Prevalence and structure of adding machines for cellular automata

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Consider the collection of left permutive cellular automata Φ with no memory, defined on the space S of all doubly infinite sequences from a finite alphabet. There exists ˜Z, a dense subset of S, such that Φ: cl{Φ^n(x): n ⩾ 0} → cl{Φ^n(x): n ⩾ 0} is topologically conjugate to an odometer for all x ∈ ˜Z so long as Φ^m is not the identity map for any m. Moreover, Φ generates the same odometer for all x ∈ ˜Z. The set ˜Z is a dense Gδ subset with full measure of a particular subspace of S.

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1. Introduction

Let f : X → X be a continuous map on the compact metric space X. An important object of study in dynamical systems is the minimal set. Minimal sets are those sets which are minimal, nonempty, closed and invariant with respect to f. Periodic orbits are the simplest examples of minimal sets. Infinite minimal sets, however, can be considerably more interesting. Examples of infinite minimal sets with interesting geometry include the irrational rotation of the circle, the Sturmian minimal set [8], the Morse minimal set [7], and the substitution minimal set [6].

We are interested in adding machines. These are a particular type of minimal set sometimes referred to as solenoids or odometers. Adding machines are always Cantor sets, and play an important role in one-dimensional dynamics. They are a basic component of Blokh’s spectral decomposition theorem [2]. Adding machines are also fundamental to Nitecki’s study of piecewise monotone maps [11]. Block and Keesling provide topological characterizations of adding machines in [3], where they also characterize topological conjugacy of adding machines. This last characterization is critical to our work.

Another body of research that relates to our work involves the generic behavior of continuous self-maps. Agronsky, Bruckner and Laczkovich show in [1] that given a generic continuous self-map f of the unit interval I there is a residual set of points x in I for which the ω-limit set ω(x, f) is a Cantor set. Using a much different approach, Lehning extends these results to continuous self-maps of any compact n-dimensional manifold [10]. In [14], Steele goes a step further by showing that, on the interval, a generic ordered pair (x, f) generates an adding machine as its ω-limit set. The work of D’Aniello, Darji and Steele [5] is an extension of all this. There, the authors show that not only is ω(x, f) an adding machine for the typical point (x, f) ∈ M × f(M, M), where M is an n-manifold or the Cantor space, but that a residual set of points in M × f(M, M) generate the same adding machine, up to homeomorphism.

The goal of this work is to develop results analogous to those of [5], applied to cellular automata.

Stanislaw Ulam and John von Neumann developed the notion of a cellular automaton in the 1940s. This was part of their work on self-reproducing machines. More recently, Stephen Wolfram, John Holland, and Christopher Langton have...
continued the study of cellular automata [12,13]. Cellular automata are used to model many physical and natural biological systems [15].

Coven, Pivato and Yassawi show that for non-trivial one-dimension cellular automata $\Phi$ defined on the space $S$ of all doubly infinite sequences with entries from a finite alphabet, there are many sequences $x$ such that $\Phi : cl(\Phi^n(x) : n \geq 0) \rightarrow cl(\Phi^n(x) : n \geq 0)$ is topologically conjugate to an odometer:

**Theorem 1.1.** (See [4, Theorem 1.]) Let $\Phi$ be a left permutive cellular automaton with no memory, defined on the space of all doubly infinite sequences with entries from a finite alphabet. If

1. $\{\Phi^n(x) : n \geq 0\}$ is infinite, and
2. $(x_1, x_2, \ldots)$ is $R$-fixed,

then $\Phi : cl(\Phi^n(x) : n \geq 0) \rightarrow cl(\Phi^n(x) : n \geq 0)$ is topologically conjugate to an odometer.

They use Theorem 1.1 to show that given an appropriate cellular automaton $\Phi$, the collection of sequences $x$ that generate an odometer with respect to $\Phi$, is large:

**Lemma 1.2.** (See [4, Lemma, p. 818.]) Let $\Phi$ be a left permutive cellular automaton with no memory, defined on the space of all doubly infinite sequences with entries from a finite alphabet. Then $\Phi^n$ has positive anticipation for every $n \geq 1$ if and only if $\Phi^m$ is not the identity map for every $m \geq 1$. In particular, if $\Phi$ is not one-to-one, then $\Phi^n$ has positive anticipation for every $n \geq 1$.

