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The Wente problem for a singular elliptic equation

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1. Introduction and statement of the results

The Jacobian $\det \nabla u$ of some vector field u belonging to $W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ plays a particular role in various partial differential equation arising from calculus of variations, mechanics, and geometry. Although $\det \nabla u$ is only integrable, but due to its special structure this quantity has some suitable regularity properties. Recently in [8], Coifman et al. have shown that $\det \nabla u$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$, a strict subspace of $L^1(\mathbb{R}^n)$.

In this paper, we shall focus on the case $n = 2$. Let Ω be a smooth and bounded domain in $\mathbb{R}^2$. Given $\alpha \not\equiv 0$ a non-negative measurable function and let $u = (a, b)$ and ξ be functions defined on Ω. Consider the following problem:

$$\begin{cases} -\operatorname{div}(\alpha \nabla \varphi) = \xi \det \nabla u = \xi(a_{x_1} b_{x_2} - a_{x_2} b_{x_1}) & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $x = (x_1, x_2)$ and a, b are two functions belonging to some weighted Sobolev spaces which will be defined later. For $i = 1, 2$, a_{x_i} denote the partial derivative with respect to the variable x_i. If $\Omega = \mathbb{R}^2$, we consider the limit condition $\lim_{|x| \to +\infty} \varphi(x) = 0$, where $|x| = r = (x_1^2 + x_2^2)^{1/2}$. In the case $\alpha = \xi = \text{const}$, problem (1) is the classical Wente problem

$$\begin{cases} -\Delta \psi = \det \nabla u = a_{x_1} b_{x_2} - a_{x_2} b_{x_1} & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

The classical Wente problem arises in the study of constant mean curvature immersions. The function ξ in (1) plays the role of the mean curvature of the surface $(\varphi, a, b)(\Omega)$. When $u = (a, b) \in H^1(\Omega, \mathbb{R}^2)$, it is proved in [5,12,13] that ψ, the solution of (2), is in

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$L^\infty(\Omega)$. In particular, this provides control of $\nabla\psi$ in $L^2(\Omega)$ and continuity of ψ by simple arguments. We also have

$$\|\psi\|_\infty + \|\nabla\psi\|_2 \leqslant C_0(\Omega)\|\nabla a\|_2\|\nabla b\|_2. \tag{3}$$

Denote by

$$C_\infty(\Omega) = \sup_{\nabla a, \nabla b \neq 0} \frac{\|\psi\|_\infty}{\|\nabla a\|_2\|\nabla b\|_2} \tag{4}$$

and

$$C_2(\Omega) = \sup_{\nabla a, \nabla b \neq 0} \frac{\|\nabla\psi\|_2}{\|\nabla a\|_2\|\nabla b\|_2}. \tag{5}$$

It is proved in [1,11,13] that $C_\infty(\Omega) = 1/(2\pi)$ and in [9] that $C_2(\Omega) = \sqrt{3/(16\pi)}$ (see also [10]).

In [2], we give a generalization of problem (2) in Higher dimensions. We suppose that $u \in W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ and we replace the operator $-\Delta$ in (2) by $(-\Delta)^{n/2}$. We proved that ψ is in $L^\infty(\mathbb{R}^n)$, for $1 \leqslant k \leqslant n/2$, $\nabla^k\psi$ is in $L^{n/k}(\mathbb{R}^n)$ and we also

$$\|\psi\|_\infty + \|\nabla^k\psi\|_{n/k} \leqslant C\|\nabla u\|_n^n.$$

Moreover, we give the best constant involving the L^∞ norm. We can imagine another generalization of problem (2) by using for example the nonlinear operator $-\Delta_n\psi = -\operatorname{div}(|\nabla\psi|^{n-2}\nabla\psi)$ which is conformally invariant, that is why we are interested in this problem. We will give in the appendix a negative answer to this problem. Finally, in [3] we always deal with problem (2) on $\mathbb{R}^2$, but we suppose that the functions a and b (like in [6]) belong to some radial weighted Sobolev spaces. We prove some similar results depending on the value of the weight on zero.

Bethuel and Ghidaglia, also Chanillo and Li have showed in [4] and [7], that if we consider the problem

$$\begin{cases} -\sum_{i,j=1}^2 \partial_i(a_{ij}\partial_j\phi) = a_{x_1}b_{x_2} - a_{x_2}b_{x_1} & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where $a_{ij} = a_{ji} \in L^\infty(\Omega)$ and for some $\lambda > 0$, we have the ellipticity condition

$$\lambda^{-1}|\xi|^2 \leqslant \sum_{i,j=1}^2 a_{ij}(x)\xi_i\xi_j \leqslant \lambda|\xi|^2,$$

then $\phi \in C(\bar{\Omega}) \cap H_0^1(\Omega)$ and

$$\|\phi\|_\infty + \|\nabla\phi\|_2 \leqslant C_1\|\nabla a\|_2\|\nabla b\|_2,$$

where C_1 is independent of Ω.

