Attractors for reaction–diffusion equations on thin domains whose linear part is non-self-adjoint

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Received 4 April 2003
Available online 11 September 2004

Abstract
For a bounded smooth domain $\Omega \subset \mathbb{R}^{N_x+N_y}$ let $\Omega_\varepsilon$, $0 < \varepsilon$, be a family of domains squeezed in $y \in \mathbb{R}^{N_y}$ direction. On $\Omega_\varepsilon$ we consider a reaction–diffusion equation with nonsymmetrical linear part. We show that under natural conditions on the nonlinearity the generated semi-flows have global attractors which in a certain sense have limits, as $\varepsilon \downarrow 0$.

MSC: 35K; 35B

Keywords: Reaction–diffusion equations; Non-self-adjoint; Thin domain; Attractors

1. Introduction

Reaction–diffusion equations play an important role in a wide field of applications, as for example population ecology, neurobiology, chemical reactions, combustions, etc. For an understanding of the dynamical behavior of these equations, equilibrium solutions—or in a wider sense attractors—are especially important. The attractors depend on the shape of the underlying domain $\Omega$. Of particular interest is squeezing $\Omega$ in one ore more directions, getting so-called thin domains. In the limit $\Omega$ collapses to a lower dimensional set, giving rise to a singular perturbation problem.
We shall show from a dynamical viewpoint that attractors (and semi-flows) of a reaction–diffusion equation with non-symmetrical linear part on thin domains have a limit.

To be more precise let \( \Omega \subset \mathbb{R}^N = \mathbb{R}^{N_x} + \mathbb{R}^{N_y} \) be a fixed smooth domain and write \( z = (x, y) \), \( x \in \mathbb{R}^{N_x} \), \( y \in \mathbb{R}^{N_y} \), for a generic point \( z \) in \( \Omega \). Squeeze \( \Omega \) in \( y \)-direction, i.e. for \( \varepsilon > 0 \) let \( T_\varepsilon : \mathbb{R}^{N_x} + \mathbb{R}^{N_y} \to \mathbb{R}^{N_x} + \mathbb{R}^{N_y} \), \( (x, y) \mapsto (x, \varepsilon y) \) and set

\[
\Omega_\varepsilon := \{(x, y) \in \mathbb{R}^{N_x} \times \mathbb{R}^{N_y} : (x, \frac{1}{\varepsilon} y) \in \Omega \} = T_\varepsilon(\Omega).
\]

On \( \Omega_\varepsilon \) consider the reaction–diffusion equation

\[
v_t = \sum_{k=1}^{N} \frac{\partial}{\partial z_k} \left( \sum_{l=1}^{N} \alpha_{l,k}(z) \frac{\partial v}{\partial z_l} \right) - \sum_{l=1}^{N} \beta_l(z) \frac{\partial v}{\partial z_l} - \gamma(z)v + f(z, v), \quad z \in \Omega_\varepsilon, \quad (1.1)
\]

\[
0 = \sum_{l,k=1}^{N} \alpha_{l,k}(z) \frac{\partial v}{\partial z_l} v_{z,k}, \quad z \in \partial \Omega_\varepsilon, \quad (1.2)
\]

where \( t > 0 \), \( v_\varepsilon = (v_{\varepsilon,1}, \ldots, v_{\varepsilon,N}) \) is the outer normal to \( \partial \Omega_\varepsilon \), \( \alpha_{l,k}, \beta_l, \gamma, f \) are smooth functions, and the \( \alpha_{l,k} \) are uniformly elliptic (but \( \alpha_{l,k} \neq \alpha_{k,l} \) is possible), i.e. there exists an \( \alpha_0 > 0 \) such that

\[
\text{Re} \sum_{l,k=1}^{N} \alpha_{l,k}(z) \bar{\zeta}_l \bar{\zeta}_k \geq \alpha_0 |\zeta|^2, \quad \forall z \in \mathbb{R}^{N}, \quad \zeta \in \mathbb{C}^{N}. \quad (1.3)
\]

\( f \) satisfies some natural growth and dissipative conditions to make the corresponding Nemitsky operator locally Lipschitz and guarantee the existence of attractors \( \tilde{A}_\varepsilon \).

It is well known that Eqs. (1.1) and (1.2) define a (local) semi-flow \( \tilde{\pi}_\varepsilon \). The question arises as to what happens to these semi-flows as \( \varepsilon \downarrow 0 \). And, if the semi-flows \( \tilde{\pi}_\varepsilon \) have global attractors \( \tilde{A}_\varepsilon \), how do they behave in the limit?

This problem was first considered by Hale and Raugel [11] for the case of the Laplacian and \( \Omega \) being the ordinate set of a smooth function \( g \), i.e. if \( \omega \subset \mathbb{R}^{N_x} \) is a domain and

\[
\Omega = \{(x, y) \in \mathbb{R}^{N_x} \times \mathbb{R} : x \in \omega, 0 < y < g(x)\}.
\]

They prove that there exists a semi-flow \( \tilde{\pi}_0 \) and that, in some sense, the family of attractors \( (\tilde{A}_\varepsilon)_{\varepsilon \geq 0} \) is upper-semi-continuous at \( \varepsilon = 0 \).

There is a variety of papers concerned with thin domains. We would like to mention a few which have a more direct influence on our article.
Prizzi and Rybakowski generalized Hale and Raugel's result in [14] to general Lipschitz domains \( \Omega \subset \mathbb{R}^{N_x+N_y} \), which e.g. may have holes or multiple branches. The corresponding limit equation is an abstract parabolic equation defined on a subspace \( H^1_\epsilon(\Omega) \) of \( H^1(\Omega) \). For a wide class of domains \( \Omega \subset \mathbb{R}^2 \) (so-called nicely decomposable domains) they described the limit problem explicitly. It is a system of second-order differential equations on a graph, coupled by a compatibility condition and a Kirchoff type balance condition. They also proved—under certain natural conditions on the nonlinearity \( f \)—for a general Lipschitz domain in \( \mathbb{R}^{N_x+N_y} \) the existence of the limit semi-flow \( \bar{\pi}_0 \) in a strong sense, and the upper-semi-continuity of the family of attractors \( \bar{A}_\epsilon \). In the second paper [15] they show these attractors to be contained in inertial manifolds of finite dimension.

In general, for \( N_x,N_y > 1 \), there does not seem to be an explicit description of the limit problem. In [8] together with Prizzi we show how the limit can be characterized for some special domains, where \( N_x = 2, \ N_y = 1 \).

Antoci and Prizzi [2] investigated unbounded thin domains collapsing onto a lower dimensional subspace. They also prove the convergence of the corresponding semi-flows in a strong sense and the existence and upper-semi-continuity of attractors.

Write (1.1) and (1.2) as an abstract equation \( v_t = -\bar{A}_\epsilon v + \hat{f}_\epsilon(v) \). In [8,14] (and other papers) the basic fact from which the convergence of the semi-flows and ultimately also the upper-semi-continuity of the attractors follows, is the convergence (in a certain sense) of the eigenvalues and eigen-vectors of \( \bar{A}_\epsilon \), which in all these papers is supposed to be self-adjoint. For an unbounded domain this technique does not work since the spectrum of \( \bar{A}_\epsilon \) contains a continuous part. In [2] Antoci and Prizzi used the convergence of the resolvents instead (their operators \( \bar{A}_\epsilon \) are still self-adjoint).

Here we investigate the case \( \bar{A}_\epsilon \) is not self-adjoint. There may not be even a complete system of (generalized) eigenvectors, so it is impossible to use the convergence of eigenvalues and eigen-vectors. But one can use the convergence of the resolvents extending the ideas of [2] to show the convergence of the semi-flows in a strong sense, and given these semi-flows have attractors \( \bar{A}_\epsilon, \ 0 \leq \epsilon \leq 1 \), their upper-semi-continuity at \( \epsilon = 0 \). Note however, that unlike in [2] our semi-flows are rather on \( L^{p_0}(\Omega) \) (\( p_0 \) as in condition H1 of Section 3) than \( H^1(\Omega) \) (see also the comments of the last section).

Before we can state precisely our main result, we need some notations.

Let \( N_x,N_y \in \mathbb{N} \) be fixed numbers, \( N = N_x + N_y \), and \( \Omega \subset \mathbb{R}^{N_x} \times \mathbb{R}^{N_y} \) be a bounded, non-empty, Lipschitz domain. We shall write \( z = (x,y) \in \Omega, \ x \in \mathbb{R}^{N_x}, \ y \in \mathbb{R}^{N_y} \) for points in \( \Omega \).

Let \( \Omega_\epsilon \) denote the squeezed domain

\[
\Omega_\epsilon := \{ (x,y) \in \mathbb{R}^{N_x} \times \mathbb{R}^{N_y} : (x,\frac{1}{\epsilon}y) \in \Omega \} = T_\epsilon(\Omega).
\]

Here, as in the whole article, unless stated otherwise, \( \epsilon \) denotes a number in the interval \( ]0,1[ \).
In (1.1) let \( \alpha_{l,k} \in C^1(\mathbb{R}^N, \mathbb{R}), \beta_l, \gamma \in C(\mathbb{R}^N, \mathbb{R}), f \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}) \). We assume \( \alpha_{l,k} \) to be uniformly elliptic, i.e. (1.3) holds. Assume also \( \gamma \) to be sufficiently big, that is if

\[
C_\beta := \sup_{z \in \mathbb{R}^N, l=1,\ldots,N} (|\beta_l(z)|),
\]

we suppose

\[
\gamma(z) > C_\gamma := N \frac{C_\beta^2}{2z_0} + \frac{1}{2} z_0.
\]

(1.4)

Note that adding a suitable multiple of \( u \) to \( f \), \( \gamma \) always satisfies (1.4) (but \( f \) may not satisfy condition H2 of Section 3 any longer).

We are interested in the behavior of the system of reaction–diffusion equations on \( \Omega_\varepsilon \) given by (1.1) and (1.2) as \( \varepsilon \downarrow 0 \).

We make a transformation onto the fixed domain \( \Omega \). Note that if \( \varepsilon \) and \( \varepsilon = (\varepsilon_x, \varepsilon_y) \) are the outer normals for \( \Omega_\varepsilon \) at \((x, \varepsilon y)\) and \( \Omega \) at \((x, y)\), respectively, then

\[
v_\varepsilon = \frac{1}{|(v_x, 1)}(v_{1,1}, \ldots, v_{1,N}, \varepsilon v_{N+1,1}, \ldots, v_{N,N}).
\]

Thus (1.1) and (1.2) become via \( u(x, y) := v(x, \varepsilon y) = v \circ T_\varepsilon(x, y) \)

\[
u_t(x, y) = \sum_{k=1}^{N_x} \frac{\partial}{\partial x_k} \left( \sum_{l=1}^{N_x} \alpha_{l,k}(x, \varepsilon y) \frac{\partial u}{\partial x_l} + \frac{1}{\varepsilon} \sum_{l=1}^{N_y} \alpha_{N_x+l,k}(x, \varepsilon y) \frac{\partial u}{\partial y_l} \right)
\]

\[
+ \frac{1}{\varepsilon} \sum_{k=1}^{N_y} \frac{\partial}{\partial y_k} \left( \sum_{l=1}^{N_x} \alpha_{l,N_x+l,k}(x, \varepsilon y) \frac{\partial u}{\partial x_l} + \frac{1}{\varepsilon} \sum_{l=1}^{N_y} \alpha_{N_x+l,N_x+l,k}(x, \varepsilon y) \frac{\partial u}{\partial y_l} \right)
\]

\[
- \sum_{l=1}^{N_x} \beta_l(x, \varepsilon y) \frac{\partial u}{\partial x_l} - \frac{1}{\varepsilon} \sum_{l=1}^{N_y} \beta_{N_x+l}(x, \varepsilon y) \frac{\partial u}{\partial y_l}
\]

\[-\gamma(x, \varepsilon y) u + f(x, \varepsilon y, u) \text{ on } \Omega,
\]

(1.5)

\[
0 = B_\varepsilon u|_{\partial \Omega},
\]

(1.6)
where

\[
B_\varepsilon u := \sum_{l,k=1}^{N_x} a_{l,k}(x,\varepsilon y) \frac{\partial u}{\partial x_l} v_{y_k} + \frac{1}{\varepsilon} \sum_{l=1}^{N_y} \sum_{k=1}^{N_y} a_{N_x+1,l}(x,\varepsilon y) \frac{\partial u}{\partial y_l} v_{y_k} + \frac{1}{\varepsilon^2} \sum_{l,k=1}^{N_y} a_{N_x+1,N_x+1}(x,\varepsilon y) \frac{\partial u}{\partial y_l} v_{y_k}.
\]

Note that \( u \in L^2(\Omega) \) (resp. \( H^1(\Omega) \)) iff \( v \in L^2(\Omega_\varepsilon) \) (resp. \( H^1(\Omega_\varepsilon) \)).