In particular, if $\Phi$ is a left permutive cellular automaton with no memory, $\Phi^n$ has positive anticipation for any $n$, and $(x_1, x_2, x_3, \ldots)$ is $R$-periodic, then one can find a dense $G_\delta$ subset $\mathcal{E}$ of $\mathcal{T} = \{y \in S : x_i = y_i$ for all $i \geq 1\}$ such that $(\Phi, x)$ generates an odometer whenever $x \in \mathcal{E}$.

After presenting the notation and recalling those previously known results that are necessary in the sequel, we concentrate on extending the results of [4] in a couple of ways. This occurs in Section 3, which addresses the frequency with which odometers occur, and describes their structure. With Theorem 3.4 we develop a subset $\mathcal{E} \subseteq \mathcal{T}$, different from that found in [4], so that

1. $\mathcal{E}$ is a dense $G_\delta$ in $\mathcal{T}$,
2. there is a (natural) measure $\mu$ on $\mathcal{T}$ so that $\mu(\mathcal{T} \setminus \mathcal{E}) = 0$, and
3. if $x \in \mathcal{E}$, then $(\Phi, x)$ generates an odometer.

An immediate consequence of Theorem 3.4 is the existence of $\mathcal{E}$, a dense subset of $S$, such that $(\Phi, x)$ generates an odometer for all appropriate $\Phi$ whenever $x \in \mathcal{E}$. This is, in part, Corollary 3.5. Moreover, Proposition 3.3 allows us to identify the odometer generated by $\Phi$ and $x$. Theorems 3.4 and Corollary 3.5 show that given a particular cellular automaton $\Phi$, the odometer generated by $(\Phi, x)$ is the same for all $x \in \mathcal{E}$ (as well as $x$ in the set $\mathcal{E}$ found in (3) above) and identify that odometer using the characterization of adding machines found in [3].

**2. Preliminaries**

**Adding machines**

To fully understand the results of [3], we define the $\alpha$-adic adding machine $\Delta_\alpha$.

Let $\alpha = (j_1, j_2, \ldots)$ be a sequence of integers where each $j_i \geq 2$. Let $\Delta_\alpha$ denote all sequences $(x_1, x_2, \ldots)$ where $x_i \in \{0, 1, \ldots, j_i - 1\}$ for each $i$. We put a metric $d_\alpha$ on $\Delta_\alpha$ given by

$$d_\alpha((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = \sum_{i=1}^{\infty} \frac{\delta(x_i, y_i)}{2^i},$$

where $\delta(x_i, y_i) = 1$ if $x_i \neq y_i$ and $\delta(x_i, y_i) = 0$ if $x_i = y_i$. Addition in $\Delta_\alpha$ is defined as follows. Set

$$(x_1, x_2, \ldots) + (y_1, y_2, \ldots) = (z_1, z_2, \ldots),$$

where $z_1 = x_1 + y_1 \mod j_1$ and $z_2 = x_2 + y_2 + t_1 \mod j_2$.

Here, $t_1 = 0$ if $x_1 + y_1 < j_1$ and $t_1 = 1$ if $x_1 + y_1 \geq j_1$. So, we carry a one in the second case. Continue adding and carrying in this way for the whole sequence.

We define $f_\alpha : \Delta_\alpha \rightarrow \Delta_\alpha$ by

$$f_\alpha((x_1, x_2, \ldots)) = (x_1, x_2, \ldots) + (1, 0, 0, \ldots)$$

and refer to $f_\alpha : \Delta_\alpha \rightarrow \Delta_\alpha$ as the adding machine map.
Then \((\Delta_\alpha, f_\alpha)\) is a dynamical system known in various contexts as a solenoid, adding machine or \(\alpha\)-odometer. The next two theorems summarize the main results of Block and Keesling.

**Theorem 2.1.** (See [3, Theorem 2.3].) Let \(\alpha = (j_1, j_2, \ldots)\) be a sequence of integers where each \(j_i \geq 2\). Let \(m_i = j_1 j_2 \ldots j_i\) for each \(i\). Let \(f : X \to X\) be a continuous map of a compact topological space \(X\). Then \(f\) is topologically conjugate to \(f_\alpha\) if and only if (1)-(3) hold.

1. For each positive integer \(i\), there is a cover \(P_i\) of \(X\) consisting of \(m_i\) pairwise disjoint, nonempty, clopen sets which are cyclically permuted by \(f\).
2. For each positive integer \(i\), \(P_{i+1}\) partitions \(P_i\).
3. If \(W_1 \supset W_2 \supset W_3 \supset \cdots\) is a nested sequence with \(W_i \in P_i\) for each \(i\), then \(\bigcap_{i=1}^{\infty} W_i\) consists of a single point.

