$, where $t \geqslant 2$, $R_1 = R_y$, and $V(C) \setminus \{x\} \subseteq V(R_t)$. We note that $R_1 = R_y$, which is a component in $G[y \to x; Z_y] - x$ with $V(R_1) \subseteq V(R)$ (so R_1 is adjacent only to y and x in G). Since G is 2-connected, there must be $x' \in N_C(R) \setminus \{x\}$ joined to some vertex y' of $R - R_1$ in G. Clearly, $y' \in R_t$. Let $D = R_t - V(C)$. Then, $N_{R-D}(D) = \{y\}$ and $N_D(y) \cap N_D(x) = \emptyset$. Let $A = N_D(y)$. If $G[y \to x; A]$ is 2-connected, let $G' = G[y \to x; A]$; if $G[y \to x; A]$ is not 2-connected, then $N_C(R - D) = \{x\}$, and we let $G' = G[y \to x; A] + yx'$. In either case, G' is 2-connected, and by Lemma 2.3(ii), C is a locally maximal cycle in G'. This proves Lemma 2.4. □

3. Proof of the theorem

Theorem 3.1. *Let C be a locally maximal cycle of length c in a 2-connected graph G on n vertices. If $\frac{2}{3}n + 1 \leqslant c \leqslant n - 1$, then*

$$e(G) \leqslant \max\{f(n, 2, c), f(n, \lfloor c/2 \rfloor, c)\}.$$

Proof. Suppose that $R_1, R_2, \ldots, R_m$ are the components of $G - C$, $m \geqslant 1$. Repeatedly applying Lemma 2.4 to each R_i (note that since the set A is nonempty, each time Lemma 2.4(ii) is applied, the number of edges not incident with C strictly decreases), we have a 2-connected graph G' in which $e(G) \leqslant e(G')$, C remains a locally maximal cycle, and for each component R of $G' - C$, $N_R(x) = V(R)$ for every $x \in N_C(R)$. For simplicity, we may simply assume that G has been chosen to be the final graph after repeatedly applying Lemma 2.4, and so

$$N_{R_i}(x) = V(R_i) \text{ for every } x \in N_C(R_i), \quad 1 \leqslant i \leqslant m. \tag{3.1}$$

Let $n_i = |V(R_i)|$ and $k_i = |N_C(R_i)|$, $1 \leqslant i \leqslant m$. For any i, $1 \leqslant i \leqslant m$, suppose that $N_C(R_i) = \{x_1, x_2, \ldots, x_{k_i}\}$. Let P_{jt} be a longest path from x_j to x_t with all internal vertices in R_i. By (3.1), for all $j \neq t$, P_{jt} have the same length, denoted by d_i, which is 2 plus the length of a longest path in R_i. So, R_i contains no path of length more than $d_i - 2$. It follows from a result of Erdös and Gallai [2, Theorem 2.6] that

$$e(R_i) \leqslant \frac{d_i - 2}{2} n_i, \quad 1 \leqslant i \leqslant m.$$

Let H be the subgraph induced by $V(C)$. Then,

$$e(G) \leqslant e(H) + \sum_{i=1}^{m} (e(R_i) + n_i k_i) \leqslant e(H) + \frac{1}{2} \sum_{i=1}^{m} n_i (d_i - 2 + 2k_i).$$

Choose α such that $d_\alpha + 2k_\alpha = \max\{d_i + 2k_i, 1 \leqslant i \leqslant m\}$ and let $d = d_\alpha$ and $k = k_\alpha$. It follows, using $\sum_{i=1}^{m} n_i = n - c$, that

$$e(G) \leqslant e(H) + \frac{d - 2 + 2k}{2}(n - c). \tag{3.2}$$

Let $R = R_\alpha$ and $X = N_C(R) = \{x_1, x_2, \ldots, x_k\}$. Then $C - X$ consists of k segments $S_1, S_2, \ldots, S_k$, where S_i is the segment of C from x_i^+ to x_{i+1}^-. Set $s_i = |V(S_i)|$, $1 \leqslant i \leqslant k$. We first prove several lemmas that deal with the estimation of the number of edges between S_i and S_j. □

Lemma 3.2. *For $i \neq j$, let $S_i = a_1 a_2 \cdots a_p$ and $S_j = b_1 b_2 \cdots b_q$, where $p = s_i$ and $q = s_j$.*

(i) *If $a_r b_\ell \in E(G)$, then*

$$(r - 1) + (\ell - 1) \geqslant d - 1 \quad \text{and} \quad (p - r) + (q - \ell) \geqslant d - 1.$$