Note that (1) is invariant by conformal transformations. More precisely, let $T : \Omega \to \Omega'$ be a conformal transformation, if we denote by $\tilde{a} = a \circ T$, $\tilde{b} = b \circ T$, $\tilde{\alpha} = \alpha \circ T$, and $\tilde{\xi} = \xi \circ T$, then the solution $\tilde{\varphi}$ of (1) if we replace a by $\tilde{a}$, b by $\tilde{b}$, α by $\tilde{\alpha}$ and ξ by $\tilde{\xi}$ is $\tilde{\varphi} = \varphi \circ T$.

In the following, we will suppose that $\Omega = B_1$ the unit disc of $\mathbb{R}^2$. We say that α and ξ satisfy (A_1) if

(A_1) α and ξ are a radial functions (i.e., $\alpha(x)=\beta(r)$ and $\xi(x)=\zeta(r)$, where $r=|x|$). We suppose also that $\beta>0$ in $(0,1]$ and β,ζ in $C^3((0,1])$.

We need to introduce some assumptions. We suppose that α and ξ satisfy (A_1). We say that α and ξ satisfy (A_2)–(A_4) if we have

(A_2) $$\lim_{r\to 0} r\beta(r)\int_r^1 \beta^{-1}(t)\,\frac{dt}{t}=\lim_{r\to 0} r\zeta(r)\int_r^1 \beta^{-1}(t)\,\frac{dt}{t}=0;$$

(A_3) $$\omega(x)\equiv\omega(\alpha,\xi)(x)=-r\partial_r\left[\zeta(r)\int_r^1 \beta^{-1}(t)\,\frac{dt}{t}\right]$$

is a positive function on $(0,1]$;

(A_4) We have $\omega\in C^2((0,1])$ by (A_1), we assume that $\Delta(\sqrt{\omega})\geqslant 0$ on $(0,1]$.

Remark 1. (1) In the case of the classical Wente problem (i.e., $\alpha=\xi=\text{const}$), the quantity $\omega\equiv 1$.

(2) Under assumptions (A_3) and (A_4), we have

$$h_\omega(x)=|x|^2\left(\frac{1}{2}\Delta\omega-\frac{1}{4}|\nabla\omega|^2\omega^{-1}\right)\omega^{-1}\geqslant 0.$$

(3) If we denote by G the Green function associated to the operator $-\operatorname{div}(\alpha\nabla\cdot)$ on B_1, i.e.,

$$\begin{cases} -\operatorname{div}(\alpha\nabla G)=\delta_0 & \text{in } B_1,\\ G=0 & \text{on } \partial B_1. \end{cases} \tag{7}$$

Under hypothesis (A_1) and (A_2), we have

$$G(x)=G(r)=\frac{1}{2\pi}\int_r^1 \beta^{-1}(t)\,\frac{dt}{t}. \tag{8}$$

Let $\omega\not\equiv 0$ be a non-negative function and we suppose that $\omega\in L^1_{\mathrm{loc}}(B_1)$. Define

$$\|f\|_{2,\omega}=\left(\int_{B_1}|f|^2\omega\,dx\right)^{1/2}. \tag{9}$$

We then define the space $H_\omega(B_1)$ which is the closure of $\mathcal{D}(B_1)$ endowed with the norm $\|\nabla\cdot\|_{2,\omega}$. Here $\mathcal{D}(B_1)$ is the space of C^∞, compact supported functions. Denote by

$$V=\big\{(a,b)\in H_\omega(B_1)\times H_\omega(B_1)\text{ such that } a\not\equiv\text{const and } b\not\equiv\text{const}\big\}. \tag{10}$$

Our first result is the following

Theorem 1. *Let φ be the solution of* (1). *Suppose that α and ξ satisfy* (A_1)–(A_3) *and ω defined in* (A_3) *is in $C([0,1])$ with $\omega(0)>0$; then*

$$\sup_{(a,b)\in V}\frac{|\varphi(0)|}{\|\nabla a\|_{2,\omega}\|\nabla b\|_{2,\omega}}=\frac{1}{2\pi}. \tag{11}$$