Moreover, in this case statement (4) also holds.

4. \(X\) is metrizable and if \(\text{mesh}(P_i)\) denotes the maximum diameter of an element of the cover \(P_i\), then \(\text{mesh}(P_i) \to 0\) as \(i \to \infty\).

Fix \(\alpha = (j_1, j_2, \ldots)\). Let \(M_\alpha\) denote a function whose domain is the set of all prime numbers and which maps to the extended natural numbers \(\{0, 1, 2, \ldots, +\infty\}\). The function \(M_\alpha\) is defined by

\[
M_\alpha(p) = \sum_{i=1}^{\infty} n_i
\]

where \(n_i\) is the power of the prime \(p\) in the prime factorization of \(j_i\).

The following theorem is a beautiful characterization of adding machines up to topological conjugacy due to Block and Keesling [3].

**Theorem 2.2.** (See [3, Corollary 2.8].) Let \(\alpha, \beta \in (\mathbb{N} \setminus \{1\})^\mathbb{N}\). Then \(f_\alpha\) and \(f_\beta\) are topologically conjugate if and only if \(M_\alpha = M_\beta\).

This allows us to precisely state the results from [5].

**Theorem 2.3.** (See [5, Theorem 3.5].) Let \(M\) be an \(n\)-dimensional manifold or the Cantor space, and call an \(\alpha\)-adic adding machine of type \(\infty\) if \(M_\alpha(p) = \infty\) for all \(p\). The set \(\{(x, f) \in M \times C(M, M) : \omega(x, f)\) is an odometer of type \(\infty\}\) is residual in \(M \times C(M, M)\).

**Cellular automata**

**Definition 2.4.** Let \(\mathcal{A}\) be a finite alphabet with \(S\) the space of all doubly infinite sequences with entries from \(\mathcal{A}\). We define the following metric on \(S\):

\[
d((\ldots, x_{-1}, x_0, x_1, \ldots), (\ldots, y_{-1}, y_0, y_1, \ldots)) = \sum_{i=0}^{\infty} \frac{\delta(x_i, y_i)}{2^i} + \sum_{i=1}^{\infty} \frac{\delta(x_{-i}, y_{-i})}{2^i},
\]

where, for \(i \in \mathbb{Z}\), \(\delta(x_i, y_i) = 0\) if \(x_i = y_i\) and \(\delta(x_i, y_i) = 1\) if \(x_i \neq y_i\).

**Definition 2.5.** (See [4, p. 816].) A **cellular automaton** is a continuous, shift-commuting self-map, defined on the compact metric space of all doubly infinite sequences with entries from a finite alphabet. As a consequence of the Curtis–Hedlund–Lyndon Theorem [9] every cellular automaton \(\Phi\) is given by a local rule \(\phi\): for some \(r \geq 0\), for all \(x\), and for all \(i, -\infty < i < \infty\),

\[
[\Phi(x)]_i = \phi(x_{i-r}, x_{i-r+1}, \ldots, x_{i+r}).
\]

A cellular automaton \(\Phi\) has anticipation \(r > 0\) if and only if there exist \(\bar{t}_r, \bar{t}_{r+1}, \ldots, \bar{t}_{r-1}\) such that \(\phi(\bar{t}_r, \bar{t}_{r+1}, \ldots, \bar{t}_{r-1}, \cdot)\), as a function of \(t_r\), is not the constant function.

A cellular automaton \(\Phi\) has memory if and only if \(\phi(t_{-r}, t_{-r+1}, \ldots, t_r)\), depends only on \(t_0, t_1, \ldots, t_r\). In this case we omit \(t_{-r}, t_{-r+1}, \ldots, t_{-1}\) and we write \(\phi(t_0, t_1, \ldots, t_r)\).

Finally, \(\phi\), with no memory and anticipation \(r > 0\), is left permutive if and only if, for every \(\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_r, \phi(\cdot, \bar{t}_1, \bar{t}_2, \ldots, \bar{t}_r)\), as a function of \(t_0\), is a permutation of the alphabet.

In what follows, given a cellular automaton \(\Phi\), by \(\Phi_R\) we denote the one-sided cellular automaton, defined on the space of all one-sided sequences by the same local rule as \(\Phi\).