Also, (1.5) and (1.6) define a flow \( \pi_\varepsilon \) iff (1.1) and (1.2) define a corresponding flow \( \tilde{\pi}_\varepsilon \). \( \pi_\varepsilon \) has an attractor iff \( \pi_\varepsilon \) has one. So it is sufficient to investigate Eqs. (1.5) and (1.6).

We shall write (1.5) as an abstract equation. In order to do so, we need some notations.

For convenience we shall write \( L^2 \) instead of \( L^2(\Omega, \mathbb{R}) \) \( L^2(\Omega, \mathbb{C}) \) in Section 2, respectively. If the underlying set is not \( \Omega \) we shall always mention it explicitly.

Other functional spaces are treated likewise. Let \((.,.)_{L^2}, \| \cdot \|_{L^2}, \| \cdot \|_{H^1}\) denote the usual scalar products and norms on the Hilbert spaces \( L^2 \) and \( H^1 \), respectively.

The sesquilinear operator corresponding to the right-hand side of (1.5) (see \( a_\varepsilon \) defined below) has a limit as \( \varepsilon \downarrow 0 \), if it remains bounded, that is if \( \partial u_{y_k} = 0 \) for all \( k = 1,\ldots,N_y \). This leads one to define

\[
H^1_s := \{ u \in H^1 : \partial_{y_j} u = 0, \ \forall j = 1,\ldots,N_y \}
\]

and \( L^2_s \) as the closure of \( H^1_s \) in \( L^2 \). Both \( H^1_s \) and \( L^2_s \) are infinite-dimensional Hilbert-spaces with the usual scalar products \((.,.)_{H^1}\) and \((.,.)_{L^2}\) (see [14]).

We shall need also the infinite-dimensional closed space \( L^0_s := \overline{H^1_{s,L^0}} \) and \( L^\infty_s := L^\infty \cap L^0_s \). Here \( p_0 > 2 \) is as in condition H1 of Section 3.

Denote by \( A : \mathbb{R}^N \rightarrow \mathbb{R}^{(N+1)\times(N+1)} \), \( A_{1,1} : \mathbb{R}^N \rightarrow \mathbb{R}^{N_x \times N_x} \), \( A_{1,2} : \mathbb{R}^N \rightarrow \mathbb{R}^{N_y \times N_x} \), \( A_{2,1} : \mathbb{R}^N \rightarrow \mathbb{R}^{N_y \times N_y} \), \( A_{2,2} : \mathbb{R}^N \rightarrow \mathbb{R}^{N_y \times N_y} \) the following maps:

\[
A := \begin{pmatrix}
A_{1,1} & A_{1,2} & b_x \\
A_{2,1} & A_{2,2} & b_y \\
0 & 0 & \gamma
\end{pmatrix} = \begin{pmatrix}
a_{1,1} & \cdots & a_{1,N} & \beta_1 \\
\vdots & \ddots & \vdots & \vdots \\
a_{N,1} & \cdots & a_{N,N} & \beta_N \\
0 & \cdots & 0 & \gamma
\end{pmatrix},
\]

where
By (1.3) and (1.4)

$$\text{Re } \xi^T A(z) \xi \geq \frac{1}{2} |\xi_0|^2, \quad \forall \xi \in \mathbb{C}^{N+1}, \ z \in \mathbb{R}^N. \quad (1.7)$$

So $A_{2,2}(z) \in \mathbb{R}^{N_y \times N_y}$ is invertible for all $z$, the map $A_{2,2}^{-1}(z)$ being $C^1$.

Define the sesquilinear forms $a_\varepsilon : H^1 \times H^1 \to \mathbb{C}$, $a_0 : H^1_s \times H^1_s \to \mathbb{C}$ by

$$a_\varepsilon(u, v) = \int_\Omega \left( \sum_{l,k=1}^{N_y} \alpha_{l,k}(x, \varepsilon y) \frac{\partial u}{\partial x_l} \frac{\partial v}{\partial x_k} + \frac{1}{\varepsilon} \sum_{l=1}^{N_x} \sum_{k=1}^{N_y} \alpha_{l,N_x+k}(x, \varepsilon y) \frac{\partial u}{\partial x_l} \frac{\partial v}{\partial y_k} + \sum_{l=1}^{N_x} \beta_l(x, \varepsilon y) \frac{\partial u}{\partial x_l} + \frac{1}{\varepsilon} \sum_{l=1}^{N_x} \beta_{N_x+l}(x, \varepsilon y) \frac{\partial u}{\partial y_l} + \gamma(x, \varepsilon y) u \right) v \, dx \, dy$$

$$= \int_\Omega (D_x u, \frac{1}{\varepsilon} D_y u, u) A(x, \varepsilon y)(D_x \tilde{v}, \frac{1}{\varepsilon} D_y \tilde{v}, \tilde{v})^T \, dx \, dy,$$

$$a_0(u, v) = \int_\Omega (D_x u, -D_x u A_{1,2} A_{2,2}^{-1}, u) A(x, 0)(D_x \tilde{v}, -D_x \tilde{v} A_{1,2} A_{2,2}^{-1}, \tilde{v})^T \, dx \, dy$$

$$= \int_\Omega (D_x u (A_{1,1}(x, 0) - A_{1,2}(x, 0) A_{2,2}^{-1}(x, 0) A_{2,1}(x, 0)) \nabla_x \tilde{v}$$

$$+ D_x u (b_x(x, 0) - A_{1,2}(x, 0) A_{2,2}^{-1}(x, 0) b_y(x, 0)) \tilde{v} + \gamma(x, 0) u \tilde{v}) \, dx \, dy.$$

They generate operators

$$A_\varepsilon : D(A_\varepsilon) \subset H^1 \to L^2, \quad A_0 : D(A_0) \subset H^1_s \to L^2_s.$$

$A_\varepsilon$ and $A_0$ will be shown in Section 2 to be sectorial operators with compact resolvents.

Multiplying Eq. (1.5) with boundary condition (1.6) by $\varphi \in H^1$ and integrating by parts we get

$$(u_t, \varphi)_{L^2} = -a_\varepsilon(u, \varphi) + (\hat{f}_\varepsilon(u), \varphi)_{L^2},$$

where for $\varepsilon \in [0, 1]$ the Nemitsky operator $\hat{f}_\varepsilon$ is defined by

$$\hat{f}_\varepsilon(u)(x, y) := f(x, \varepsilon y, u(x, y)). \quad (1.8)$$
Hence written as an abstract Eqs. (1.5), (1.6) become

\[ u_t = -A_k u + \hat{f}_k(u), \quad t > 0. \]  

(1.9)

Posing suitable growth and dissipativity conditions on \( f \) (see conditions H1 and H2 in Section 3 and H3 in Section 4) (1.9) defines a semi-flow \( \pi_{\varepsilon} \) with attractor \( A_{\varepsilon} \).

We shall show that these semi-flows converge to a semi-flow \( \pi_0 \) with attractor \( A_0 \) which are defined by the limit equation

\[ u_t = -A_0 u + \hat{f}_0(u), \quad t > 0. \]  

(1.10)

We shall often write \( u_0\pi_{\varepsilon}t \) and \( u_0\pi_0t \) for \( \pi_{\varepsilon}(t,u_0) \) and \( \pi_0(t,u_0) \), respectively.

For \( 0 \leq d \leq 1 \) define equivalent norms on \( H^1 \) by

\[ \|u\|_{\varepsilon,d}^2 := \|u\|_{L^2}^2 + \|\nabla_x u\|_{L^2}^2 + \varepsilon^{-2d}\|\nabla_y u\|_{L^2}^2. \]

The semi-flows \( \pi_{\varepsilon} \) converge in a strong sense to the limit semi-flow \( \pi_0 \) (see Proposition 3.2). Here strong means with respect to \( \|\cdot\|_{\varepsilon,d}, \ 0 \leq d < 1 \).

Our main result is the upper-semi-continuity of the attractors \( A_{\varepsilon} \):

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^N \) be a bounded \( C^2 \) domain and assume \( f \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}) \) satisfies conditions H1 and H2 (see Section 3). Define the operator \( \hat{f}_k \) as in (1.8).

Let \( A_{\varepsilon}, \ 0 \leq \varepsilon \leq 1 \), be as before.

Then \( u_0\pi_{\varepsilon}t := u_\varepsilon(t), \ u_\varepsilon(t) \) the solution of Eq. (1.9) with initial condition \( u_\varepsilon(0) = u_0 \in L^{p_0} \), \( p_0 \) as in H1, defines a global semi-flow on \( L^{p_0} \), for \( 0 < \varepsilon \leq 1 \).

Similarly, \( u_0\pi_0t := u_0(t), \ u_0(t) \) the solution of Eq. (1.10) with initial condition \( u_0(0) = u_0 \in L^{p_0} \), defines a global semi-flow on \( L^{p_0} \).

For all \( 0 \leq \varepsilon \leq 1 \), the semi-flows \( \pi_{\varepsilon} \) have global attractors \( A_{\varepsilon} \) which attract bounded sets of \( L^{p_0} \) for \( \varepsilon > 0 \), and \( L^{p_0}_\varepsilon \) for \( \varepsilon = 0 \). \( A_{\varepsilon} \) is compact and connected in \( L^{p_0} \) and \( L^{p_0}_\varepsilon \), respectively. Moreover, \( A_{\varepsilon} \) is the \( \omega \)-limit set with respect to \( \pi_{\varepsilon} \)

\[ A_{\varepsilon} = \omega(\{u \in L^{p_0} : \|u\|_{L^{p_0}} \leq \delta_f\}) \subset \{u \in H^1 : \|u\|_{L^{p_0}} \leq \delta_f, \|u\|_{\varepsilon,1} \leq \tilde{\delta}_f\}, \]

for \( \varepsilon > 0 \), and

\[ A_0 = \omega(\{u \in L^{p_0}_\varepsilon : \|u\|_{L^{p_0}} \leq \delta_f\}) \subset \{u \in H^1_\varepsilon : \|u\|_{L^{p_0}} \leq \delta_f, \|u\|_{H^1} \leq \tilde{\delta}_f\}, \]

for \( \varepsilon = 0 \). Here \( \delta_f, \tilde{\delta}_f \) are as in Propositions 3.1 and 3.3; both are independent of \( \varepsilon \geq 0 \).

The family of attractors \( A_{\varepsilon} \) is upper-semi-continuous at \( \varepsilon = 0 \), i.e. for \( 0 \leq d < 1 \)

\[ \lim_{\varepsilon \downarrow 0} \sup_{u \in A_{\varepsilon}} \inf_{v \in A_0} \|u - v\|_{\varepsilon,d} = 0. \]
The semi-flows $\pi_\varepsilon$ can also be defined on $H^1$ and $C(\bar{\Omega})$ (see Corollary 3.2 and Proposition 4.2).

Theorem 1.1 will be proved in Section 3.

This article is organized as follows: In Section 2 we show the convergence of the linear semi-flows $e^{-A_\varepsilon t}$ to $e^{-A_0 t}$ in $\| \cdot \|_{\varepsilon,d}, \ 0 \leq d < 1$. Section 3 treats the nonlinear cases over $L^p_0$: existence of semi-flows and their attractors, convergence of the semi-flows and upper-semi-continuity of the attractors. Section 4 is concerned with the case that $f$ does not satisfy the growth condition H1. In it we define semi-flows on $C(\bar{\Omega})$ and (in a certain sense) on $L^\infty_s$, respectively. Section 5 contains some comments on the difficulties of the non-self-adjoint case.