As a consequence of the proof of Theorem 1, we have

Corollary 1. *Let φ be the solution of* (1); *then for every non-negative function α and for every function ξ* (*theses functions are not necessarily radially symmetric*), *we denote by $\omega(x)=-2\pi r\partial_r(\xi(x)G(x))$. Suppose that ω is a non-negative function, and there exists $x^0\in B_1$ such that ω is continuous and positive in a neighborhood of x^0; then we have*

$$\sup_{(a,b)\in V}\frac{\|\varphi\|_\infty}{\|\nabla a\|_{2,\omega}\|\nabla b\|_{2,\omega}}\geqslant\frac{1}{2\pi}. \tag{12}$$

Consider the space

$$\mathcal{D}_\omega(B_1)=\Big\{f\in\mathcal{D}(B_1)\text{ such that }\lim_{r\to 0}r\omega'(r)f^2(x)=0\Big\} \tag{13}$$

and define the space $\tilde{H}_\omega(B_1)$ which is the closure of $\mathcal{D}_\omega(B_1)$ endowed with the norm $\|\nabla\cdot\|_{2,\omega}$. Suppose that ($A_4$) holds; we introduce the space

$$\tilde{V}=\big\{a,b\in\tilde{H}_\omega(B_1)\text{ such that }a\not\equiv\text{const and }b\not\equiv\text{const}\big\}. \tag{14}$$

We prove

Theorem 2. *Let φ be the solution of* (1). *If α and ξ satisfy* (A_1)–(A_4), *then*

$$\sup_{(a,b)\in\tilde{V}}\frac{|\varphi(0)|}{\|\nabla a\|_{2,\omega}\|\nabla b\|_{2,\omega}}\leqslant\frac{1}{2\pi}\frac{1}{\Big(1+\inf\limits_{B_1}h_\omega\Big)^{1/2}}. \tag{15}$$

We now consider the symmetry case. Let

$$V_r=\big\{(a,b)\in V,\,(a,b)(x)=\omega^{-1/2}(x)g\big(|x|\big)x\big\}, \tag{16}$$

where $g:B_1\to\mathbb{R}$ is some regular function. We still suppose that α and ξ satisfy (A_1), we say that α and ξ satisfy (A_5)–(A_7), if

(A_5) $\lim\limits_{r\to 0}r^3\omega'(r)\omega^{-1}(r)=0$;
(A_6) $\lim\limits_{r\to 0}r\beta(r)=\lim\limits_{r\to 0}r^2\omega^{-1}(r)\zeta(r)=0$;
(A_7) $\forall r\in(0,1]$, we have $\zeta(r)\geqslant 0$.

We have

Theorem 3. *Suppose that α and ξ satisfy* (A_1)–(A_5), *$h_w\in L^\infty(B_1)$, and $(a,b)\in V_r$. If φ is the solution of* (1), *then*

$$\frac{1}{2\pi}\frac{1}{\left(1+\sup_{B_1} h_\omega\right)} \leqslant \sup_{(a,b)\in V_r}\frac{|\varphi(0)|}{\|\nabla a\|_{2,\omega}\|\nabla b\|_{2,\omega}} \leqslant \frac{1}{2\pi}\frac{1}{\left(1+\inf_{B_1} h_\omega\right)}. \tag{17}$$

Moreover, if we assume that α and ξ satisfy (A_6) *and* (A_7)*, then $\varphi \in L^\infty(B_1)$ and we have*

$$\sup_{(a,b)\in V_r}\frac{\|\varphi\|_\infty}{\|\nabla a\|_{2,\omega}\|\nabla b\|_{2,\omega}} \leqslant \frac{1}{2\pi}\frac{\max\left\{1, \sup_{B_1}\left(\xi/(\alpha\omega)\right)\right\}}{\left(1+\inf_{B_1} h_\omega\right)}. \tag{18}$$

Remark 2. (1) A family of examples α and ξ satisfying (A_1)–(A_7) is

$$\alpha(x) = \xi(x) = |x|^s \quad \text{with } s > -1.$$

In this case, we have

$$\omega(x) = |x|^s \quad \text{and} \quad h_\omega = \frac{s^2}{4}.$$

(2) When we study the classical Wente problem, we note that the supremum of

$$\frac{\|\varphi\|_\infty}{\|\nabla a\|_2\|\nabla b\|_2}$$

is the same on V and V_r, but here using Theorem 3 and Corollary 1, we observe a gap phenomenon. In particular, for the case $\alpha(x) = \xi(x) = |x|^s$, then $\omega(x) = |x|^s$ and $-1 < s \leqslant 0$, we get by (17) and (18),

$$\sup_{(a,b)\in V_r}\frac{\|\varphi\|_\infty}{\|\nabla a\|_{2,\omega}\|\nabla b\|_{2,\omega}} = \frac{1}{2\pi}\frac{1}{1+s^2/4}.$$