(ii) *For a_r, a_{p-t} with $r + t \leqslant d - 1$ (so $r \leqslant p - t$), if there are distinct b_ℓ, b_m such that $a_r b_\ell$, $a_{p-t} b_m \in E(G)$ (or $a_r b_m$, $a_{p-t} b_\ell \in E(G)$), then $|m - \ell| \geqslant d + 1 - r - t$.*

Proof. (i) Since $a_r b_\ell \in E(G)$, we have a cycle $C' = a_r b_\ell b_{\ell+1} \cdots b_q x_{j+1} x_{j+1}^+ \cdots x_i P x_j x_j^- \cdots x_{i+1} a_p a_{p-1} \cdots a_r$ of length $c + (d-1) - (r-1) - (\ell-1)$ with $|E(C') \cap E(C, G-C)| = 2$, where P is a path of length d from x_i to x_j with all its internal vertices in R. By the choice of C, $(r-1) + (\ell - 1) \geqslant d - 1$. By symmetry, $(p - r) + (q - \ell) \geqslant d - 1$, as required.

(ii) Without loss of generality, suppose that $a_r b_\ell$, $a_{p-t} b_m \in E(G)$. Let P be a path of length d from x_i to x_{i+1} with all internal vertices in R. Then

$$C' = x_i x_i^- \cdots x_{j+1} b_q b_{q-1} \cdots b_m a_{p-t} a_{p-t-1} \cdots a_r b_\ell b_{\ell-1} \cdots x_{i+1} P x_i$$

is cycle of length

$$c + (d - 1) - (r + t - 1) - (m - \ell - 1) = c + d + 1 - r - t - (m - \ell)$$

with $|E(C') \cap E(C, G - C)| = 2$. By the choice of C, $|m - \ell| \geqslant d + 1 - r - t$, as required. □

Lemma 3.3. *For $i \neq j$, let $S_i = a_1a_2 \cdots a_p$ and $S_j = b_1b_2 \cdots b_q$, where $p = s_i$ and $q = s_j$.*

(i) *For $a_r,\ a_{p-t}$ with $r + t \leqslant d - 1$,*

$$e(\{a_r, a_{p-t}\}, S_j) \leqslant q - (d - r - t).$$

(ii) *For each m, $1 \leqslant m \leqslant q - 1$,*

$$e(b_m b_{m+1}, S_i) \leqslant 2p - d + 1$$

with equality only if $p = d - 1$, and $e(b_m, S_i) = p$ or $e(b_{m+1}, S_i) = p$.

(iii) *$e(\{a_1, b_1\}, S_i) \leqslant p$ and $e(\{a_p, b_q\}, S_i) \leqslant p$.*

Proof. (i) Let B be the set of vertices in S_j which are joined to both a_r and a_{p-t}. If $B \neq \emptyset$, then for each $b_i \in B$, by Lemma 3.2(ii),

$$e(\{a_r, a_{p-t}\}, b_{i+j}) = 0 \quad \text{and} \quad e(\{a_r, a_{p-t}\}, b_{i-j}) = 0$$

for all j, $1 \leqslant j \leqslant d - r - t$, which implies that there are at least $(|B| + 1)(d - r - t)$ vertices in S_j, none of which is joined to either a_r or a_{p-t}. It follows that

$$\begin{aligned} e(\{a_r, a_{p-t}\}, S_j) &\leqslant 2|B| + (q - |B| - (|B| + 1)(d - r - t)) \\ &= q - (d - r - t) - |B|(d - 1 - r - t). \end{aligned}$$

But $r + t \leqslant d - 1$, and thus we may suppose that $B = \emptyset$.

If $e(a_{p-t}, S_j) = 0$, then $e(\{a_r, a_{p-t}\}, S_j) = e(a_r, S_j)$. By Lemma 3.2(i), none of the first $d - r$ vertices of S_j is joined to a_r, and hence $e(a_r, S_j) \leqslant q - (d - r) \leqslant q - (d - r - t)$. Therefore, we may assume that $e(a_{p-t}, S_j) > 0$, and similarly, $e(a_r, S_j) > 0$.

Let $a_{p-t}b_\ell,\ a_rb_m \in E(G)$ and choose b_ℓ and b_m as close to each other as possible, so that none of the vertices (in S_j) between b_ℓ and b_m is joined to a_r or a_{p-t}. By Lemma 3.2(ii), $|m - \ell| \geqslant d + 1 - r - t$. It follows that there are at least $d - r - t$ vertices that are not joined to a_r or a_{p-t}. Therefore, $e(\{a_r, a_{p-t}\}, S_j) \leqslant q - (d - r - t)$, as required.