2. The linear case

Throughout this section we will assume the functional spaces to be complex spaces and $\Omega$ to be a bounded Lipschitz domain.

In this section we treat the linear equation

$$u_t = -A_\varepsilon u, \quad t > 0. \tag{2.1}$$

We shall prove the convergence of the linear semi-flows $e^{-A_\varepsilon t}$ to $e^{-A_0 t}$ in $\| \cdot \|_{\varepsilon,d}, \ 0 \leq d < 1$.

We start by showing $A_\varepsilon$ and $A_0$ to be indeed sectorial operators with compact resolvents.

By the continuity of the coefficients and inequality (1.7), there is a $C > 0$, independent of $\varepsilon \geq 0$, such that for all $u \in H^1$ ($u \in H^1_s$ if $\varepsilon = 0$):

$$C \| u \|_{\varepsilon,1}^2 \geq |a_\varepsilon(u,u)| \geq \Re a_\varepsilon(u,u) \geq \frac{1}{2} \gamma_0 \| u \|_{\varepsilon,1}^2. \tag{2.2}$$

$a_\varepsilon$ is sesquilinear, bounded and coercive, $H^1$ is compactly and densely embedded in $L^2$, so with Proposition 9 and Remark 11, Section 3, Chapter VI of [5] follows: $D(A_\varepsilon)$ is dense in $L^2$ and $H^1$, $A_\varepsilon : D(A_\varepsilon) \subset L^2 \to L^2$ is closed, $A_\varepsilon(D(A_\varepsilon)) = L^2$, $A_\varepsilon^{-1} : L^2 \to H^1$ exists and is continuous. The same conclusions hold for $A_0$ (substituting $H^1$ and $L^2$ by $H^1_s$ and $L^2_s$, resp.). Hence $0 \in \rho(A_\varepsilon)$ and $A_\varepsilon$ has compact resolvent, for all $0 \leq \varepsilon \leq 1$.

Note that this implies $A_\varepsilon$ has only pointspectrum with finite multiplicities, $0 \leq \varepsilon$.

Note also that there is a $\varepsilon_1 > 0$, independent of $\varepsilon$, such that $\Re \sigma(A_\varepsilon) \geq 2\varepsilon_1$, for all $0 \leq \varepsilon$.

With $C$ as in inequality (2.2), we have

$$\left| \frac{\Im (A_\varepsilon u,u)}{\Re (A_\varepsilon u,u)} \right| \leq \frac{2C}{\gamma_0} : \tilde{C}, \quad \forall 0 \neq u \in D(A_\varepsilon), \ 0 \leq \varepsilon \leq 1. \tag{2.3}$$
Hence the numerical range \( \{ (A_\varepsilon u, u) : u \in D(A_\varepsilon) \} \) is contained in the sector

\[
|\arg(\lambda)| \leq \frac{1}{2} \pi,
\]

where \( \theta_1 > \arctan \tilde{C} \) is independent of \( \varepsilon \geq 0 \). We find (eventually decreasing \( \varepsilon_1 > 0 \))

\[
\| (\lambda - A_\varepsilon)^{-1} \|_{L^2} \leq \frac{M_0}{|\lambda| + \delta_0}, \quad \text{Re } \lambda < 2 \varepsilon_1 \text{ or } |\arg(\lambda)| > \theta_1, \quad 0 \leq \varepsilon,
\]

where \( M_0, \delta_0 > 0 \) are independent of \( \varepsilon \) and \( A_\varepsilon \) is sectorial for all \( 0 \leq \varepsilon \) (see e.g. Exercise 6, Section 1.3 of [12]). In particular the fractional power spaces \( X^{\beta}_\varepsilon, 0 \leq \beta, \) where \( X^0_\varepsilon = L^2 \) (\( L^2_\varepsilon \) if \( \varepsilon = 0 \)), \( X^1_\varepsilon = D(A_\varepsilon) \).

It is well known that \( A_\varepsilon \) is the infinitesimal generator of an analytic semigroup \( e^{-A_\varepsilon t} \) on \( L^2 \) (resp. on \( L^2_\varepsilon \) if \( \varepsilon = 0 \)). Moreover, since \( -A_\varepsilon \) is dissipative and \( \text{Re } \sigma(A_\varepsilon) \geq 2 \varepsilon_1 > 0, \quad 0 \leq \varepsilon \leq 1 \), \( e^{-(A_\varepsilon - \varepsilon_1)t} \) is a contraction semigroup (see e.g. Theorem 7, Section 3, Chapter XVII A [6]). This in turn gives

\[
\| e^{-A_\varepsilon t} u \|_{L^2} \leq e^{-\varepsilon_1 t} \| u \|_{L^2}, \quad \forall u \in L^2, \quad 0 \leq \varepsilon \leq 1.
\]

We need a similar estimate for \( \| e^{-A_\varepsilon t} u \|_{\varepsilon, 1} \). With (2.2) and (2.6) we have for \( 0 \leq \varepsilon \),

\[
\| e^{-A_\varepsilon t} u \|_{\varepsilon, 1}^2 \leq \frac{2}{\varepsilon_0} \text{Re } a_\varepsilon (e^{-A_\varepsilon t} u, e^{-A_\varepsilon t} u) \leq \frac{2}{\varepsilon_0} |(A_\varepsilon e^{-A_\varepsilon t} u, e^{-A_\varepsilon t} u)_{L^2}| \leq C t^{-\frac{1}{2}} e^{-\varepsilon_1 t} \| u \|_{L^2}^2,
\]

where the constant \( C > 0 \) can be chosen independent of \( \varepsilon \) (see e.g. proof of 1.3.4 [12] using (2.4) and (2.5)).

From this we immediately get for a constant \( C > 0 \) independent of \( \varepsilon \)

\[
\| e^{-A_\varepsilon t} u \|_{\varepsilon, 1} \leq C t^{-\frac{1}{2}} e^{-\varepsilon_1 t} \| u \|_{L^2}, \quad \forall u \in L^2, \quad 0 \leq \varepsilon \leq 1.
\]

In a certain sense boundedness of \( (u_n)_n \subset D(A_\varepsilon_n) \), \( \varepsilon_n \downarrow 0 \), induces a weak convergence of \( a_{\varepsilon_n}(u_n, \cdot) \) to \( a_0(u_0, \cdot) \). More precisely, we have

**Lemma 2.1.** Let \( \varepsilon_n \downarrow 0 \) and \( (u_n)_n \subset D(A_\varepsilon_n) \). Assume that \( \| u_n \|_{\varepsilon, 1} \) and \( \| A_\varepsilon u_n \|_{L^2} \) are bounded. Then there is a subsequence, called \( (u_n)_n \) too, such that \( u_n \rightharpoonup u_0 \in H^1_\varepsilon \) weakly in \( H^1_\varepsilon \) and

\[
a_{\varepsilon_n}(u_n, v) \to a_0(u_0, v), \quad \forall v \in H^1_\varepsilon.
\]

**Proof.** In this proof (and others to follow) we shall write \( A_n, a_n, \ldots \) for \( A_{\varepsilon_n}, a_{\varepsilon_n}, \ldots \). \( (u_n)_n \) is bounded in \( H^1_\varepsilon \), hence taking a subsequence there is an \( u_0 \in H^1_\varepsilon \) and \( u_n \rightharpoonup u_0 \)
weakly. Moreover, \((\frac{1}{\varepsilon_n} \| \nabla_y u_n \|_{L^2})_n\) is bounded, so \(u_0 \in H^1_s\) and for suitable \(v_l \in L^2\) (taking again a subsequence) \(\frac{1}{\varepsilon_n} \frac{\partial u_n}{\partial y_l} \rightharpoonup v_l\) weakly in \(L^2\), \(l = 1, \ldots, N_y\).

Now for all \(v \in H^1\)

\[
0 \leftarrow (A_n u_n, \varepsilon_n v)_{L^2} = a_n(u_n, \varepsilon_n v) \\
\quad \rightarrow \int_{\Omega} (D_x u_0, v_1, \ldots, v_{N_y}, u_0) A(x, 0)(0, \ldots, 0, D_y \tilde{v}, 0)^T \, dx \, dy \\
\quad = \sum_{k=1}^{N_y} \left( \sum_{l=1}^{N_x} a_{l,N_x+k}(x,0) \frac{\partial u_0(x,y)}{\partial x_l} \\
\quad + \sum_{l=1}^{N_y} a_{N_x+l,N_x+k}(x,0) v_l(x,y), \frac{\partial v(x,y)}{\partial y_k} \right)_{L^2}. \tag{2.10}
\]

Define a closed subspace of \(L^2\) by taking those functions which are locally functions of \(x\) only:

\[
V_1 := \{ v \in L^2 : \text{for all } (x_0, y_0) \in \Omega \exists \delta > 0 \text{ such that for a.a. } (x, y) \text{ with} \\
| (x, y) - (x_0, y_0) | < \delta, \ v(x, y) = \tilde{v}(x_0, y_0)(x) \}.
\]

We can decompose \(L^2\) with respect to \(V_1\):

\[
L^2 = V_1 \oplus V_1^\perp
\]

and write \(v_l = v_{l,s} + w_{l,s}, \ v_{l,s} \in V_1, \ w_{l,s} \in V_1^\perp\), for all \(l\).

Note that for \(v \in H^1_s\) we have \(v, \frac{\partial v}{\partial x_j} \in V_1\) (for a proof see [14]) and thus also \(a_{l,k}(x,0) v(x,y), a_{l,k}(x,0) \frac{\partial v}{\partial x_j} \in V_1\), for all \(l, k, j\).

We claim

\[
h_k(x,y) := \sum_{l=1}^{N_x} a_{l,N_x+k}(x,0) \frac{\partial u_0(x,y)}{\partial x_l} + \sum_{l=1}^{N_y} a_{N_x+l,N_x+k}(x,0) v_{l,s}(x,y) = 0,
\]

for all \(k = 1, \ldots, N_y\).

Assume the claim to be false. Then there exists a \(\delta > 0\) and \((x_0, y_0) \in \Omega\) such that the ball \(B((x_0, y_0), \delta) \subset \Omega\) and

\[
\int_{B((x_0,y_0), \frac{1}{2}\delta)} |h_k|^2 \, dx \, dy > 0
\]

for a suitable \(k\).
Now $h_k \in V_1$ so without loss of generality $h_k(x, y) = h_k(x, y_0)$ on $B((x_0, y_0), \delta)$. Let $\chi(s) = 1$, $s \leq 0$, $\chi(s) = 0$, $s \geq 1$ be a $C^\infty$ cut-off-function and define

$$w_k(x, y) := h_k(x, y)\chi\left(\frac{4}{\delta^2}(|x - x_0|^2 + |y - y_0|^2) - 1\right)y_k.$$  

Then the $y$ derivatives of $w_k$ exist and for $1 \leq l \leq N_y$, $l \neq k$

$$\frac{\partial}{\partial y_l} w_k(x, y) = h_k(x, y_0)\chi\left(\frac{4}{\delta^2}(|x - x_0|^2 + |y - y_0|^2) - 1\right)\frac{8}{\delta^2}(y_l - y_{0,l})y_k$$

$$= -\frac{\partial}{\partial y_l} w_k(x, y_1, \ldots, y_{l-1}, 2y_{0,l} - y_l, y_{l+1}, \ldots, y_{N_y}).$$

Analogously,

$$\frac{\partial}{\partial y_k} w_k(x, y) = -\frac{\partial}{\partial y_k} w_k(x, y_1, \ldots, y_{k-1}, 2y_{0,k} - y_k, y_{k+1}, \ldots, y_{N_y})$$

$$+ 2h_k(x, y_0)\chi\left(\frac{4}{\delta^2}(|x - x_0|^2 + |y - y_0|^2) - 1\right)$$

from which

$$\int_\Omega h_k \frac{\partial w_k}{\partial y_l} \, dx \, dy$$

$$= \int_{B((x_0, y_0), \delta)} h_k \frac{\partial w_k}{\partial y_l} \, dx \, dy$$

$$= \begin{cases} 0, & l \neq k, \\ \int_{B((x_0, y_0), \delta)} h_k^2(x, y_0)\chi\left(\frac{4}{\delta^2}(|x - x_0|^2 + |y - y_0|^2) - 1\right) \, dx \, dy > 0, & l = k \end{cases}$$

follows. But this contradicts (2.10). Our claim has been proved.