(3) We can prove some similar result to (18) without supposing assumption (A_7). Indeed, following the proof of Theorem 3 and using (29) and (30), we have

$$\sup_{(a,b)\in V_r}\frac{\|\varphi\|_\infty}{\|\nabla a\|_{2,\omega}\|\nabla b\|_{2,\omega}} \leqslant \frac{1}{2\pi}\frac{\left(1+\sup_{B_1}\left(|\xi|/(\alpha\omega)\right)\right)}{\left(1+\inf_{B_1} h_\omega\right)}. \tag{19}$$

Corollary 2. *There exist α and ξ satisfying* (A_1)–(A_7) *such that*

$$\sup_{(a,b)\in V_r}\frac{\|\varphi\|_\infty}{\|\nabla a\|_{2,\omega}\|\nabla b\|_{2,\omega}} < \sup_{(a,b)\in V}\frac{\|\varphi\|_\infty}{\|\nabla a\|_{2,\omega}\|\nabla b\|_{2,\omega}}. \tag{20}$$

2. Proofs of results

2.1. Proofs of Theorem 1 and Corollary 1

We will suppose that a and b belong to $\mathcal{D}(B_1)$. The general case can be obtained by approximating a and b by C^∞ functions with compact support. The proof of (11) follows similar arguments used in [1,11,13]. In fact, we have

$$\varphi(0) = \int_{B_1} G(x)\xi(x)(a_{x_1}b_{x_2} - a_{x_2}b_{x_1})\,dx = \int_0^1 \int_0^{2\pi} G(r)\zeta(r)(a_r b_\theta - a_\theta b_r)\,d\theta\,dr$$
$$= \int_0^1 \int_0^{2\pi} G(r)\zeta(r)\big[(ab_\theta)_r - (ab_r)_\theta\big]\,d\theta\,dr = \frac{1}{2\pi}\int_0^1 \int_0^{2\pi} \frac{1}{r}\omega(r)(a-\bar{a})b_\theta\,d\theta\,dr,$$

where we have used

$$\omega(r) = -2\pi r\partial_r\big(G(r)\zeta(r)\big) \quad \text{and} \quad \bar{a}(r) = \frac{1}{2\pi}\int_0^{2\pi} a(re^{i\sigma})\,d\sigma.$$

Using

$$\int_0^{2\pi} |a-\bar{a}|^2\,d\theta \leqslant \int_0^{2\pi} a_\theta^2\,d\theta,$$

we have

$$\big|\varphi(0)\big| \leqslant \frac{1}{2\pi}\int_0^1 \frac{1}{r}\left(\int_0^{2\pi} a_\theta^2\,d\theta\right)^{1/2}\left(\int_0^{2\pi} b_\theta^2\,d\theta\right)^{1/2}\omega\,dr$$
$$\leqslant \frac{1}{2\pi}\left(\int_0^1\int_0^{2\pi} \frac{1}{r}a_\theta^2\omega\,d\theta\,dr\right)^{1/2}\left(\int_0^1\int_0^{2\pi} \frac{1}{r}b_\theta^2\omega\,d\theta\,dr\right)^{1/2}$$
$$\leqslant \frac{1}{2\pi}\left(\int_{B_1} |\nabla a|^2\omega\,dx\right)^{1/2}\left(\int_{B_1} |\nabla b|^2\omega\,dx\right)^{1/2}$$
$$\leqslant \frac{1}{2\pi}\|\nabla a\|_{2,\omega}\|\nabla b\|_{2,\omega}.$$

The second inequality is valid by (A_1) and (A_3). So we get

$$\sup_{(a,b)\in V} \frac{|\varphi(0)|}{\|\nabla a\|_{2,\omega}\|\nabla b\|_{2,\omega}} \leqslant \frac{1}{2\pi}. \tag{21}$$

Now we turn to prove the inverse inequality of (21) which gives also the proof of Corollary 1. Here we do not need to suppose (A_1). First, we mention that by the conformal transformations, we can suppose that $x^0 = 0$. Let g be a function defined in $(0,\infty)$ such that $\lim_{r\to 0} r\log r g(r) = 0$, $g(r) = 0$ if $r \geqslant 1$ and $(a,b)(x) = g(|x|)x \in V$.