(ii) We first consider the case that there is a_r such that $e(a_r, b_mb_{m+1}) = 2$. Choose such a_r as close to a_1 or a_p as possible. We may assume that $r - 1 \leqslant p - r$. By the choice of a_r, none of the first and the last $r - 1$ vertices of S_i can be joined to both b_m and b_{m+1}, which gives that $e(b_mb_{m+1}, S_i) \leqslant 2p - 2(r - 1)$. If $r - 1 \geqslant \frac{d}{2}$, then $e(b_mb_{m+1}, S_i) \leqslant 2p - d$, and we are done. Suppose therefore that $r - 1 \leqslant \frac{d-1}{2}$, that is $r \leqslant \frac{d+1}{2}$. By Lemma 3.2(ii), none of the last $d - r$ vertices of S_i can be joined to b_m or b_{m+1}, that is, $e(b_mb_{m+1}, a_i) = 0$ for all i, $p - (d - r) + 1 \leqslant i \leqslant p$. It follows that

$$e(b_mb_{m+1}, S_i) \leqslant 2p - (r - 1) - 2(d - r) = 2p - d - (d - r - 1).$$

If d is odd (so $d \geqslant 3$), then, since $r \leqslant \frac{d+1}{2}$, we have $d - r - 1 \geqslant \frac{d-3}{2} \geqslant 0$; if d is even, then $r \leqslant \frac{d}{2}$, and we have $d - r - 1 \geqslant \frac{d-2}{2} \geqslant 0$; In either case, we have that $e(b_mb_{m+1}, S_i) \leqslant 2p - d$.

Next we consider the case that

$$e(b_mb_{m+1}, a_i) \leqslant 1 \quad \text{for all } i, \quad 1 \leqslant i \leqslant p. \tag{3.3}$$

Then,

$$e(b_m b_{m+1}, S_i) \leqslant p = 2p - d - (p - d).$$

Thus, $e(b_m b_{m+1}, S_i) \leqslant 2p - d + 1$, with equality only if $p = d - 1$, and all equalities hold in (3.3), which implies, by Lemma 3.2(ii), that either $e(b_m, S_i) = p$ or $e(b_{m+1}, S_i) = p$.

(iii) Let $A = \{a_i : a_1 a_{i+1} \in E(G), 1 \leqslant i \leqslant p - 1\}$. If there is $a_i \in A$ such that $a_i b_1 \in E(G)$, then $C' = b_1 a_i a_{i-1} \cdots a_1 a_{i+1} a_{i+2} \cdots a_p x_{i+1} \cdots x_j P x_i x_i^- \cdots b_q b_{q-1} \cdots b_1$ is a cycle of length $c + d - 1$ and $|E(C') \cap E(C, G - C)| = 2$, contradicting the choice of C. Thus, $A \cap N_{S_i}(b_1) = \emptyset$, which implies that $e(b_1, S_i) \leqslant p - |A| = p - e(a_1, S_i)$, and so $e(b_1, S_i) + e(a_1, S_i) \leqslant p$. By symmetry, $e(b_q, S_i) + e(a_p, S_i) \leqslant p$. This completes the proof of Lemma 3.3. □

Lemma 3.4. *Suppose that $d \geqslant 3$. For $i \neq j$,*

$$e(S_i, S_j) \leqslant \begin{cases} (s_j - 1)(s_i - 1) - \frac{s_j(d-2)}{2} + \frac{s_j - 2}{2} & \text{if } s_i = d - 1 \text{ and } s_j \geqslant 2d - 1, \\ (s_j - 1)(s_i - 1) - \frac{s_j(d-2)}{2} & \text{otherwise.} \end{cases}$$

Proof. Let $S_i = a_1 a_2 \cdots a_p$ and $S_j = b_1 b_2 \cdots b_q$, where $p = s_i$ and $q = s_j$. By Lemma 3.3(i) (S_i and S_j interchange, $r = 1$ and $t = 0$),

$$e(\{b_1, b_q\}, S_i) \leqslant p - (d - 1). \tag{3.4}$$

Without loss of generality, we may assume that $e(b_1, S_i) \leqslant e(b_q, S_i)$, and so

$$e(b_q, S_i) \geqslant \frac{1}{2} e(\{b_1, b_q\}, S_i).$$

Then

$$\begin{aligned} e(\{b_1, b_2, b_q\}, S_i) &= e(\{b_1, b_q\}, S_i) + e(\{b_2, b_q\}, S_i) - e(b_q, S_i) \\ &\leqslant \frac{1}{2} e(\{b_1, b_q\}, S_i) + e(\{b_2, b_q\}, S_i). \end{aligned}$$