With the claim we find

$$v_s(x, y) = \begin{pmatrix} v_{1,s}(x, y) \\ \vdots \\ v_{N_y,s}(x, y) \end{pmatrix} = -(A_{2,2}^{-1})^T(x, 0)A_{1,2}^T(x, 0)\nabla_x u_0(x, y).$$

(2.11)
Now let \( v \in H^1_s \) be arbitrary. Then

\[
\begin{align*}
\text{a}_n(u_n, v) &= \int \Omega \left( D_x u_n, \frac{1}{\epsilon_n} D_y u_n, u_n \right) A(x, \epsilon_n y) (D_x \tilde{v}, 0, \tilde{v})^T \, dx \, dy \\
&\rightarrow \int \Omega \left( D_x u_0, v_1, \ldots, v_N, u_0 \right) A(x, 0) (D_x \tilde{v}, 0, \tilde{v})^T \, dx \, dy \\
&= \int \Omega \left( D_x u_0, \epsilon_n v, u_0 \right) A(x, 0) (D_x \tilde{v}, 0, \tilde{v})^T \, dx \, dy \\
&= \int \Omega \left( D_x u_0, -D_x u_0 A_{1,2}(x, 0) A_{2,2}^{-1}(x, 0), u_0 \right) A(x, 0) (D_x \tilde{v}, 0, \tilde{v})^T \, dx \, dy \\
&= a_0(u_0, v).
\end{align*}
\]

This concludes the proof of the lemma. \( \square \)

Eqs. (2.6) and (2.8) are the key ingredients which enable us to apply the technique from [2]. Following closely their ideas we shall now prove first the convergence of the resolvents and then that of the semigroups \( e^{-A_{\epsilon} t} \) to \( e^{-A_0 t} \).

**Lemma 2.2.** Let \( \epsilon_n \downarrow 0, \lambda \in \mathbb{C} \) with \( \text{Re} \lambda > -2 \chi_1 \) or \( |\arg(\lambda)| < \pi - \theta al \) and \( L^2 \ni w_n \rightarrow w_0 \in L^2_x \) in \( \| \cdot \|_{L^2} \). Set \( u_n = (A_{\epsilon_n} + \lambda I)^{-1} w_n \), \( u_0 = (A_0 + \lambda I)^{-1} w_0 \).

Then there is a \( C > 0 \), independent of \( \lambda \), such that for all \( 0 \leq d < 1 \) \( (\|u_n\|_{\epsilon_n,d}) \rightarrow C \) and \( \|u_n - u_0\|_{\epsilon_n,d} \rightarrow 0 \), \( n \rightarrow \infty \).

**Proof.** As before we write \( A_n, a_n, \ldots \) for \( A_{\epsilon_n}, a_{\epsilon_n}, \ldots \). By inequalities (2.2) and (2.5) there are \( C_1, C_2 > 0 \), independent of \( \lambda \), such that

\[
\begin{align*}
\|u_n\|_{\epsilon,1}^2 &\leq C_1 \text{Re} \ a_n(u_n, u_n) = C_1 \text{Re}((A_n u_n, u_n)_{L^2} + (\lambda u_n, u_n)_{L^2} - \lambda \|u_n\|_{L^2}^2) \\
&\leq C_1 (\|w_n\|_{L^2} - \text{Re} \lambda \|u_n\|_{L^2}) \|u_n\|_{L^2} \\
&\leq C_2 \|w_n\|_{L^2} \|u_n\|_{L^2}.
\end{align*}
\]

We can apply Lemma 2.1. There is a subsequence, called \((u_n)\) too, and a \( \tilde{u}_0 \in H^1_s \) with \( u_n \rightharpoonup \tilde{u}_0 \) weakly in \( H^1 \) and \( a_n(u_n, v) \rightharpoonup a_0(\tilde{u}_0, v) \) for all \( v \in H^1_s \).

But this immediately implies \( \tilde{u}_0 = (A_0 + \lambda I)^{-1} w_0 = u_0 \).

Also, as in the proof of Lemma 2.1, there is a \( v \in (L^2)^N \) such that \( \frac{1}{\epsilon_n} D_y u_n \rightharpoonup v \) weakly in \( L^2 \). As in this lemma let \( V_1 \) be the space of all \( L^2 \)-functions which are locally functions of \( x \) only and write \( L^2 = V_1 \oplus V_1^\perp \). If \( v = v_s + v_\perp, v_s \in V_1^N, v_\perp \in (V_1^\perp)^N \), then Eq. (2.11) holds.
We get

\[
\begin{align*}
\int_\Omega \left( D_x u_n, \frac{1}{\varepsilon_n} D_y u_n, u_n \right) A(x, \varepsilon_n y) \left( D_x \bar{u}_n, \frac{1}{\varepsilon_n} D_y \bar{u}_n, \bar{u}_n \right)^T dxdy & + \left( -D_x u_0, D_x u_0 A_{1,2}(x,0) A_{2,2}^{-1}(x,0), \bar{u}_0 \right)^T dxdy + \lambda \| u_0 \|_{L^2}^2
\end{align*}
\]

and thus by (1.7) and (2.11)

\[
\begin{align*}
\| D_x u_n - D_x u_0 \|_{L^2}^2 & + \frac{1}{\varepsilon_n} \| D_y u_n + D_x u_0 A_{1,2}(x,0) A_{2,2}^{-1}(x,0) \|_{L^2}^2 + \| u_n - u_0 \|_{L^2}^2 \\
& \leq C \left| \int_\Omega \left( D_x u_n - D_x u_0, \frac{1}{\varepsilon_n} D_y u_n + D_x u_0 A_{1,2}(x,0) A_{2,2}^{-1}(x,0), u_n - u_0 \right) \times A(x, \varepsilon_n y) \left( D_x \bar{u}_n - D_x \bar{u}_0, \frac{1}{\varepsilon_n} D_y \bar{u}_n \right)^T dxdy + D_x \bar{u}_0 A_{1,2}(x,0) A_{2,2}^{-1}(x,0), \bar{u}_n - \bar{u}_0 \right)^T dxdy \\
& \leq C \| a_n(u_n, u_n) \|_{a_0(u_0, u_0)} + \int_\Omega \left( D_x u_n, \frac{1}{\varepsilon_n} D_y u_n, u_n \right) \times A(x, \varepsilon_n y)(-D_x \bar{u}_0, D_x \bar{u}_0 A_{1,2}(x,0) A_{2,2}^{-1}(x,0), -\bar{u}_0)^T dxdy \\
& \quad \rightarrow a_0(u_0, u_0) \| A(x,0) (-D_x \bar{u}_0, D_x \bar{u}_0 A_{1,2}(x,0) A_{2,2}^{-1}(x,0), -\bar{u}_0)^T \|_{L^2} \text{ in } L^2 \\
& + \int_\Omega (D_x u_0, D_x u_0 A_{1,2} A_{2,2}^{-1}, -u_0) A(x, \varepsilon_n y) \left( D_x \bar{u}_n, \frac{1}{\varepsilon_n} D_y \bar{u}_n, \bar{u}_n \right)^T dxdy \\
& \quad \rightarrow a_0(u_0, u_0) \| A(-D_x \bar{u}_0, D_x \bar{u}_0 A_{1,2} A_{2,2}^{-1}, -\bar{u}_0)^T \|_{L^2} \text{ dxdy} = -a_0(u_0, u_0) \\
& + \int_\Omega (D_x u_0, D_x u_0 A_{1,2} A_{2,2}^{-1}, -u_0) A(-D_x \bar{u}_0, D_x \bar{u}_0 A_{1,2} A_{2,2}^{-1}, -\bar{u}_0)^T dxdy \\
& \quad \rightarrow a_0(u_0, u_0) \\
& \rightarrow 0.
\end{align*}
\]
Thus $D_xu_n \to D_xu_0$ in $L^2$ and $\frac{1}{\varepsilon_n} \| D_yu_n \|_{L^2} \to 0$, as $\frac{1}{\varepsilon_n} \| D_yu_n \|_{L^2}$ is bounded.

Note that by Lemma 2.2 $(A_{\varepsilon_n} + \lambda)^{-1} \to (A_0 + \lambda)^{-1}$ pointwise on $L^2_s$, if $\varepsilon_n \downarrow 0$. Lemma 2.2 allows to prove the convergence of the linear semigroups. This convergence is in $\| . \|_{\varepsilon,d}$ and not only in $\| . \|_{L^2}$, as would be the case adapting the Trotter–Kato Theorem (see e.g. in [18]).

**Proposition 2.1.** Let $\varepsilon_n \downarrow 0$ and $L^2 \ni u_n \to u_0 \in L^2_s$ in $\| . \|_{L^2}$.

Then for all $0 \leq d < 1$

$$\| e^{-A_{\varepsilon_n}t} u_n - e^{-A_0t} u_0 \|_{\varepsilon_n,d} \to 0, \quad n \to \infty$$

uniformly on $[t_1, \infty[ \subset ]0, \infty[.$

**Proof.** As before we shall write $A_n, a_n, \ldots$ for $A_{\varepsilon_n}, a_{\varepsilon_n}, \ldots$.

By (2.8)

$$\| e^{-A_{\varepsilon_n}t} u_n - e^{-A_0t} u_0 \|_{n,1} \leq \| e^{-A_{\varepsilon_n}t} u_n - e^{-A_n t} u_0 \|_{n,1} + \| e^{-A_n t} u_0 - e^{-A_0 t} u_0 \|_{n,1}$$

$$\leq C t^{-\frac{1}{2}} e^{-\frac{1}{2}t} \| u_n - u_0 \|_{L^2} + \| e^{-A_n t} u_0 - e^{-A_0 t} u_0 \|_{n,1},$$

hence it is sufficient to show for $0 \leq d < 1$ fixed

$$\| e^{-A_{\varepsilon_n}t} u_0 - e^{-A_0t} u_0 \|_{n,d} \to 0$$

uniformly on $[t_1, \infty[, \quad t_1 > 0$.

If $\Gamma$ is defined as in Fig. 1, then we have for $0 < t_1 \leq t$ and $u_0 \in L^2_s$

$$\| e^{-A_{\varepsilon_n}t} u_0 - e^{-A_0t} u_0 \|_{n,d}$$

$$= \frac{1}{2\pi} \left\| \int_\Gamma ((\dot{\lambda}I + A_{\varepsilon_n})^{-1} - (\dot{\lambda}I + A_0)^{-1}) e^{\gamma t} u_0 \, d\lambda \right\|_{n,d}$$

$$\leq \frac{1}{2\pi} \left( \int_0^\infty \left( \int_{s_1(1+tan^2(\theta-\pi))} ((se^{\theta i} + A_{\varepsilon_n})^{-1} - (se^{\theta i} + A_0)^{-1}) u_0 \|_{n,d} \\ + \| ((se^{\theta i} + A_{\varepsilon_n})^{-1} - (se^{\theta i} + A_0)^{-1}) u_0 \|_{n,d} \right) e^{t s \cos \theta} \, ds \\
+ \int_{s_1(tan(\theta-\pi))} ((-s_1 - is + A_{\varepsilon_n})^{-1} - (-s_1 - is + A_0)^{-1}) u_0 \|_{n,d} \\
\times e^{-s_1 t} \, ds \right).$$
By Lemma 2.2 the integrand tends pointwise to zero and is bounded, so the Lebesgue-dominated convergence proves our conclusion. □

**Remark 2.1.** If \( \Omega \subset \mathbb{R}^2 \) is a nicely decomposed domain (for the exact definition see [14]), then \( A_0 \) can be characterized explicitly.