Let $\rho > 0$, denote by $(a^\rho, b^\rho)(x) = (a,b)(x/\rho) = g(r/\rho)(x/\rho)$ and φ^ρ the solution of (1) corresponding to a^ρ and b^ρ. Since α and ξ are not supposed symmetric, then we define ω by $\omega(x) \equiv \omega(\alpha,\xi)(x) = -2\pi r\partial_r(\xi(x)G(x))$. Throughout, we will omit the variable r of radial functions g, g' for the simplicity. Note that

$$G(x)\xi(x) = -\frac{\omega(0)}{2\pi}\log r\big(1 + o_r(1)\big).$$

As in [1], we have after a change of variables

$$\varphi^\rho(0) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^1 r\omega(\rho x) g^2(r)\,dr\,d\theta \xrightarrow{\rho\to 0} \frac{\omega(0)}{2}\int_0^1 r g^2(r)\,dr.$$

Furthermore,

$$\|\nabla a^\rho\|_{2,w}^2 = \int_0^{2\pi}\int_0^1 \left[r g^2 + r^3 g'^2 \cos^2\theta + r^2 (g^2)'\cos^2\theta\right]\omega(\rho x)\,dr\,d\theta.$$

Then we have

$$\|\nabla a^\rho\|_{2,\omega}^2 \xrightarrow{\rho\to 0} \pi\omega(0)\int_0^1 r^3 g'^2\,dr.$$

In the same way, we prove that

$$\|\nabla b^\rho\|_{2,\omega}^2 \xrightarrow{\rho\to 0} \pi\omega(0)\int_0^1 r^3 g'^2\,dr.$$

Finally choosing $g_\varepsilon(r) = r^{\varepsilon/2-1} - 1$ with $\varepsilon > 0$, we have

$$\lim_{\rho\to 0}\left|\varphi_\varepsilon^\rho(0)\right| = \frac{\omega(0)}{2\varepsilon}\bigl(1 + o_\varepsilon(1)\bigr)$$

and

$$\lim_{\rho\to 0}\left\|\nabla a_\varepsilon^\rho\right\|_{2,\omega}^2 = \lim_{\rho\to 0}\left\|\nabla b_\varepsilon^\rho\right\|_{2,\omega}^2 = \frac{\pi\omega(0)}{\varepsilon}\bigl(1 + o_\varepsilon(1)\bigr).$$

We deduce that

$$\lim_{\rho\to 0}\frac{|\varphi_\varepsilon^\rho(0)|}{\|\nabla a_\varepsilon^\rho\|_{2,\omega}\|\nabla b_\varepsilon^\rho\|_{2,\omega}} \to \frac{1}{2\pi} \quad \text{as } \varepsilon \to 0. \qquad \square$$

2.2. Proof of Theorem 2

Here also, we suppose that a and b belong to $\mathcal{D}_\omega(B_1)$. The proof of (15) follows similar arguments used in [3]. Denote by $\tilde a(x) = \omega(x)^{1/2} a(x)$ and $\tilde b(x) = \omega(x)^{1/2} b(x)$; then we have

$$\varphi(0) = \int_{B_1} G(x)\xi(x)(a_{x_1} b_{x_2} - a_{x_2} b_{x_1})(x)\,dx.$$

By an easy computation, we have

$$\begin{aligned}
\det\nabla u &= \omega^{-1}(\tilde a_{x_1}\tilde b_{x_2} - \tilde a_{x_2}\tilde b_{x_1}) - \frac{\omega^{-2}}{2}\left[\tilde a(\omega_{x_1}\tilde b_{x_2} - \omega_{x_2}\tilde b_{x_1}) - \tilde b(\omega_{x_1}\tilde a_{x_2} - \omega_{x_2}\tilde a_{x_1})\right] \\
&= \frac{\omega^{-1}}{r}(\tilde a_r\tilde b_\theta - \tilde a_\theta\tilde b_r) - \frac{\omega^{-2}}{2r}\left[\tilde a(\omega_r\tilde b_\theta - \omega_\theta\tilde b_r) - \tilde b(\omega_r\tilde a_\theta - \omega_\theta\tilde a_r)\right].
\end{aligned}$$