By Lemma 3.3(i) (S_i and S_j interchange and $r + t = 2$),

$$e(\{b_2, b_q\}, S_i) \leqslant p - (d - 2).$$

It follows from (3.4) that

$$e(\{b_1, b_2, b_q\}, S_i) \leqslant \frac{1}{2}(p - d + 1) + p - (d - 2) = \frac{3}{2}(p - d + 1) + 1. \tag{3.5}$$

If $p \geqslant d$ or if there is no vertex $b \in S_j$ with $e(b, S_i) = p$, then by Lemma 3.3(ii) (without equalities),

$$e(b_m b_{m+1}, S_i) \leqslant 2p - d, \qquad 1 \leqslant m \leqslant q - 1.$$

Therefore, if q is even,

$$e(S_j - \{b_1, b_q\}, S_i) \leqslant \frac{q - 2}{2}(2p - d),$$

which together with (3.4) gives that

$$e(S_i, S_j) \leqslant \frac{q-2}{2}(2p-d) + p - (d-1) = (q-1)(p-1) - \frac{q}{2}(d-2);$$

if q is odd (so $q \geqslant 3$),

$$e(S_j - \{b_1, b_2, b_q\}, S_i) \leqslant \frac{q-3}{2}(2p-d),$$

which together with (3.5) gives that

$$\begin{aligned} e(S_i, S_j) &\leqslant \frac{q-3}{2}(2p-d) + \frac{3}{2}(p-d+1) + 1 \\ &= (q-1)(p-1) - \frac{q}{2}(d-2) - \frac{p-3}{2}, \end{aligned}$$

and since $p \geqslant d \geqslant 3$, the required result follows.

Suppose therefore that $p = d-1$ and there is $b_\ell \in S_j$ such that $e(b_\ell, S_i) = p$. By Lemma 3.2(i), $\ell - 1 \geqslant d-1$ and $q - \ell \geqslant d-1$, which gives that $q \geqslant 2d-1$. By Lemma 3.3(ii), we have now that

$$e(b_m b_{m+1}, S_i) \leqslant 2p - d + 1, \qquad 1 \leqslant m \leqslant q-1$$

and thus, if q is even,

$$e(S_j - \{b_1, b_q\}, S_i) \leqslant \frac{q-2}{2}(2p-d) + \frac{q-2}{2},$$

if q is odd,

$$e(S_j - \{b_1, b_2, b_q\}, S_i) \leqslant \frac{q-3}{2}(2p-d) + \frac{q-3}{2}.$$

Since (3.4) and (3.5) still hold, if q is even,

$$e(S_i, S_j) \leqslant (q-1)(p-1) - \frac{q}{2}(d-2) + \frac{q-2}{2},$$

if q is odd,

$$e(S_i, S_j) \leqslant (q-1)(p-1) - \frac{q}{2}(d-2) + \frac{q-3}{2}.$$

Consequently,

$$e(S_i, S_j) \leqslant (q-1)(p-1) - \frac{q}{2}(d-2) + \frac{q-2}{2}.$$

This completes the proof of Lemma 3.4. □

Lemma 3.5. *Suppose that $d \geqslant 3$. For $i \neq j$, if $s_j = d-1$, let F be the subgraph induced by $V(S_i) \cup V(S_j)$, then*

$$\sum_{x \in S_i} d_F(x) \leqslant s_i(s_i - 1) + \frac{s_j - 2}{2}(2s_i - d) + 2.$$

Proof. As before, let $S_i = a_1a_2\cdots a_p$ and $S_j = b_1b_2\cdots b_q$, where $p = s_i$ and $q = s_j$. Let $b_r \in S_j$. If $r \leqslant \frac{d}{2}$, by Lemma 3.2(i), b_r is not joined to any of the first $\lceil \frac{d}{2} \rceil$ vertices of S_i, which gives that $e(b_r, S_i) \leqslant p - \frac{d}{2}$. If $r \geqslant \frac{d+1}{2}$, since $q = d-1$, we have that $q - r \leqslant \frac{d-3}{2}$, and again by Lemma 3.2(i), b_r is not joined to any of the last $\lceil \frac{d+1}{2} \rceil$ vertices of S_i, which gives that $e(b_r, S_i) \leqslant p - \frac{d+1}{2}$. Consequently,

$$e(b_r, S_i) \leqslant p - \frac{d}{2} \quad \text{for each } r, \quad 1 \leqslant r \leqslant q.$$

So

$$\sum_{r=2}^{q-1} e(b_r, S_i) \leqslant (q-2)(p - \frac{d}{2}) = \frac{q-2}{2}(2p-d).$$