Roughly speaking, a nicely decomposed domain can be divided along vertical lines (i.e. \( x = \text{constant} \)) into a finite number of domains \( \Omega_j \), where for each \( \Omega_j \) the cross sections \( \Omega_{j,x} = \{x\} \times \mathbb{R} \cap \Omega_j \) are connected. If the original domain \( \Omega \) has been cut along a line \( \{c\} \times \mathbb{R} \), we say that the related \( \Omega_j \) join each other at \( c \) (again see exact definition in [14]). Denote by \( \sigma^- (c) \) resp. \( \sigma^+ (c) \) those \( \Omega_j \) which join at \( c \) from the left and right, respectively.

On each \( \omega_j \) any function \( u \in H^1_s \) only depends on \( x \): \( u(x,y) = v(x) \) a.e., for a suitable \( v \in L^2_{\text{loc}}(\text{proj}_x \Omega_j) \). If \( u \in H^1_s \), then additionally \( u_x(x,y) = u'(x) \) a.e.

Now if \( p_j(x) := |\Omega_{j,x}|, \ u \in H^1_s, \ \omega \in L^2_s, \ \omega|\Omega_j(x,y) = u_j(x), \ \omega|\Omega_l(x,y) = \omega_j(x) \), then \( A_0 u = \omega \) is equivalent to the system of ordinary differential equations

\[
p_j \omega_j = - \left( p_j u'_j \left( \frac{x_{1,1}(x,0) x_{2,1}(x,0)}{x_{2,2}(x,0)} \right) \right) + u'_j \left( \frac{x_{1,2}(x,0) \beta_2(x,0)}{x_{2,2}(x,0)} \right) + \gamma(x,0) u_j, \quad j = 1, 2, \ldots
\]

coupled by a compatibility and Kirchhoff-type balance condition

\[
u_j(c) = u_l(c) \quad \text{if} \ \Omega_j \text{ and } \Omega_l \text{ join at } c.
\]
\[
\sum_{\sigma^{-}(c)} p_j(c)u_j'(c) \left( \alpha_{1,1}(c, 0) - \frac{\alpha_{1,2}(c, 0) \alpha_{2,1}(c, 0)}{\alpha_{2,2}(c, 0)} \right) = \sum_{\sigma^{+}(c)} p_j(c)u_j'(c) \left( \alpha_{1,1}(c, 0) - \frac{\alpha_{1,2}(c, 0) \alpha_{2,1}(c, 0)}{\alpha_{2,2}(c, 0)} \right) .
\]

3. The nonlinear case: \( L^{p_0} \)

In all of this section let \( \Omega \) be a bounded \( C^2 \) domain. Consider the following hypotheses:

(H1) \( |\frac{\partial}{\partial s} f(z, s)| \leq C_{f,1}(1 + |s|^{p_0 - 1}), \quad z \in \mathbb{R}^N, \quad s \in \mathbb{R}, \)

(H2) \( sf(z, s) \leq 0, \quad z \in \mathbb{R}^N, \quad |s| \geq C_{f,2}, \)

where \( C_{f,1}, C_{f,2} > 0, \ p_0 > 2 \) and if \( N \geq 3 \) additionally \( p_0 < 2^* = \frac{2N}{N-2} \).

Obviously, it suffices if conditions H1, H2 hold only for those \( z = (x, y) \) which lie in at least one \( \Omega_\varepsilon, \ 0 \leq \varepsilon \leq 1. \)

In this section we will assume that all functions are real valued, i.e. all the functional spaces are over \( \mathbb{R} \), and that \( f \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}) \) satisfies condition H1.

We shall show that the nonlinear abstract Eq. (1.9)

\[
\frac{du}{dt} = -A_\varepsilon u + \hat{f}_\varepsilon(u) \quad t > 0,
\]

induces semi-flows \( \pi_\varepsilon, \ 0 \leq \varepsilon \leq 1, \) on \( L^{p_0} (L^{p_0}_\varepsilon \text{ if } \varepsilon = 0) \) and \( \pi_\varepsilon \) converges to \( \pi_0 \) with respect to \( \| \cdot \|_{\pi,0} \) (see Proposition 3.2). If \( f \) additionally satisfies H2, then all these semi-flows are global and have attractors \( A_\varepsilon \) which are upper-semi-continuous at \( \varepsilon = 0. \)

It is well known that \( \hat{f}_\varepsilon : L^{p_0} \to L^2 \) is locally Lipschitz, i.e. for all \( C > 0 \) exists an \( L = L(C) \) (independent of \( 0 \leq \varepsilon \)) such that

\[
\| \hat{f}_\varepsilon(u) - \hat{f}_\varepsilon(v) \|_{L^2} \leq L \| u - v \|_{L^{p_0}}, \quad \| u \|_{L^{p_0}}, \| v \|_{L^{p_0}} \leq C.
\]

Note that \( \hat{f}_0|_{L^{p_0}_\varepsilon} : L^{p_0}_\varepsilon \to L^2. \)

Denote by \( X^\varepsilon, \ 0 \leq \varepsilon \leq 1, \) the fractional power spaces with respect to \( A_\varepsilon \) \( (X^0 = L^2, \ X^1 = D(A_\varepsilon) \subset H^1 \text{ if } \varepsilon > 0, \ X_0 = L^2, \ X_0^1 = D(A_0) \subset H^1_0). \) Then for \( \frac{1}{2} < \beta \leq 1 \) we have \( X^\varepsilon \subset H^1 \subset L^{p_0} \) (resp. \( X^\varepsilon_0 \subset H^1_0 \subset L^{p_0}_\varepsilon \)) continuously (see e.g. Exercise 11, Section 1.4 [12]). It is well known that Eq. (3.1) has a solution \( u_\varepsilon(t) \) for every initial value \( u_0 \in X_0^\beta \) and these solutions define a local semi-flow \( u_0 \pi_\varepsilon t := u_\varepsilon(t), \ 0 \leq \varepsilon. \) Here and in the remainder of this section \( \beta \) will always denote a number in \( [\frac{1}{2}, 1]. \)
$X_\varepsilon^\beta$ is an abstract space. We want to extend $\pi_\varepsilon$ onto $L^{p_0}$ and $L^2$, resp. Before we can do this, we need a technical lemma.

**Lemma 3.1.** Let $0 \leq \varepsilon \leq 1$, $u_0 \in X_\varepsilon^\beta$, $u_\varepsilon(t) = u_0 \pi_\varepsilon t$ be a solution of (3.1), for $0 \leq t < T$, $T$ maximal.

1. $g_{\varepsilon, p_0} : [0, T] \to \mathbb{R}$, $g_{\varepsilon, p_0}(t) := \|u_\varepsilon(t)\|_{L^{p_0}}$ is continuous. For $0 < t < T$ it is differentiable and

$$g'_{p_0, \varepsilon}(t) = p_0 \int_\Omega u_\varepsilon(t)|u_\varepsilon(t)|^{p_0-2}u_{\varepsilon, i}(t) \, dx \, dy.$$

2. If $T < \infty$ then

$$\limsup_{t \to T} \|u_\varepsilon(t)\|_{L^{p_0}} = \infty$$

3. Let $C > 1$ and assume $\|u_0\|_{L^{p_0}} \leq C - 1$. Let $T_1 > 0$ be such that $\|u_\varepsilon(t)\|_{L^{p_0}} \leq C$ for $0 \leq t \leq T_1$. Then there are constants $\tilde{C}_1, \tilde{C}_2 = \tilde{C}_2(C) > 0$, both independent of $\varepsilon \geq 0$ and $u_0$, $\tilde{C}_1$ also independent of $C$, such that

$$\|A_\varepsilon^\beta u_\varepsilon(t)\|_{L^2} \leq \tilde{C}_1 t^{-\beta} e^{-\varepsilon t} \|u_0\|_{L^2} + \tilde{C}_2, \quad 0 < t \leq T_1.$$

4. There is a $T_2 = T_2(C)$, independent of $u_0$ and $\varepsilon \geq 0$, such that one can choose $T_1 \geq T_2$ in part (3).

5. $\|e^{-A_\varepsilon t}u_0\|_{L^{p_0}} \leq \|u_0\|_{L^{p_0}}$, $0 \leq t$.

**Proof.** $g_{\varepsilon, p_0}$ is continuous because $X_\varepsilon^\beta \subset H^1 \subset L^{p_0}$. It is differentiable because $u : [0, T] \to X_\varepsilon^\beta$ is differentiable (Theorem 3.5.2 [12]) and $u \mapsto \Phi_p(u) = \|u\|_{L^p}$ is differentiable with derivative $D\Phi_p(u)v = p \int_\Omega u|u|^{p-2}v \, dx \, dy$, $v \in L^p$.

Let $C > \|u_0\|_{L^{p_0}}$ and assume $u_\varepsilon(t)$ exists and satisfies $\|u_\varepsilon(t)\|_{L^{p_0}} \leq C$ for $0 \leq t \leq T_1 = T_1(u_0, C)$. $T_1$ is supposed to be maximal (if $T_1 < \infty$).

For $t > 0$ and $u \in L^2$ ($u \in L^2$ if $\varepsilon = 0$)

$$\|A_\varepsilon^\beta e^{-A_\varepsilon t}u\|_{L^2} \leq C_2 t^{-\beta} e^{-\varepsilon t} \|u\|_{L^2}, \quad (3.3)$$

where $C_2 > 0$ is independent of $\varepsilon \geq 0$ (see e.g. Theorem 1.4.3 [12] and (2.7)).

Use this estimate in the variation-of-constants-formula to get for $0 < t \leq T_1$

$$\|A_\varepsilon^\beta u_\varepsilon(t)\|_{L^2} \leq C_2 t^{-\beta} e^{-\varepsilon t} \|u_0\|_{L^2} + \int_0^t C_2 (t-s)^{-\beta} e^{-\varepsilon (t-s)} \|f_\varepsilon(u_\varepsilon(s))\|_{L^2} \, ds \leq C_2 t^{-\beta} e^{-\varepsilon t} \|u_0\|_{L^2} + C_3(C),$$
which proves part (3). Above inequality also implies part (2).

Now we prove part (4). There is a $T_{2,0} = T_{2,0}(C)$ such that $u_0 \pi_0 t$ exists for $0 \leq t \leq T_{2,0}$, if $\|u_0\|_{L^{p_0}} \leq C - 1$. Hence we only have to prove the conclusion of part (4) for the case $\varepsilon > 0$.

We shall prove it first under the additional assumption $\hat{f}_\varepsilon : L^{p_0} \rightarrow L^\infty$, then use a continuity argument to get it for general $\hat{f}_\varepsilon$.

As a first step we show $(u_\varepsilon = u_\varepsilon(t))$

$$\int_\Omega u_\varepsilon |u_\varepsilon|^{p_0-2} A_\varepsilon u_\varepsilon \, dx \, dy \geq \frac{2}{p_0} \|u_\varepsilon\|_{L^{p_0}}^{p_0} \left[ \frac{1}{p_0} + (1 - \frac{2}{p_0}) C_\gamma \|u_\varepsilon\|_{L^{p_0}} \right] \geq 0.$$  (3.4)

$\hat{f}_\varepsilon(u_\varepsilon) \in L^\infty$ and $u_{\varepsilon,t} \in H^1$ (see Theorem 3.5.2 [12]) imply $A_\varepsilon u_\varepsilon \in L^{p_0}$, and the left-hand side in above expression is well defined.

Unfortunately $u_\varepsilon |u_\varepsilon|^{p_0-2}$ is not necessarily in $H^1$. We approximate it by $u_{\varepsilon,n} := (u_\varepsilon)_{c_n}$, where for a constant $c > 0$ we define for any $u \in L^2$

$$u_c(x, y) := \begin{cases} u(x, y), & |u(x, y)| < c, \\ \text{sign}(u(x, y)) c, & |u(x, y)| \geq c. \end{cases}$$

If $u \in H^1$ and $\|\{(x, y) : |u(x, y)| = c\}\| = 0$, then $u_c \in H^1$ too and $D_{uc}(x, y) = Du(x, y)$, $|u(x, y)| < c$, $Du_c = 0$ elsewhere.