Using (A_2) and (8), we have $\lim_{r\to 0} r\zeta(r)G(r) = 0$. So

$$\begin{aligned}
\varphi(0) &= \int_0^1 G(r)\omega^{-1}(r)\zeta(r)\int_0^{2\pi}(\tilde a_r\tilde b_\theta - \tilde a_\theta\tilde b_r)\,d\theta\,dr \\
&\quad + \frac{1}{2}\int_0^1 G(r)\omega^{-2}(r)\omega'(r)\zeta(r)\int_0^{2\pi}\tilde b^2\left(\frac{\tilde a}{\tilde b}\right)_\theta d\theta\,dr \\
&= \int_0^1 G(r)\omega^{-1}(r)\zeta(r)\int_0^{2\pi}(\tilde a_r\tilde b_\theta - \tilde a_\theta\tilde b_r)\,d\theta\,dr \\
&\quad - \int_0^1 G(r)\omega^{-2}(r)\omega'(r)\zeta(r)\int_0^{2\pi}\tilde a\tilde b_\theta\,d\theta\,dr \\
&= -\int_0^1 \Big(\big(G(r)\omega^{-1}(r)\zeta(r)\big)' + G(r)\omega^{-2}(r)\omega'(r)\zeta(r)\Big)\int_0^{2\pi}\tilde a\tilde b_\theta\,d\theta\,dr \\
&= -\int_0^1 \omega^{-1}(r)\big(\zeta(r)G(r)\big)'\int_0^{2\pi}\tilde a\tilde b_\theta\,d\theta\,dr \\
&= \frac{1}{2\pi}\int_0^1\int_0^{2\pi}\frac{1}{r}(\tilde a - \bar{\tilde a})\tilde b_\theta\,d\theta\,dr,
\end{aligned}$$

where

$$\bar{\tilde a}(r) = \frac{1}{2\pi}\int_0^{2\pi}\tilde a(re^{i\sigma})\,d\sigma.$$

We have then

$$\big|\varphi(0)\big| \leqslant \frac{1}{2\pi}\left(\int_0^1\int_0^{2\pi}\frac{1}{r}|\tilde a - \bar{\tilde a}|^2\,d\theta\,dr\right)^{1/2}\left(\int_0^1\int_0^{2\pi}\frac{1}{r}b_\theta^2\,d\theta\,dr\right)^{1/2}.$$

Next, we will evaluate $\|\nabla a\|_{2,\omega}$. We have

$$\|\nabla a\|_{2,\omega}^2 = \int_{B_1}\big|\nabla[\omega^{-1/2}\tilde a]\big|^2\omega\,dx = \int_{B_1}\frac{\tilde a^2}{4}\omega^{-2}|\nabla\omega|^2 - \tilde a\omega^{-1}\nabla\tilde a\cdot\nabla\omega + |\nabla\tilde a|^2\,dx.$$

We have

$$2\int_{B_1}\tilde a\omega^{-1}\nabla\tilde a\cdot\nabla\omega\,dx = \int_{B_1}\tilde a^2\big(|\nabla\omega|^2\omega^{-1} - \Delta\omega\big)\omega^{-1}\,dx + \int_0^{2\pi}\big[r\omega'(r)a^2\big]_0^1\,d\theta.$$

Since $a \in \mathcal{D}_\omega(B_1)$, then

$$\|\nabla a\|_{2,\omega}^2 = \int_0^1 \int_0^{2\pi} \frac{1}{r} h_\omega \tilde{a}^2 \, d\theta \, dr + \int_0^1 \int_0^{2\pi} \left(r\tilde{a}_r^2 + \frac{\tilde{a}_\theta^2}{r} \right) d\theta \, dr. \tag{22}$$

In the same way, we prove that

$$\|\nabla b\|_{2,\omega}^2 = \int_0^1 \int_0^{2\pi} \frac{1}{r} h_\omega \tilde{b}^2 \, d\theta \, dr + \int_0^1 \int_0^{2\pi} \left(r\tilde{b}_r^2 + \frac{\tilde{b}_\theta^2}{r} \right) d\theta \, dr, \tag{23}$$

where the function h_ω is that given in Remark 1. Since

$$\int_0^{2\pi} \tilde{a}^2 \, d\theta \geqslant \int_0^{2\pi} |\tilde{a} - \bar{\tilde{a}}|^2 \, d\theta \quad \text{and} \quad \int_0^{2\pi} \tilde{a}_\theta^2 \, d\theta \geqslant \int_0^{2\pi} |\tilde{a} - \bar{\tilde{a}}|^2 \, d\theta,$$

and using condition (A_4), we deduce that

$$\|\nabla a\|_{2,\omega}^2 \geqslant \left(1 + \inf_{B_1} h_\omega\right) \int_0^1 \int_0^{2\pi} \frac{1}{r} |\tilde{a} - \bar{\tilde{a}}|^2 \, d\theta \, dr$$

and

$$\|\nabla b\|_{2,\omega}^2 \geqslant \int_0^1 \int_0^{2\pi} \frac{1}{r} \tilde{b}_\theta^2 \, d\theta \, dr.$$

Finally, we get

$$\left|\varphi(0)\right| \leqslant \frac{1}{2\pi} \frac{1}{\left(1 + \inf_{B_1} h_\omega\right)^{1/2}} \|\nabla a\|_{2,\omega} \|\nabla b\|_{2,\omega}. \qquad \square$$