Therefore,

$$e(S_i, S_j) \leqslant \frac{q-2}{2}(2p-d) + e(\{b_1, b_q\}, S_i).$$

Using the fact that $d_{S_i}(a_\ell) \leqslant p-1$ for all ℓ, $2 \leqslant \ell \leqslant p-1$, we have that

$$\sum_{x \in S_i} d_{S_i}(x) \leqslant (p-2)(p-1) + e(\{a_1, a_p\}, S_i).$$

Noting that

$$\sum_{x \in S_i} d_F(x) = \sum_{x \in S_i} d_{S_i}(x) + e(S_i, S_j),$$

we obtain that

$$\begin{aligned} \sum_{x \in S_i} d_F(x) &\leqslant (p-2)(p-1) + \frac{q-2}{2}(2p-d) \\ &\quad + e(\{a_1, a_p\}, S_i) + e(\{b_1, b_q\}, S_i) \\ &= (p-2)(p-1) + \frac{q-2}{2}(2p-d) + e(\{a_1, b_1, a_p, b_q\}, S_i). \end{aligned}$$

By Lemma 3.3(iii), $e(\{a_1, b_1\}, S_i) \leqslant p$ and $e(\{a_p, b_q\}, S_i) \leqslant p$, and hence,

$$e(\{a_1, b_1, a_p, b_q\} \leqslant 2p.$$

It follows that

$$\sum_{x \in S_i} d_F(x) \leqslant p(p-1) + \frac{q-2}{2}(2p-d) + 2,$$

as required by Lemma 3.5.

Now, we return to the proof of Theorem 3.1. By (3.2), we need to estimate $e(H)$. The proof is divided into two parts, according to $d \geqslant 3$ or $d = 2$.

Part I. $d\geqslant 3$. Let $A=\{i\ :\ s_i=d-1\}$ and $B=\{i\ :\ s_i\geqslant 2d-1\}$. Set $a=|A|$, $b=|B|$, $s=\sum_{i=1}^{k} s_i$ and $s'=\sum_{i\in B} s_i$. By the definition, $s'\geqslant b(2d-1)$. We first show that

$$\sum_{i=1}^{k}\sum_{x\in S_i} d_H(x)\leqslant c^2+c+3k^2-3ck-2k-\frac{(d-2)(c-3)}{2}. \tag{3.6}$$

If $A=\emptyset$ or $B=\emptyset$, then by Lemma 3.4, for all $i\neq j$,

$$e(S_i,S_j)\leqslant (s_j-1)(s_i-1)-\frac{s_j}{2}(d-2)$$

and thus, using $\sum_{j\neq i} s_j=s-s_i$,

$$\sum_{j\neq i} e(S_i,S_j)\leqslant (s-s_i-(k-1))(s_i-1)-\frac{s-s_i}{2}(d-2). \tag{3.7}$$

Therefore,

$$\begin{aligned}\sum_{x\in S_i} d_H(x) &\leqslant \sum_{j\neq i} e(S_i,S_j)+s_i(s_i-1)+s_ik\\ &\leqslant s(s_i-1)+s_i+(k-1)-\frac{s-s_i}{2}(d-2)\end{aligned} \tag{3.8}$$

and so,

$$\begin{aligned}\sum_{i=1}^{k}\sum_{x\in S_i} d_H(x) &\leqslant s(s-k)+s+k(k-1)-\frac{s(k-1)}{2}(d-2)\\ &= s(s-k)+s+k(k-1)-\frac{(d-2)(s+k-3)}{2}\\ &\quad -\frac{(d-2)(k-2)(s-1)+(d-2)}{2}\\ &= c^2+c+3k^2-3ck-2k-\frac{(d-2)(c-3)}{2}\\ &\quad -\frac{(d-2)(k-2)(s-1)+(d-2)}{2},\end{aligned} \tag{3.9}$$

where we have used that $s=c-k$. But $d\geqslant 3$ and $k\geqslant 2$, and so (3.6) follows. In what follows, suppose therefore that $a\geqslant 1$ and $b\geqslant 1$. For a segment S_i, we distinguish the following three cases.