We can choose a sequence of constants $c_n \rightarrow \infty$ such that $\|\{(x, y) : |u_\varepsilon(x, y)| = c_n\}\| = 0$ for all $n$. Then $u_n \in H^1$ and thus $u_{n,t} |u_{n,t}|^{p_0-2} \in H^1$ too.

Because $A_\varepsilon u_\varepsilon \in L^{p_0}$

$$\left| \int_{\{z \in \Omega : |u_\varepsilon(z)| \geq c_n\}} (u_\varepsilon |u_\varepsilon|^{p_0-2} - u_\varepsilon |u_\varepsilon|^{p_0-2}) A_\varepsilon u_\varepsilon \, dx \, dy \right|$$

$$\leq 2^{p_0-1} \left( \int_{\{z \in \Omega : |u_\varepsilon(z)| \geq c_n\}} |A_\varepsilon u_\varepsilon|^{p_0} \, dx \, dy \right)^{1/p_0}$$

$$\times \left( \int_{\{z \in \Omega : |u_\varepsilon(z)| \geq c_n\}} |u_\varepsilon|^{p_0} \, dx \, dy \right)^{p_0-1/p_0} \rightarrow 0$$

holds, which in turn implies

$$\int_\Omega u_\varepsilon |u_\varepsilon|^{p_0-2} A_\varepsilon u_\varepsilon \, dx \, dy \leftarrow (A_\varepsilon u_\varepsilon, u_{n,t} |u_{n,t}|^{p_0-2})_{L^2} = a_\varepsilon(u_\varepsilon, u_n |u_n|^{p_0-2})$$

$$= \int_{\{z \in \Omega : |u_\varepsilon(z)| \leq c_n\}} \frac{4(p_0 - 1)}{p_0^2} \left( \sum_{l,k=1}^{N_\varepsilon} \frac{\partial |u_n|^{p_0/2}}{\partial x_l} \frac{\partial |u_n|^{p_0/2}}{\partial x_k} \right)$$

$$\times \left( \sum_{l,k=1}^{N_\varepsilon} \right. \left. \frac{\partial |u_n|^{p_0/2}}{\partial x_l} \frac{\partial |u_n|^{p_0/2}}{\partial x_k} \right)$$
\[
+ \frac{1}{\varepsilon} \sum_{l=1}^{N_x} \sum_{k=1}^{N_y} (\xi_{l,N_x+k} + \xi_{N_x+k,l}) \frac{\partial |u_n|_{p_0}^p}{\partial x_l} \frac{\partial |u_n|_{p_0}^p}{\partial y_k}
\]

\[
+ \frac{1}{\varepsilon^2} \sum_{l,k=1}^{N_y} \xi_{N_x+l,N_x+k} \left( \frac{\partial |u_n|_{p_0}^p}{\partial y_l} + \frac{\partial |u_n|_{p_0}^p}{\partial y_k} \right)
\]

\[
+ \frac{2}{p_0} \left( \sum_{l=1}^{N_x} \beta_l \frac{\partial |u_n|_{p_0}^p}{\partial x_l} + \frac{1}{\varepsilon} \sum_{l=1}^{N_y} \beta_{N_x+l} \frac{\partial |u_n|_{p_0}^p}{\partial y_l} \right) |u_n|_{p_0}^p - \gamma |u_n|_{p_0}^p \right) dxdy
\]

\[
+ \frac{2}{p_0} \left( \frac{\sum_{l=1}^{N_x} \beta_l \frac{\partial |u_n|_{p_0}^p}{\partial x_l} + \frac{1}{\varepsilon} \sum_{l=1}^{N_y} \beta_{N_x+l} \frac{\partial |u_n|_{p_0}^p}{\partial y_l} \right) \right) \right) \right) dxdy
\]

\[
\geq \frac{2}{p_0} a_\varepsilon (|u_n|_{p_0}^p, |u_n|_{p_0}^p) + \left( 1 - \frac{2}{p_0} \right) C_\gamma \|u_n\|_{L^p_0}^p - \frac{2}{p_0} \int_{\{z \in \Omega: |u_\varepsilon(z)| > c_n\}} \frac{\gamma c_n^p}{dxdy} \right) \rightarrow 0
\]

\[
- \int_{\{z \in \Omega: |u_\varepsilon(z)| > c_n\}} \left( \frac{\sum_{l=1}^{N_x} \beta_l \frac{\partial |u_n|_{p_0}^p}{\partial x_l} + \frac{1}{\varepsilon} \sum_{l=1}^{N_y} \beta_{N_x+l} \frac{\partial |u_n|_{p_0}^p}{\partial y_l} \right) \right) \right) \right) dxdy
\]

We already know \( Du_\varepsilon \in H^1 \) hence

\[
(*) \leq \left( \int_{\{z \in \Omega: |u_\varepsilon(z)| > c_n\}} \left( \frac{\sum_{l=1}^{N_x} \beta_l \frac{\partial |u_n|_{p_0}^p}{\partial x_l} + \frac{1}{\varepsilon} \sum_{l=1}^{N_y} \beta_{N_x+l} \frac{\partial |u_n|_{p_0}^p}{\partial y_l} \right) \right) \right) \right) \right) dxdy \rightarrow 0
\]

\[
\left( \int_{\{z \in \Omega: |u_\varepsilon(z)| > c_n\}} c_n^p dxdy \right)^{p_0-1} \rightarrow 0,
\]

which taking the limit \( n \rightarrow \infty \) shows

\[
\int_\Omega u_\varepsilon |u_\varepsilon|_{p_0-2} A_\varepsilon u_\varepsilon dxdy \geq \frac{2}{p_0} a_\varepsilon (|u_n|_{p_0}^p, |u_n|_{p_0}^p) + (1 - \frac{2}{p_0}) C_\gamma \|u_n\|_{L^p_0}^p.
\]

Eq. (2.2) now easily proves inequality (3.4).

Note for later use that if \( u_\varepsilon(t) \in L^\infty \), then \( \hat{f}_\varepsilon \) has values in \( L^\infty \) too and inequality (3.4) holds with \( p_0 \) replaced by any \( p > 2 \).
With (3.4) and part (1)

\[ g'_{\varepsilon,p_0}(t) \leq p_0 \int_{\Omega} u_\varepsilon(t)|u_\varepsilon(t)|^{p_0-2} \hat{f}_\varepsilon(u_\varepsilon(t)) \, dx \, dy - \frac{2p_0}{p_0} \|u_\varepsilon^{p_0}(t)\|_{p_0,1}^2 \]

\[ \leq p_0 \|u_\varepsilon(t)\|_{L^{\infty}(p_0-1)}^{p_0-1}(L(C)) \|u_\varepsilon(t)\|_{L^{p_0}} + \|\hat{f}_\varepsilon(0)\|_{L^2} - C_4 \|u_\varepsilon(t)\|_{L^{p_0}}^{p_0} \]

\[ \leq \|u_\varepsilon(t)\|_{L^{p_0}}^{p_0-1} (C_5(C) - C_4 \|u_\varepsilon(t)\|_{L^{p_0}}^{p_0}) \]

where \( L(C) \) is as in (3.2) and \( C_4, C_5 > 0 \) are constants independent of \( \varepsilon \) and \( \hat{f}_\varepsilon \) (apart from the value \( \hat{f}_\varepsilon(0) \)).

If \( \|u_\varepsilon(t)\|_{L^{p_0}} \geq \frac{C_4(C)}{C_4} \), then \( g'_{\varepsilon,p_0}(t) \leq 0 \), hence for all \( 0 \leq t \leq T_1 \)

\[ g'_{\varepsilon,p_0}(t) \leq \left( \frac{C_5(C)}{C_4} \right)^{p_0-1} C_5(C) =: C_6(C), \]

where \( C_6 \) is again independent of \( \varepsilon \). We get

\[ g_{\varepsilon,p_0}(t) \leq g_{\varepsilon,p_0}(0) + tC_6(C), \quad 0 \leq t \leq T_1. \] (3.5)

Part (2) implies now \( g_{\varepsilon,p_0}(t) \leq C^{p_0} \) as long as \( t \leq T_2 := C^{p_0-(C-1)p_0 \varepsilon} \), which proves part (4) in the case of bounded \( \hat{f}_\varepsilon \).

If \( \hat{f}_\varepsilon \) is not bounded, set \( \hat{f}_\lambda(u) := \hat{f}_\varepsilon(u)\chi(\frac{u}{\lambda}) \), \( 1 < \lambda, \chi \) a suitable cut-off function. \( C_4, C_5 \) and hence \( C_6 \) and \( T_2 \) are independent of \( \lambda \). The solution \( \tilde{u}_\lambda(t) \) of (3.1) with initial value \( u_0 \) and non linear \( \hat{f}_\lambda \) exists and satisfies (3.5). It is well known that \( \tilde{u}_\lambda(t) \rightarrow u_\varepsilon(t) \) in \( \|\cdot\|_{H^1} \) as \( \lambda \rightarrow \infty \), for \( t > 0 \) (e.g. Theorem 3.4.8 [12]), hence (3.5) holds for the original solution \( u_\varepsilon \) too. Part (4) follows for general \( \hat{f}_\varepsilon \) and \( \varepsilon > 0 \).

\( e^{-A_\varepsilon t} u \) is the solution of (3.1) with \( \hat{f} \equiv 0 \), so by part 1 and Eq. (3.4)

\( \frac{d}{dt}(\|e^{-A_\varepsilon t} u\|_{p_0}^{p_0}) \leq 0 \) which shows the last part of the lemma.

Lemma 3.1 enables us to extend the semi-flows \( \pi_\varepsilon \) from \( X_\varepsilon^{\beta} \) to \( L^{p_0} \) and \( L_s^{p_0} \), respectively. We will use the same symbol \( \pi_\varepsilon \) for the extensions too.

**Proposition 3.1.** Let \( \varepsilon \geq 0 \) and assume \( f \) satisfies H1.

Then Eq. (3.1) with initial condition \( u_0 \in L^{p_0} \) if \( \varepsilon > 0 \) and \( u_0 \in L^{p_0}_s \) if \( \varepsilon = 0 \) has a unique solution \( u_\varepsilon(t) \), \( u_0 \pi_\varepsilon,t := u_\varepsilon(t) \) defines a semi-flow on \( L^{p_0} \) and \( L^{p_0}_s \), respectively. Moreover, there is a \( T_1 = T_1(C) > 0 \), independent of \( \varepsilon \geq 0 \), such that \( u_0 \pi_\varepsilon,t \) exists and \( \|u_0 \pi_\varepsilon,t\|_{L^{p_0}} \leq \|u_0\|_{L^{p_0}} + 1 \) for all \( u_0 \in L^{p_0}, \|u_0\|_{L^{p_0}} \leq C, \ 0 \leq t \leq T_1 \) and \( \varepsilon \geq 0 \).

If \( f \) satisfies additionally H2 then for \( \varepsilon > 0 \) the semi-flows \( \pi_\varepsilon \) are global ones. For any \( C > 0 \) there are constants \( \delta_f, T_2 = T_2(C) > 0 \), both independent of \( \varepsilon, \delta_f \).
independent of $C$ too, such that

$$
\|u_0 \pi_{\varepsilon} t\|_{L^p} \leq \|u_0\|_{L^p} + \delta_f, \quad t \geq 0,
$$

$$
\|u_0 \pi_{\varepsilon} t\|_{L^p} \leq \delta_f, \quad t \geq T_2, \quad \|u_0\|_{L^p} \leq C.
$$

**Proof.** Let $u_0 \in L^p$ if $\varepsilon > 0$ and $u_0 \in L^p$ if $\varepsilon = 0$.

We know $X^1 \subset H^1 (H^1_s)$ if $\varepsilon = 0$ densely, hence there is a sequence $u_n \in X^\beta_\varepsilon$ such that $\|u_n - u_0\|_{L^p \to 0}$. By Lemma 3.1 part (4) there is a $T_1 > 0$ independent of $n$ and $\varepsilon \geq 0$ such that $u_n \pi_{\varepsilon} t$ exists and satisfies $\|u_n \pi_{\varepsilon} t\|_{L^p} \leq 1 + \|u_0\|_{L^p}$ for $0 \leq t \leq T_1$.