2.3. *Proof of Theorem 3*

Let $(a, b) \in V_r$, since ω satisfies (A_5), then $V_r \subset \tilde{V}$. Next, we will prove (17). In this case

$$\varphi(0) = \frac{1}{2} \int_0^1 r g^2 \, dr.$$

According to (22) and (23) it is easy to see that

$$\|\nabla a\|_{2,\omega}^2 = \|\nabla b\|_{2,\omega}^2 = \pi \left(\int_0^1 r h_\omega g^2 \, dr + \int_0^1 r^3 g'^2 \, dr \right) \tag{24}$$

and

$$\int_0^1 r^3 g'^2\,dr = \int_0^1 rg^2\,dr + \int_0^1 r(rg)'^2\,dr. \tag{25}$$

We deduce by (24) and (25) that

$$\|\nabla a\|_{2,\omega}\|\nabla b\|_{2,\omega} \geqslant \pi\Big(1+\inf_{B_1} h_\omega\Big)\int_0^1 rg^2\,dr \tag{26}$$

and then

$$\sup_{(a,b)\in V_r} \frac{|\varphi(0)|}{\|\nabla a\|_{2,\omega}\|\nabla b\|_{2,\omega}} \leqslant \frac{1}{2\pi}\frac{1}{\Big(1+\inf_{B_1} h_\omega\Big)}.$$

Also, by (24), we have

$$\frac{|\varphi(0)|}{\|\nabla a\|_{2,\omega}\|\nabla b\|_{2,\omega}} \geqslant \frac{1}{2\pi}\frac{\int_0^1 rg^2\,dr}{\sup_{B_1} h_\omega \int_0^1 rg^2\,dr + \int_0^1 r^3 g'^2\,dr}. \tag{27}$$

Taking $g_\varepsilon(r) = r^{\varepsilon-1}$, with $\varepsilon > 0$, then we can check that $(a_\varepsilon, b_\varepsilon)(x) = \omega^{-\frac{1}{2}} g_\varepsilon(|x|)x$ is in V_r, since we can approach g_ε by regular functions of compact support. Using approximation argument, (27) holds for g_ε. So, we obtain

$$\frac{|\varphi_\varepsilon(0)|}{\|\nabla a_\varepsilon\|_{2,\omega}\|\nabla b_\varepsilon\|_{2,\omega}} \geqslant \frac{1}{2\pi}\frac{1}{(\varepsilon-1)^2+\sup_{B_1} h_\omega}.$$

Let ε tends to 0; then

$$\sup_{(a,b)\in V_r} \frac{|\varphi(0)|}{\|\nabla a\|_{2,\omega}\|\nabla b\|_{2,\omega}} \geqslant \frac{1}{2\pi}\frac{1}{\Big(1+\sup_{B_1} h_\omega\Big)}$$

and the proof of (17) is completed.

Next, we turn to prove (18). By (1), we have

$$\begin{aligned}
-\frac{1}{r}\frac{d}{dr}\left[r\beta(r)\frac{d\varphi}{dr}\right] &= \frac{\zeta(r)}{2r}\frac{d}{dr}\big(r^2\omega^{-1}(r)g^2(r)\big) \\
&= \frac{1}{2r}\frac{d}{dr}\big(r^2\zeta(r)\omega^{-1}(r)g^2(r)\big) - \frac{1}{2}r\omega^{-1}(r)\zeta'(r)g^2(r).
\end{aligned}$$

By (A_5) and (A_6), we have

$$\begin{aligned}
\varphi(r) &= \frac{1}{2}\int_r^1 t\zeta(t)\beta^{-1}(t)\omega^{-1}(t)g^2(t)\,dt - \frac{1}{2}\int_r^1 \frac{1}{t\beta(t)}\int_0^t \sigma^2\zeta'(\sigma)\omega^{-1}(\sigma)g^2(\sigma)\,d\sigma\,dt \\
&= \frac{1}{2}\int_r^1 tg^2(t)\,dt - \pi G(r)\int_0^r t^2\zeta'(t)\omega^{-1}(t)g^2(t)\,dt \\
&= \frac{1}{2}\int_0^1 tg^2(t)\,dt + \int_0^r t\left[\pi t\zeta'(t)\big(G(t)-G(r)\big) - \frac{1}{2}\frac{\zeta(t)}{\beta(t)}\right]\omega^{-1}(t)g^2(t)\,dt.
\end{aligned}$$