Case 1. $i\notin A\cup B$. By Lemma 3.4, for all $j\neq i$,

$$e(S_i,S_j)\leqslant (s_j-1)(s_i-1)-\frac{s_j}{2}(d-2) \tag{3.10}$$

and as the derivation of (3.8),

$$\sum_{x\in S_i} d_H(x)\leqslant s(s_i-1)+s_i+(k-1)-\frac{s-s_i}{2}(d-2). \tag{3.11}$$

Case 2. $i \in A$. By Lemma 3.4, if $j \notin B$, we have (3.10). If $j \in B$,

$$e(S_i, S_j) \leqslant (s_j - 1)(s_i - 1) - \frac{s_j}{2}(d-2) + \frac{s_j - 2}{2}.$$

Thus,

$$\begin{aligned}\sum_{j \neq i} e(S_i, S_j) &\leqslant \sum_{j \neq i}\left[(s_j - 1)(s_i - 1) - \frac{s_j}{2}(d-2)\right] + \sum_{j \in B} \frac{s_j - 2}{2} \\ &\leqslant (s - s_i - (k-1))(s_i - 1) - \frac{s - s_i}{2}(d-2) + \frac{s' - 2b}{2}\end{aligned}$$

and as the way (3.8) is derived from (3.7), we have that,

$$\sum_{x \in S_i} d_H(x) \leqslant s(s_i - 1) + s_i + (k-1) - \frac{s - s_i}{2}(d-2) + \frac{s' - 2b}{2}. \tag{3.12}$$

Case 3. $i \in B$. Let $\ell \in A$ and let F be the subgraph induced by $V(S_i) \cup V(S_\ell)$. By Lemma 3.5,

$$\begin{aligned}\sum_{x \in s_i} d_F(x) &\leqslant s_i(s_i - 1) + \frac{s_\ell - 2}{2}(2s_i - d) + 2 \\ &\leqslant s_i(s_i - 1) + (s_\ell - 1)(s_i - 1) - \frac{s_\ell}{2}(d-2) - (s_i - d - 1).\end{aligned}$$

For all $j \notin \{i, \ell\}$, by Lemma 3.4,

$$e(S_i, S_j) \leqslant (s_j - 1)(s_i - 1) - \frac{s_j}{2}(d-2).$$

It follows that

$$\begin{aligned}\sum_{x \in S_i} d_H(x) &\leqslant \sum_{x \in S_i} d_F(x) + \sum_{j \notin \{i, \ell\}} e(S_i, S_j) + s_i k \\ &\leqslant \sum_{j \neq i}\left[(s_i - 1)(s_j - 1) - \frac{s_j}{2}(d-2)\right] \\ &\quad + s_i(s_i - 1) + s_i k - (s_i - d - 1) \\ &= s(s_i - 1) + s_i + (k-1) - \frac{s - s_i}{2}(d-2) - (s_i - d - 1).\end{aligned} \tag{3.13}$$

By (3.11), (3.12), and (3.13), we have that

$$\begin{aligned}\sum_{i=1}^{k} \sum_{x \in S_i} d_H(x) &\leqslant \sum_{i=1}^{k}\left[s(s_i - 1) + s_i + (k-1) - \frac{s - s_i}{2}(d-2)\right] \\ &\quad + \sum_{i \in A} \frac{s' - 2b}{2} - \sum_{i \in B}(s_i - d - 1).\end{aligned} \tag{3.14}$$

As seen in the derivation of (3.9) from (3.8), the first summation at the right-hand side of (3.14) is

$$c^2 + c + 3k^2 - 3ck - 2k - \frac{(d-2)(c-3)}{2} - \frac{(d-2)(k-2)(s-1) + (d-2)}{2}. \tag{3.15}$$

Clearly,

$$\begin{aligned}\sum_{i \in A} \frac{s' - 2b}{2} - \sum_{i \in B} (s_i - d - 1) &= \frac{as' - 2ab}{2} - (s' - bd - b) \\ &= \frac{s'(a-1)}{2} - \frac{b}{2}(2a-3) - \frac{s' - b(2d-1)}{2} \\ &\leqslant \frac{s'(a-1)}{2} - \frac{b}{2}(2a-3).\end{aligned} \tag{3.16}$$

Applying (3.16) and (3.15) into (3.14), and writing

$$g(d,a,k) = \frac{s'(a-1)}{2} - \frac{b}{2}(2a-3) - \frac{(d-2)(k-2)(s-1) + (d-2)}{2},$$

we have that

$$\sum_{i=1}^{k} \sum_{x \in S_i} d_H(x) \leqslant c^2 + c + 3k^2 - 3ck - 2k - \frac{(d-2)(c-3)}{2} + g(d,a,k).$$

It remains to show that $g(d,a,k) \leqslant 0$. Clearly,

$$g(d,a,k) \leqslant g(3,a,k) = \frac{s'(a-1)}{2} - \frac{b}{2}(2a-3) - \frac{(k-2)(s-1)+1}{2}.$$