For $0 < t \leq T_1$ Lemma 3.1 part (5), (2.8) and (3.2) imply

$$
\|u_n \pi_{\varepsilon} t - u_m \pi_{\varepsilon} t\|_{L^p} \\
\leq \|u_n - u_m\|_{L^p} + C_1 \int_0^t (t - s)^{-\frac{1}{2}} e^{-2s} \frac{1}{L_1} (1 + \|u_0\|_{L^p}) \|u_n \pi_{\varepsilon} s\|_{L^p} ds \\
\leq \|u_n - u_m\|_{L^p} + 2C_1 L (1 + \|u_0\|_{L^p}) \sup_{0 \leq s \leq t} \|u_n \pi_{\varepsilon} s - u_m \pi_{\varepsilon} s\|_{L^p} \sqrt{T_1}.
$$

If $0 \leq t \leq (4C_1 L (1 + \|u_0\|_{L^p}))^{-2}$ then $(u_n \pi_{\varepsilon} t)_n$ is a Cauchy-sequence in $L^p$. Denote the limit by $u_0(t)$. Then $\|u_n \pi_{\varepsilon} t - u_0(t)\|_{L^p} \to 0$ uniformly in $0 \leq t \leq (4C_1 L (1 + \|u_0\|_{L^p}))^{-2}$. Note that $C_1$ is independent of $\varepsilon \geq 0$.

By Lemma 3.1(1) is $(A_{\varepsilon}^\beta u_n \pi_{\varepsilon} t|_{L^2})_n$ bounded for each fixed $t > 0$. $X^\beta_\varepsilon$ is reflexive because $A_{\varepsilon}^\beta : L^2 \to X^\beta_\varepsilon$ (resp. $A_{\varepsilon}^{-\beta} : L^2_0 \to X^\beta_\varepsilon$ if $\varepsilon = 0$) is an isomorphism, thus $(u_n \pi_{\varepsilon} t)_n$ is weakly compact. $u_0(t) \in X^\beta_\varepsilon$ follows for $0 < t \leq T_1$.

It is obvious that $u_0 \pi_{\varepsilon} t := u_0(t)$ is a solution of (3.1) for $t > 0$ and the only thing still missing for the so defined $\pi_{\varepsilon}$ to be a semi-flow on $L^p$ and $L^p_0$, resp., is the continuity at $t = 0$.

To show this let $\delta > 0$ be arbitrary. Then there is an $n$ such that $\|u_n \pi_{\varepsilon} t - u_0 \pi_{\varepsilon} t\|_{L^p} \leq \frac{1}{2} \delta$ for all $0 \leq t \leq t_1$, some small $t_1 > 0$. Choose $t_1$ small enough to satisfy $\|u_n \pi_{\varepsilon} t - u_n\|_{L^p} \leq \frac{1}{2} \delta$ for all $0 \leq t \leq t_1$. We get the desired

$$
\|u_0 \pi_{\varepsilon} t - u_0\|_{L^p} \leq \delta, \quad 0 \leq t \leq t_1.
$$

Note that part (1) of Lemma 3.1 still holds for the extended semi-flow.

To finish the proof of the proposition we only need to show the additional statement where $f$ satisfies H2.
As in the proof of Lemma 3.1 part (4), we treat first the case \( \hat{f}_\varepsilon : L^{p_0} \to L^\infty \). Then (3.4) holds, implying

\[
g'_{s,p_0}(t) \leq p_0 \int_{\{(x,y)\in \Omega : |u_{\varepsilon}(t)| \leq C_{f,2}\}} C_{f,2}^{-1} \sup(|f(z,u)| : |u| \leq C_{f,2}, z \in \bigcup_{0 \leq \varepsilon \leq 1} \Omega_\varepsilon) \, dx \, dy
\]

\[
- \left(1 - \frac{2}{p_0}\right) C_{\gamma} \|u_{\varepsilon}(t)\|_{L^{p_0}}^{p_0} \\
\leq C_2 - C_{3} \|u_{\varepsilon}(t)\|_{L^{p_0}}^{p_0} = C_2 - C_3 g_{s,p_0}(t),
\]

(3.7)

where \( C_2, C_3 > 0 \) are independent of \( \varepsilon \). Multiply (3.7) by \( e^{C_3 t} \) and integrate to get

\[
g_{s,p_0}(t) \leq g_{s,p_0}(0) e^{-C_3 t} + C_2. \tag{3.8}
\]

If we choose a sequence \( \hat{f}_n \) approximating \( \hat{f}_\varepsilon \) as in the proof of Lemma 3.1 part (4), then \( C_2, C_3 \) are independent of \( n \). With the same argument as before (3.8) holds for general \( \hat{f}_\varepsilon \) too.

Since \( \hat{f}_\varepsilon \) maps bounded sets of \( L^{p_0} \) into bounded sets of \( L^2 \), this proves the semi-flows to be global. Setting \( \delta_f := 2C_2, T_2 := \frac{1}{C_1} \ln \frac{C_2}{C_1} \) the conclusion follows.

Note for later use that if \( f \) is exchanged by \( \tilde{f}(z,u) = f(z,u)\chi(u) \), \( \chi \) a \( C^\infty \) cut-off-function satisfying \( \chi(u) \equiv 1 \) for \( |u| \leq C_{f,2} \), then \( \delta_f = \delta_{\tilde{f}} \). Indeed, \( \tilde{f} \) satisfies H2 with \( C_{\tilde{f},2} = C_{f,2} \) and in the proof above \( C_2 \) does not change. \( \square \)

Having established the existence of the semi-flows \( \pi_{\varepsilon}, \varepsilon \geq 0 \), we can prove the convergence in a strong sense as \( \varepsilon \to 0 \):

**Proposition 3.2.** Assume \( f \) satisfies condition \( H1 \). Let \( (\varepsilon_n)_n \) be a sequence of positive numbers tending to 0, \( u_n \in L^{p_0}, u_0 \in L^{p_0} \) such that \( (\|u_n\|_{L^{p_0}})_n \) is bounded and \( u_n \to u_0 \) in \( \|\cdot\|_{L^2} \). Assume that all semi-flows \( u_n \pi_{\varepsilon_1}, \varepsilon_1 \geq 0 \), exist for \( 0 \leq t \leq T \), for some \( T > 0 \).

Then for all \( t_n \to t_0 \in ]0, T[ \) and \( 0 \leq d < 1 \) we have

\[
\|u_n \pi_{\varepsilon_n} t_n - u_0 \pi_{0} t_0\|_{\varepsilon_n,d} \to 0, \quad n \to \infty.
\]

**Proof.** First assume \( T \leq T_1 = T_1(2 \sup_n \|u_n\|_{L^{p_0}} + \sup_{0 \leq t \leq T} \|u_0 \pi_{0} t\|_{L^{p_0}} + 1), T_1 \) as in Proposition 3.1. Then \( \|u_n \pi_{\varepsilon_n} t\|_{L^{p_0}} \) is bounded uniformly in \( 0 \leq t \leq T, 0 \leq \varepsilon \leq 1 \). An obvious adaptation of the proof of Theorem 5.1 [14]—the main difference being the change from \( \|\cdot\|_{\varepsilon,1} \) to \( \|\cdot\|_{\varepsilon,d}, 0 < d < 1 \)—proves our proposition in this particular case.

If \( T > T_1 \), we iterate above argument: substitute \( u_n \) and \( u_0 \) by \( u_n \pi_{\varepsilon}(T_1 - \delta) \) and \( u_0 \pi_{0}(T_1 - \delta) \), resp., for a small \( \delta > 0 \). We can use the same \( T_1 \) again (eventually
restricting \( n \) to big enough values) to prove convergence on \([T_1 - \frac{1}{2} \delta, \min(T, 2T_1 - \delta)]\). Thus after a finite number of steps Proposition 3.2 has been proven. \( \square \)

**Remark 3.1.** (a) The assumption that all semi-flows \( u_n \pi_\varepsilon t, 0 \leq \varepsilon \), exist for \( 0 \leq t \leq T \) is not really necessary. By Proposition 3.1 there is always some \( T > 0 \) such that this condition is satisfied.

(b) Note that in our situation we do not have the slightly stronger convergence in \( \| \cdot \|_{\varepsilon, 1} \) as for example in [2,8,14].

In fact, if in Proposition 3.2 the convergence were in \( \| \cdot \|_{\varepsilon, 1} \), then

\[
a_{\varepsilon n}(u_n \pi_{\varepsilon n} t, u_n \pi_{\varepsilon n} t) \to \int_\Omega (D_x u_0 \pi_0 t, 0, u_0)A(x, 0)(D_x u_0 \pi_0 t, 0, u_0)^T dx dy
\]

\[
\neq a_0(u_0 \pi_0 t, u_0 \pi_0 t)
\]

in general.

Proposition 3.2 is the counterpart to Theorem 5.1 of [14]. Analogously to their Corollary 5.2 (and with the same proof) in our situation we have

**Corollary 3.1.** Let \( f \) satisfy condition H1, \( \varepsilon_n \downarrow 0 \), \( C_1 > 0 \), \( 0 < d < 1 \) and \( C_2 \) the constant in \( \| \cdot \|_{L^p} \leq C_2 \| \cdot \|_{H^1} \). Suppose \( \sigma_n \) is a full solution of \( \pi_{\varepsilon n} \) satisfying

\[
\sup_{t \in \mathbb{R}, n \in \mathbb{N}} \| \sigma_n(t) \|_{\varepsilon_n, d} < C_1.
\]

Assume that whenever \( u_0 \in L^{p_0}_s \), \( \| u_0 \|_{L^p} < C_1 C_2 \), then \( u_0 \pi_0 t \) exists for all \( t \geq 0 \).

Then there is a subsequence, called \( (\varepsilon_n)_n \) too, and a full solution \( \sigma_0 : \mathbb{R} \to L^{p_0}_s \) of \( \pi_0 \) such that

\[
\| \sigma_n(t) - \sigma_0(t) \|_{\varepsilon_n, d} \to 0 \quad \forall t \in \mathbb{R}.
\]

We proceed to questions concerning the attractors, starting with the existence.

**Proposition 3.3.** Let \( f \) satisfy conditions H1 and H2 and \( 0 \leq \varepsilon \).

Then the semi-flows \( \pi_\varepsilon \) are global ones and have global attractors \( \mathcal{A}_\varepsilon \) which attract bounded sets of \( L^{p_0} \) and \( L^{p_0}_s \), resp. \( \mathcal{A}_\varepsilon \) is compact and connected. Moreover, \( \mathcal{A}_\varepsilon \) is the \( \omega \)-limit set with respect to \( \pi_\varepsilon \)

\[
\mathcal{A}_\varepsilon = \omega(\{ u \in L^{p_0} : \| u \|_{L^p} \leq \delta_f \}) \subset \{ u \in H^1 : \| u \|_{L^p} \leq \delta_f, \| u \|_{\varepsilon, 1} \leq \tilde{\delta}_f \}, \quad \varepsilon > 0,
\]

\[
\mathcal{A}_\varepsilon = \omega(\{ u \in L^{p_0}_s : \| u \|_{L^p} \leq \delta_f \}) \subset \{ u \in H^1_s : \| u \|_{L^p} \leq \delta_f, \| u \|_{H^1} \leq \tilde{\delta}_f \}, \quad \varepsilon = 0,
\]
where $\delta_f$ is as in Proposition 3.1, $\tilde{\delta}_f = 2(1 + \frac{e^{-x_1}}{x_1})C \sup(\|\tilde{f}_e(u)\|_{L^2} : \|u_0\|_{L^p_0} \leq 2\delta_f, 0 \leq \varepsilon \leq 1) < \infty$, $C$ as in (2.8). $\delta_f$ and $\tilde{\delta}_f$ are independent of $\varepsilon \geq 0$.