Using (A_3), we get

$$\pi t\zeta'(t) \leqslant \frac{\zeta(t)}{2\beta(t)G(t)} \quad \text{for all } t \in (0,1), \tag{28}$$

and the following estimates hold:

$$\varphi(r) \leqslant \frac{1}{2}\int_0^1 tg^2(t)\,dt - \frac{1}{2}\int_0^r t\omega^{-1}(t)\frac{\zeta(t)G(r)}{\beta(t)G(t)}g^2(t)\,dt, \tag{29}$$

$$\varphi(r) \geqslant \frac{1}{2}\int_r^1 tg^2(t)\,dt - \frac{1}{2}\int_0^r t\omega^{-1}(t)\frac{\zeta(t)G(r)}{\beta(t)G(t)}g^2(t)\,dt. \tag{30}$$

We deduce by (A_7) that

$$\|\varphi\|_\infty \leqslant \frac{1}{2}\max\left\{1, \sup_{B_1}\frac{\xi}{\alpha\omega}\right\}\int_0^1 tg^2(t)\,dt. \tag{31}$$

Using (26), we have

$$\frac{\|\varphi\|_\infty}{\|\nabla a\|_{2,\omega}\|\nabla b\|_{2,\omega}} \leqslant \frac{1}{2\pi}\frac{\max\left\{1, \sup_{B_1}\big(\xi/(\alpha\omega)\big)\right\}}{\left(1+\inf_{B_1} h_\omega\right)}.$$

The proof of Theorem 3 is completed. □

Appendix A

Lemma A.1. *Let $u \in W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$, consider the following problem*:

$$\begin{cases} -\Delta_n\psi = -\operatorname{div}\big(|\nabla\psi|^{n-2}\nabla\psi\big) = \det\nabla u \quad \text{in } \mathbb{R}^n, \\ \lim_{|x|\to+\infty}\psi(x) = 0. \end{cases} \tag{A.1}$$

Then, we have

$$\sup_{\nabla u\neq 0}\frac{\|\psi\|_\infty^{n-1}}{\|\nabla u\|_n^n}=+\infty. \tag{A.2}$$

A.1. Proof of Lemma A.1

The operator $-\Delta_n$ is nonlinear, then we cannot give an integral representation formula of the solution ψ. But we hope that ψ is in $L^\infty(\mathbb{R}^n)$ since the right-hand side of problem (A.1) is in the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$. Also, for the operator $-\Delta_n$ there exists some function which plays the "same" role as a Green function

$$G(x)=-\frac{1}{\sigma_n^{n-1}}\log|x|$$

(we have $-\Delta_n G=\delta_0$ in $\mathcal{D}'(\mathbb{R}^n)$). Moreover the function $\log r$ is in BMO which is the dual of $\mathcal{H}^1(\mathbb{R}^n)$. For all these reasons, we could think to prove some inequality of the type

$$\|\psi\|_\infty^{n-1}\leqslant C\|\nabla u\|_n^n.$$

The power $(n-1)$ is due to some homogeneity considerations (if we replace u by λu then the solution of (A.1) is $\lambda^{n/(n-1)}\psi$). Unfortunately, the best constant is $+\infty$. Indeed, if we choose $u_\varepsilon(x)=g_\varepsilon(r)x$ with $g_\varepsilon(r)=r^{\varepsilon-1}e^{-(n-1)/nr}$ with $\varepsilon>0$, then

$$\det\nabla u_\varepsilon=\frac{1}{nr^{n-1}}\frac{d}{dr}\big(r^n g_\varepsilon^n(r)\big).$$

Problem (A.1) is equivalent to

$$-\frac{1}{r^{n-1}}\frac{d}{dr}\left[r^{n-1}\left(\frac{d\psi_\varepsilon}{dr}\right)^{n-1}\right]=\frac{1}{nr^{n-1}}\frac{d}{dr}\big(r^n g_\varepsilon^n(r)\big).$$

We have then

$$\psi_\varepsilon(r)=\frac{1}{n^{1/(n-1)}}\int_r^{+\infty}t^{1/(n-1)}g_\varepsilon^{n/(n-1)}(t)\,dt.$$

So

$$\|\psi_\varepsilon\|_\infty=\frac{n-1}{n^{n/(n-1)}\varepsilon}\big(1+o_\varepsilon(1)\big).$$

By an easy computation, we have $|\nabla u_\varepsilon|^2=ng_\varepsilon^2(r)+2rg_\varepsilon(r)g_\varepsilon'(r)+r^2g_\varepsilon'^2(r)$ and then

$$\|\nabla u_\varepsilon\|_n^n=\frac{\sigma_n(n-1)^{n/2}}{n\varepsilon}\big(1+o_\varepsilon(1)\big).$$

We easily see that

$$\lim_{\varepsilon\to 0}\frac{\|\psi_\varepsilon\|_\infty^{n-1}}{\|\nabla u_\varepsilon\|_n^n}=+\infty. \qquad\square$$

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