If $k = 2$, then $a = 1$ and $b = 1$, and we have that $g(3,1,2) = 0$. If $a \geqslant 2$, then, using $k \geqslant a + 1$ and $s \geqslant s' + 1$, we have that

$$g(3,a,k) \leqslant -\frac{b}{2} - \frac{1}{2} < 0.$$

Therefore we assume that $k \geqslant 3$ and $a = 1$. Then,

$$g(3,1,k) = \frac{b}{2} - \frac{(k-2)(s-1)+1}{2} \leqslant \frac{b - (s-1) - 1}{2}.$$

Since $s \geqslant b+1$, we have that $g(3,1,k) < 0$. In each case, $g(d,a,k) \leqslant 0$. This proves (3.6).

By the fact that

$$2e(H) = \sum_{x \in H} d_H(x) \leqslant k(c-1) + \sum_{i=1}^{k} \sum_{x \in S_i} d_H(x),$$

it follows from (3.6) that

$$2e(H) \leqslant c^2 + c + 3k^2 - 2ck - 3k - \frac{(d-2)(c-3)}{2}$$

and so

$$e(H) \leqslant \frac{1}{2}(c^2 + c + 3k^2 - 2ck - 3k) - \frac{(d-2)(c-3)}{4}.$$

It follows from (3.2) that

$$e(G) \leqslant \frac{1}{2}(c^2 + c + 3k^2 + 2kn - 4ck - 3k) + \frac{(d-2)}{4}(2n - 3c + 3).$$

Since $c \geqslant \frac{2}{3}n + 1$,

$$e(G) \leqslant \frac{1}{2}(c^2 + c + 3k^2 + 2kn - 4ck - 3k) = f(n, k, c).$$

Since $2 \leqslant k \leqslant c/2$, we have that $f(n, k, c) \leqslant \max\{f(n, 2, c), f(n, \lfloor c/2 \rfloor, c)\}$, and the theorem follows. This completes the proof of Part I.

Part II. $d = 2$. Let w be the unique vertex of R (so $d(w) = k$) and G' the subgraph induced by $V(C) \cup \{w\}$. Then G' is non-hamiltonian. Choose a cycle C' (in G') of length c such that $e(C', G' - C')$ as large as possible, among all cycles of length c in G'. Suppose that u is the unique vertex of $G' - C'$ and X is the set of neighbors of u in G'. Set $x = |X|$. Then, $k \leqslant x \leqslant \frac{c}{2}$, and $C' - X$ consists of x segments $S_1, S_2, \ldots, S_x$. For simplicity, we consider these segments as same as those in Part I above, with k replaced by x. As before, define $s_i = |V(S_i)|$, $1 \leqslant i \leqslant x$. Set $Y = \{S_i : s_i = 1, 1 \leqslant i \leqslant x\}$ and $y = |Y|$. Let $B = \{S_1, S_2, \ldots, S_x\} \setminus Y$ and $b = |B|$. For simplicity, we may assume that $B = \{S_1, S_2, \ldots, S_b\}$ (so $s_i \geqslant 2$ for each i, $1 \leqslant i \leqslant b$) and let

$$s = \sum_{i=1}^{b} s_i, \qquad \text{and so,} \qquad c = s + x + y.$$

Let $F = G' - (Y \cup \{u\})$. We shall show that for each i, $1 \leqslant i \leqslant b$,

$$\sum_{v \in S_i} d_F(v) \leqslant (s_i - 1)(c - x) + x + s_i - 1. \tag{3.17}$$

Let $S_i = a_1 a_2 \cdots a_p$ with $p = s_i$, and for any $j \neq i$, $1 \leqslant j \leqslant b$, $S_j = b_1 b_2 \cdots b_q$ with $q = s_j$. By Lemma 3.3(i) (with $r = 1$ and $t = 0$),

$$e(\{a_1, a_p\}, S_j) \leqslant q - 1 \tag{3.18}$$

and for each ℓ, $1 \leqslant \ell \leqslant p - 1$, by Lemma 3.2(ii) ($S_i$ and S_j interchange, $r = 1$ and $t = 0$), $e(a_\ell a_{\ell+1}, \{b_1, b_q\}) \leqslant 2$, which implies that

$$e(a_\ell a_{\ell+1}, S_j) \leqslant 2q - 2, \quad 1 \leqslant \ell \leqslant p - 1. \tag{3.19}$$