**Definition 2.6.** Let \(\mathcal{A}\) be a finite alphabet. An \(n\)-block from \(\mathcal{A}\) is \((x_0, x_1, \ldots, x_{n-1})\), where \(x_i \in \mathcal{A}\), \(0 \leq i \leq n - 1\).
Lemma 2.7. Let \( \Phi \) be a left permutative cellular automaton with no memory, defined on the space \( S \) of all doubly infinite sequences with entries from a finite alphabet \( A = \{0, 1, \ldots, b - 1\} \). If \( \Phi^n(x) : n \geq 0 \) is infinite and \((x_1, x_2, \ldots)\) is \( \Phi_R \)-periodic, then \( \Phi : cl(\Phi^n(x) : n \geq 0) \rightarrow cl(\Phi^n(x) : n \geq 0) \) is topologically conjugate to an \( \alpha \)- odometer, where \( \alpha = (p_1, p_2, p_3, \ldots) \) and \( 2 \leq p_1 \leq b \) is prime for all \( i \).

Proof. Suppose \((x_1, x_2, \ldots)\) is \( \Phi_R \)-periodic of period \( m \). Since \( \Phi^n(x) : n \geq 0 \) is infinite, it follows that \( \Phi^{mn}(x) : n \geq 0 \) is infinite, too. Since \( \Phi^n(x) : n \geq 0 \) is infinite and \((x_1, x_2, \ldots)\) is \( \Phi_R \)-fixed, Theorem 1.1 indicates that \( \Phi^m : cl(\Phi^{mn}(x) : n \geq 0) \rightarrow cl(\Phi^{mn}(x) : n \geq 0) \) is topologically conjugate to an odometer, say of type \( \alpha = (s_1, s_2, s_3, \ldots) \). But this implies that \( \Phi : cl(\Phi^n(x) : n \geq 0) \rightarrow cl(\Phi^n(x) : n \geq 0) \) is topologically conjugate to an odometer of type \((m, s_1, s_2, s_3, \ldots)\) \cite[Corollary, p. 817]{4}.

We note that \( m \leq b \) and \( s_i \leq b \) for all \( i \). From Corollary 2.8 of [3] one sees that each \( p_j \) is a divisor of at least one element of the sequence \((m, s_1, s_2, s_3, \ldots)\). Thus, \( 2 \leq p_i \leq b \). 

3. Existence and structure of odometers

Suppose \( \Phi \) is a left permutative cellular automaton, \( \Phi^n \) has positive anticipation for any \( n \), and \((z_1, z_2, z_3, \ldots)\) is \( \Phi_R \)-periodic. Proposition 3.1 shows that \( (\Phi, x) \) generates an odometer whenever \( x \) contains every \( n \)-block, and \( x_i = z_i \) for all \( i \geq 1 \).

Proposition 3.2 considers the sets \( T = \{y \in S: y_i = z_i \text{ for every } i \geq 1\} \) and \( Z = \{x \in T: x \text{ contains every } n \text{-block}\} \), and shows that \( Z \) is large in \( T \) both topologically and in measure.

Moreover, Proposition 3.3 indicates that \((\Phi, x)\) generates a unique odometer for all \( x \) in \( Z \), and that it is possible to describe this odometer using the characterization theorem found in [3]. Theorem 3.4 and Corollary 3.5 bring together and summarize the results from these propositions.

Proposition 3.1. Let \( \Phi \) be a left permutative cellular automaton with no memory, defined on the space \( S \) of all doubly infinite sequences with entries from a finite alphabet. If \( \Phi^n \) has positive anticipation for every \( n \), and \((z_1, z_2, \ldots)\) is \( \Phi_R \)-periodic, then \( \Phi^{mn}(x) : m \geq 0 \) is infinite whenever \( x \in T = \{x \in S: x_i = z_i \text{ for every } i \geq 1\} \) and \( x \) contains every \( n \)-block.

Proof. Say \((z_1, z_2, \ldots)\) is \( \Phi_R \)-periodic of period \( n \), and suppose that \( \Phi^{jn}(x) : j \geq 0 \) is finite. We consider two possibilities.

First, let us assume that \( x \) is \( \Phi^n \)-periodic, say of period \( p \). Then \( x \in Fix(\Phi^{mp}) \). But \( \Phi^{mp} \) has positive anticipation by hypothesis, so that \( \Phi^{mp} \) is not the identity map, and \( x \) contains every \( n \)-block.

Now, let us assume that \( x \) is eventually periodic with respect to \( \Phi^n \), but not periodic. Then, there exists some \( y \in \omega(x, \Phi^n) \) so that \( \Phi^{-n}(y) \) contains at least two elements. But \( \Phi^n \) is a homeomorphism from \( y \in S: y_i = z_i \text{ for every } i \geq 1 \) to itself (see the proof of [4, Theorem 2, p. 818]).

We conclude that \( \Phi^{jn}(x) : j \geq 0 \) is infinite. Since \( \Phi^{jn}(x) : j \geq 0 \subseteq \Phi^{mn}(x) : m \geq 0 \), our conclusion follows.

Proposition 3.2. Let \( A \) be a finite alphabet, and take \( \{z_i\}_{i=1}^{\infty} \) to be a sequence of elements from \( A \). We let \( T = \{y \in S: y_i = z_i \text{ for every } i \geq 1\} \) be a subset of the space \( S \) of all doubly infinite sequences with entries from \( A \), and set \( Z = \{x \in T: x \text{ contains every } n \text{-block}\} \).

Then

1. \( Z \) is a dense \( G_\delta \) subset of \( T \).
2. there exists a measure \( \mu \) on \( T \) so that \( \mu(T \setminus Z) = 0 \).

Proof. We first show that \( Z \) is a dense \( G_\delta \) subset of \( T \). Since the collection of all \( n \)-blocks is countable, it suffices to show that those elements of \( T \) which contain a particular \( n \)-block comprise a dense, open subset of \( T \). Take \( x \in T, \epsilon > 0 \) and \((z_{n-1}, z_{n-2}, \ldots, z_0)\) an \( n \)-block of elements from \( A \). We show that \( \{y \in T: y \text{ contains the } n \text{-block } (z_{n-1}, z_{n-2}, \ldots, z_0)\} \) is dense and open.

Let \( m \in \mathbb{N} \) so that \( \frac{1}{mp} < \epsilon \). If \( y \in T \) so that \( y_i = x_i \) whenever \( i > -m \), then \( d(y, x) < \epsilon \). We need now only note that the set \( \{y \in T: y_i = z_i \text{ for all } i \geq -m \text{ and } y_{-m-i} = z_i \text{ for } 0 \leq i \leq n - 1\} \) is open in \( T \).

We now show that there exists a measure \( \mu \) on \( T \) so that \( \mu(T \setminus Z) = 0 \). Suppose that the alphabet \( A \) contains \( b \) elements, so that we may take \( A = \{0, 1, \ldots, b - 1\} \). To each \( x \in T \) associate \( \psi(x) = y \in [0, 1] \), where \( y = x_0x_{-1}x_{-2}x_{-3} \ldots \) is a number in base \( b \). It follows that the map \( \psi : T \rightarrow [0, 1] \) is surjective, but not one-to-one. In fact, if \( x \neq y \) in \( T \) and \( \psi(x) = \psi(y) \) in \( [0, 1] \), then \( \psi(x) = x_0x_1x_2 \ldots x_{-1}(b - 1)(b - 1) \ldots \) and \( \psi(y) = x_0x_1x_2 \ldots x_{-1}(q + 1)00000 \ldots (\frac{b}{b} + \sum_{i=-m}^{n-1} \frac{1}{b^{i+1}} = \frac{b}{b^0}) \), for some natural number \( n \). We note that this is possible for only countably many points in \( T \).

Now, let \( R \subset T \). Take \( \mu^\ast(R) \) to be the Lebesgue outer measure of the \( \psi \)-image of \( R \), \( \lambda^\ast(\psi(R)) \). Since the map \( \psi : T \rightarrow [0, 1] \) is surjective, it follows that \( \mu(T) = 1 \). Moreover, \( \mu(T \setminus Z) = 0 \) as \( \mu(T \setminus Z) = \lambda(\psi(T \setminus Z)) = \lambda^\ast(\psi(T \setminus Z)) = 0 \).

Proposition 3.3. Let \( \Phi \) be a left permutative cellular automaton with no memory, defined on the space \( S \) of all doubly infinite sequences with entries from a finite alphabet \( A = \{0, 1, \ldots, b - 1\} \). Suppose \( \Phi^n(x) : n \geq 0 \) is infinite, \((x_1, x_2, \ldots)\) is \( \Phi_R \)-periodic, \( x \) contains
every $n$-block and $\Phi : cl(\Phi^n(x); \ n \geq 0) \to cl(\Phi^n(x); \ n \geq 0)$ is topologically conjugate to the $\alpha$-odometer. Then, for every prime $p$, $2 \leq p \leq b$, either $M_\alpha(p) = 0$ or $M_\alpha(p) = \infty$.