Case 1. p is even. By (3.19),

$$e(S_i - \{a_1, a_p\}, S_j) \leqslant \frac{p-2}{2}(2q - 2) = (p - 2)(q - 1).$$

Combining with (3.18) yields that

$$e(S_i, S_j) \leqslant (p - 1)(q - 1) = (s_i - 1)(s_j - 1).$$

This holds for all $j \neq i$, $1 \leqslant j \leqslant b$, and thus,

$$\sum_{j \neq i} e(S_i, S_j) \leqslant (s_i - 1)(s - s_i - b + 1).$$

Noting that

$$\sum_{v \in S_i} d_F(v) = \sum_{j \neq i} e(S_i, S_j) + \sum_{v \in S_i} d_{S_i}(v) + e(S_i, X),$$

we have that

$$\begin{aligned} \sum_{v \in S_i} d_F(v) &\leqslant (s_i - 1)(s - s_i - b + 1) + s_i(s_i - 1) + x s_i \\ &= (s_i - 1)(s - b + x) + x + s_i - 1 \end{aligned} \tag{3.20}$$

and (3.17) follows from the fact that $s - b + x = c - y - b = c - x$.

Case 2. p is odd (so $p \geqslant 3$).

If $d_F(a_1) \geqslant x + 1$, then $a_2 b_1 \notin E(G)$, for otherwise there is a cycle C'' with $V(C'') = (V(C) \cup \{w\}) \setminus \{a_1\}$, contradicting the choice of C', and thus, $e(a_2, S_j) \leqslant q - 1$, which together with (3.18) gives that

$$e(\{a_1, a_2, a_p\}, S_j) \leqslant 2(q - 1).$$

By (3.19),

$$e(S_i - \{a_1, a_2, a_p\}, S_j) \leqslant \frac{p - 3}{2}(2q - 2) = (p - 3)(q - 1).$$

It follows that

$$e(S_i, S_j) \leqslant (p - 1)(q - 1) = (s_i - 1)(s_j - 1),$$

which holds for all $j \neq i$, $1 \leqslant j \leqslant b$, and as above we obtain (3.17).

If $d_F(a_1) \leqslant x$, by (3.19),

$$e(S_i - \{a_1\}, S_j) \leqslant \frac{p - 1}{2}(2q - 2) = (p - 1)(q - 1) = (s_i - 1)(s_j - 1),$$

for all $j \neq i$, $1 \leqslant j \leqslant b$, and thus,

$$\sum_{j \neq i} e(S_i - \{a_1\}, S_j) \leqslant (s_i - 1)(s - s_i - b + 1).$$

Therefore,

$$\begin{aligned} \sum_{v \in S_i} d_F(v) &\leqslant d_F(a_1) + \sum_{j \neq i} e(S_i - \{a_1\}, S_j) + (s_i - 1)(s_i - 1) + x(s_i - 1) \\ &\leqslant x + (s_i - 1)(s - s_i - b + 1) + (s_i - 1)(s_i - 1) + x(s_i - 1), \\ &= (s_i - 1)(s - b + x) + x, \end{aligned}$$

which is less than the right-hand side of (3.20), and as there, (3.17) follows.

Summing (3.17) over all i, $1 \leqslant i \leqslant b$, we obtain that

$$\sum_{i=1}^{b} \sum_{v \in S_i} d_F(v) \leqslant (s-b)(c-x) + xb + s - b.$$

Then

$$\begin{aligned} 2e(F) &= \sum_{i=1}^{b} \sum_{v \in S_i} d_F(v) + \sum_{v \in X} d_F(v) \\ &\leqslant (s-b)(c-x) + xb + s - b + x(c-1-y) \\ &= c^2 + c + 3x^2 - 2cx - 2yx - 3x. \end{aligned}$$

By the choice of C', $d_{G'}(v) \leqslant x$ for each vertex $v \in Y$, and by Lemma 3.2(i), Y is an independent set in G. Therefore,

$$e(G') = e(F) + \sum_{v \in Y} d_{G'}(v) + d_{G'}(u) \leqslant e(F) + xy + x$$

and so

$$e(G') \leqslant \frac{1}{2}(c^2 + c + 3x^2 - 2cx - x).$$

Since

$$e(G) \leqslant e(G') + k(n-c-1) \leqslant e(G') + x(n-c-1),$$

we have that

$$e(G) \leqslant \frac{1}{2}(c^2 + c + 3x^2 + 2xn - 4cx - 3x) = f(n,x,c).$$

Again, since $2 \leqslant x \leqslant c/2$, we have that $f(n,x,c) \leqslant \max\{f(n,2,c), f(n, \lfloor c/2 \rfloor, c)\}$, and the theorem follows. This completes the proof of the theorem. □

Since a longest cycle is locally maximal, we see that Theorem 3.1 together with Theorem 1.2 confirms Conjecture 1.1.

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