Proof. We show that if $M_\alpha(p) > 0$, then $M_\alpha(p) = \infty$. Fix $p$, let $k \in \mathbb{Z}$ with $q \in \mathbb{N}$ minimal so that $(x_{-k}, \ x_{-k+1}, \ldots)$ is $\Phi^q$-fixed and $p$ divides $q$. Then $[\Phi^q(x)]_{-k} = x_{-k}$, and if $\Phi$ has $r$-anticipation, then $[\Phi^q(x)]_{-k}$ is a function of the $qr + 1$-block $(x_{-k}, x_{-k+1}, \ldots, x_{-k+qr})$.

Since $x$ contains every $n$-block, $x$ contains the $qr + 1$-block $(x_{-k}, x_{-k+1}, \ldots, x_{-k+qr})$ infinitely often. Pick $l$ so that $(x_{-i}, x_{-i+1}, \ldots, x_{-i+qr}) = (x_{-k}, x_{-k+1}, \ldots, x_{-k+qr})$ and $-l + qr < -k$. Take $\tilde{q}$ in $\mathbb{N}$ minimal so that $(x_{-i}, x_{-i+1}, \ldots)$ is $\Phi^\tilde{q}$ fixed. It follows that $p$ divides $\frac{q}{\tilde{q}}$. □

Theorem 3.4. Let $\Phi$ be a left permutative cellular automaton with no memory, defined on the space $S$ of all doubly infinite sequences with entries from a finite alphabet, such that $\Phi^n$ has positive anticipation for all $n$. Suppose $(z_1, z_2, z_3, \ldots)$ is $\Phi^k$-periodic, $T = \{y \in S; \ y_i = z_i \text{ for all } i \geq 1\}$, and $\mathcal{Z} = \{x \in T; \ x \text{ contains every } n \text{-block}\}$. Then

1. $\mathcal{Z}$ is a dense $G_\delta$ subset of $T$.
2. There is a measure $\mu$ on $T$ so that $\mu(T \setminus \mathcal{Z}) = 0$.
3. If $x \in \mathcal{Z}$, then $\Phi : cl(\Phi^n(x); \ n \geq 0) \to cl(\Phi^n(x); \ n \geq 0)$ is topologically conjugate to an odometer, and
4. $(\Phi, x)$ generates a unique $\alpha$-odometer for all $x \in \mathcal{Z}$. Moreover, for any prime $p$, $2 \leq p \leq b$, either $M_\alpha(p) = 0$ or $M_\alpha(p) = \infty$.

Proof. Statements (1) and (2) follow from Proposition 3.2, statement (3) follows from Proposition 3.1, and statement (4) follows from Proposition 3.3. □

Corollary 3.5. Let $A$ be a finite alphabet with $S$ the space of all doubly infinite sequences with entries from $A$. Fix $k \in \mathbb{Z}$ and $c \in A$, and set

$$
\mathcal{T}_{c,k} = \{y \in S; \ y_i = c \text{ for all } i \geq k\},
$$

$$
\mathcal{Z}_{c,k} = \{x \in \mathcal{T}_{c,k}; \ x \text{ contains every } n \text{-block}\},
$$

and

$$
\tilde{S} = \bigcup_{k \in \mathbb{Z}} \bigcup_{x \in A} \mathcal{Z}_{c,k}.
$$

Then

1. $\tilde{S}$ is dense in $S$.
2. If $\Phi$ is a left permutative cellular automaton with no memory and $\Phi^n$ has positive anticipation for all $n$, and $x \in \tilde{S}$, then $(\Phi, x)$ generates a unique $\alpha$-odometer for all $x \in \mathcal{Z}$. Moreover, for any prime $p$, $2 \leq p \leq b$, either $M_\alpha(p) = 0$ or $M_\alpha(p) = \infty$.

Proof. Since $\bigcup_{k \in \mathbb{Z}} \bigcup_{x \in A} \mathcal{T}_{c,k}$ is dense in $S$, and $\mathcal{Z}_{c,k}$ is dense in $\mathcal{T}_{c,k}$ by Proposition 3.2, one sees that $\tilde{S} = \bigcup_{k \in \mathbb{Z}} \bigcup_{x \in A} \mathcal{Z}_{c,k}$ is dense in $S$. This proves part (1).

Moreover, $(c, c, \ldots)$ is $\Phi^k$-periodic for every left permutative cellular automaton $\Phi$ with no memory and positive anticipation for all $n$, so that part (2) follows from Proposition 3.1. Part (3) is a consequence of Proposition 3.3. □

References