**Proof.** If $\varepsilon > 0$ then by Proposition 3.1 is $\pi_\varepsilon$ a global semi-flow, it is bounded by $\|u_0\|_{L^p_0} + \delta_f$ and $\{\|u_0\|_{L^p_0} \leq \delta_f\}$ is an absorbing set. Applying Proposition 3.2 to $u_\varepsilon = u_0 \in L^p_0$, $u_\varepsilon \pi_\varepsilon t$ is bounded in $\|\cdot\|_{L^p_0}$ by $\|u_0\|_{L^p_0} + \delta_f$, hence global too. Moreover, $\{u \in L^p_\varepsilon : \|u_0\|_{L^p_0} \leq \delta_f\}$ is a bounded absorbing set for $\pi_\varepsilon$.

For $t > 0$ is $u \mapsto u \pi_\varepsilon t$ bounded continuous $L^p_0 \to H^1 (L^p_0 \to H^1_\varepsilon$ if $\varepsilon = 0)$, hence compact $L^p_0 \to L^p_0 (L^p_0 \to L^p_\varepsilon$ if $\varepsilon = 0)$. It follows the $\omega$-limit set $\omega(\{\|u_0\|_{L^p_0} \leq \delta_f\})$ is an attractor bounded in $\|\cdot\|_{L^p_0}$ by $\delta_f$ (see e.g. [16] Theorem 1.1).

The only thing we still have to show is the boundedness of $A_\varepsilon$ in $\|\cdot\|_{\varepsilon,1}$.

Let $u_\varepsilon \in L^p_0$ ($u_\varepsilon \in L^p_0$ if $\varepsilon = 0$), $\|u_0\|_{L^p_0} \leq \delta_f$, then $\|u_\varepsilon \pi_\varepsilon t\|_{L^p_0} \leq 2\delta_f$ for all $t$. Hence by (2.8) for $t > 1$

$$\|u_\varepsilon \pi_\varepsilon t\|_{\varepsilon,1} \leq C(t^{-\frac{1}{2}} e^{-x_1 t} \|u_0\|_{L^2} + \int_0^t (t-s)^{-\frac{1}{2}} e^{-(t-s)x_1} \|f(x, \varepsilon y, u_\varepsilon s)\|_{L^2} ds)$$

$$\leq Ct^{-\frac{1}{2}} e^{-x_1 t} \|u_0\|_{L^2} + 2(1+ e^{-x_1})^{-\frac{1}{2}} \tilde{\delta}_f \left( \int_{t-1}^t (t-s)^{-\frac{1}{2}} ds + \int_0^{t-1} e^{-(t-s)x_1} ds \right)$$

$$\to \tilde{\delta}_f,$$

as $t \to \infty$, which concludes the proof. $\square$

We now prove Theorem 1.1. We only have to show the upper-semi-continuity.

As before write $A_n, \pi_n, \pi_{\varepsilon n}, \pi_\varepsilon, \pi_{\varepsilon n}$, etc. Let $0 < d < 1$.

Assume the attractors are not upper-semi-continuous at $\varepsilon = 0$. Then there are a sequence $\varepsilon_n \downarrow 0$, $u_n \in A_n$ and a $\tilde{\delta} > 0$ such that

$$\inf_{v \in \mathcal{A}_0} \|u_n - v\|_{n,d} > \tilde{\delta}, \quad \forall n. \quad (3.9)$$

$A_n$ consists of fully bounded solutions, hence there are solutions $\sigma_n$ to $\pi_n$ and $\sigma_n(0) = u_n$. By Proposition 3.3 $\|\sigma_n(t)\|_{\varepsilon,1} \leq \delta_f$.

We can apply Corollary 3.1: without loss of generality, there is a full solution $\sigma_0$ to $\pi_0$ and $\|\sigma(t) - \sigma_0(t)\|_{n,d} \to 0$, for all $t \in \mathbb{R}$. As a full solution $\sigma(t) \in \mathcal{A}_0$. Now $u_n = \sigma_n(0) \to \sigma_0(0)$ contradicts (3.9).

**Corollary 3.2.** Let $\Omega$ be a $C^\infty$ domain and $f$ as in Theorem 1.1. Then $\pi_\varepsilon$ restricted to $H^1$ is a semi-flow on $H^1$. It is global and has $\mathcal{A}_\varepsilon$ from Theorem 1.1 as an attractor.
$A_\varepsilon$ is compact and connected in $H^1$ and

$$A_\varepsilon = \omega(\{u \in H^1 : \|u\|_{\varepsilon,1} \leq \tilde{\delta}_f\}).$$

**Proof.** It is well known that $D(A_\varepsilon) = \{u \in H^2 : B_\varepsilon u = 0 \text{ on } \partial \Omega\}$ and $H^1$ is the complex interpolation space $H^1 = [L^2, D(A_\varepsilon)]_\frac{1}{2}$ (see e.g. 4.3.3 [17]).

We can apply Theorems 7.1.2 and 7.1.10 [13] to see that (3.1) with initial value $u_0 \in H^1$ has a solution $u_\varepsilon(t)$ and $(u_0, t) \mapsto u_\varepsilon(t)$ defines a (local) semi-flow on $H^1$. By Theorem 1.1 is is global and by Proposition 3.1 for every $C > 0$ there is $T_1 = T_1(C) > 0$ such that $\|u_\varepsilon(t)\|_{L^p} \leq \delta_f$ for $t \geq T_1$, $\|u_\varepsilon(t)\|_{L^p} \leq C + \delta_f$, for $t \geq 0$ and $\|u_0\|_{L^p}$. This and inequality (2.8) yield for $t \geq T_1$, $\|u_0\|_{\varepsilon,1} \leq \frac{C}{C_1}$, $C_1$ the constant in $\|\cdot\|_{L^p} \leq C_1 \|\cdot\|_{H^1}$.

$$\|u_\varepsilon(t)\|_{\varepsilon,1} \leq C_2 (t^{-\frac{1}{2}} e^{-2\lambda_1 t} \|u_0\|_{L^2} + \int_0^{T_1} (t-s)^{-\frac{1}{2}} e^{-2\lambda_1 (t-s)} \sup_{\|u\|_{L^p} \leq C+\delta_f} \|f_\varepsilon(u)\|_{L^2} \, ds)$$

$$+ \int_{T_1}^t (t-s)^{-\frac{1}{2}} e^{-2\lambda_1 (t-s)} \sup_{\|u\|_{L^p} \leq \delta_f} \|f_\varepsilon(u)\|_{L^2} \, ds)$$

$$< C_2 (t^{-\frac{1}{2}} e^{-2\lambda_1 t} C + C_3(C) e^{-2\lambda_1 t} \tilde{\delta}_f),$$

where the constants $C_2, C_3 > 0$ are independent of $t$, $u_0$, and $\tilde{\delta}_f$ as in Proposition 3.3. That is $\{u \in H^1 : \|u\|_{\varepsilon,1} \leq \tilde{\delta}_f\}$ is an absorbing set. Since $u_0 \in H^1 \mapsto u_\varepsilon(t) \in D(A_{\varepsilon,\beta}^\beta)$, $\beta < \tilde{\beta} < 1$, is continuous for $t > 0$ and $X_{\varepsilon,\beta}^\beta \subset X_{\varepsilon,\beta}^\beta$ compactly, $u_0 \in H^1 \mapsto u_\varepsilon(t) \in H^1$ is a compact map for $t > 0$. Hence $\pi_\varepsilon$ as a semi-flow on $H^1$ has an attractor as stated above. □

4. The nonlinear case: $C(\tilde{\Omega})$

In this section we shall always suppose $\Omega$ to be a bounded $C^2$-domain, $f$ satisfies condition H2 (but not necessarily H1), and all functions are real valued, i.e. all the functional spaces are over $\mathbb{R}$.

We want to extend the results of the previous section to the case that $f$ does not satisfy H1. This can be done (in a way) working in $C(\tilde{\Omega})$.

Consider the following hypotheses:

(H3) $sf(z, s) \leq -C_{f,3}|s|^{\beta_f}, \quad z \in \mathbb{R}^N, \ |s| \geq C_{f,2},$

where $C_{f,3} > 0$ and $\beta_f > 2$.

Obviously it suffices if the condition above holds only for those $z = (x, y)$ which lie in at least one $\Omega_\varepsilon$, $0 \leq \varepsilon \leq 1$. 
Condition H3 is somewhat unusual. It will only be used to prove that after a finite time semi-flows on \( L^{p_0} \) become bounded in \( \| \cdot \|_\infty \) (see Corollary 4.1).

We consider the nonlinear abstract Eq. (3.1)
\[
    u_t = -A_\v u + \hat{f}_\v(u) \quad t > 0,
\]
(4.1)
where \( f \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}) \) satisfies H2.

It is well known that \( \hat{f}_\v : L^\infty \to L^\infty \) is locally Lipschitz, i.e. for all \( C > 0 \) exists an \( L = L(C) \) (independent of \( 0 \leq \v < 1 \)) such that
\[
    \| \hat{f}_\v(u) - \hat{f}_\v(v) \|_\infty \leq L \| u - v \|_\infty, \quad \| u \|_\infty, \| v \|_\infty \leq C.
\]

Set
\[
    D(A_\v, \infty) := \left\{ u \in \bigcap_{p \geq 1} W^{2,p}_\text{loc}(\tilde{\Omega}) : u, A_\v u \in C(\tilde{\Omega}), B_\v u \big|_{\partial \Omega} = 0 \right\},
\]
\[
    A_\v, \infty : D(A_\v, \infty) \to C(\tilde{\Omega}), \quad A_\v, \infty u = A_\v u,
\]
then \( A_\v, \infty \) is sectorial (see e.g. Corollary 3.1.24 [13], note that formally we can write \( A_\v \) as a differential operator on \( H^2 \) with symmetric coefficients and unsymmetric boundary condition), \( D(A_\v, \infty) \) is dense in \( C(\tilde{\Omega}) \) and \( D(A_\v, \infty) \subset C^{1+\v} (\tilde{\Omega}) \) for \( 0 < \v < 1 \). For every \( u_0 \in C(\tilde{\Omega}) \) there is a solution \( u(t) = u(t; u_0, \v), 0 \leq t < T = T(u_0, \v) \) of (4.1) with \( u(0) = u_0 \) satisfying
\[
    u(\cdot, u_0, \v) \in C^{1+\v}([\delta_1, \delta_2], C(\tilde{\Omega})) \cap C([\delta_1, \delta_2], D(A_\v, \infty)) \cap C([0, \delta_2], C(\tilde{\Omega})),
\]
for all \( 0 < \delta_1 < \delta_2 < T, \quad 0 < \v < 1 \) (see e.g. Propositions 7.1.3 and 7.1.10 [13]).

The abstract theorems mentioned above do not apply to the limit case \( \v = 0 \). It is not clear if (4.1) with \( \v = 0 \) defines a semi-flow on a suitable closed subspace of \( C(\tilde{\Omega}) \). But, using functions like \( g_{\v, p_0} \) we can restrict \( \pi_0 \) to \( L^\infty_\v = L^{p_0}_\v \cap L^\infty \), the restriction will be denoted by \( \pi_{0, \v} \) (see Proposition 4.1).

We start with the counterpart of Lemma 3.1 part (1) which will then be used to show \( \pi_{\v, \infty} \) are global semi-flows with attractors.

**Lemma 4.1.** Let \( u_\v(t) = u_0 \pi_{\v, \infty} t, u_0 \in C(\tilde{\Omega}) \), be a solution of (4.1), for \( 0 \leq t \leq T < \infty \).

Then \( g_{\v, p} : [0, T] \to \mathbb{R}, \quad g_{\v, p}(t) := \| u_\v(t) \|_{L^p}^p \) is continuous. For \( 0 < t < T \) it is differentiable and
\[
    g_{p, \v}'(t) = p (\| u_\v(t) \|_{L^p}^{p-2} u_\v(t), u_{\v, t}(t))_{L^2},
\]
for all \( 2 \leq p < \infty \).
Proof. The conclusion follows directly from the continuity, respectively, the differentiability of \( t \mapsto u_\varepsilon(t) \). □

Lemma 4.2. \( \pi_{\varepsilon,\infty} \) is a global semi-flow on \( C(\bar{\Omega}) \) and for every \( C > 0 \) there are constants \( T = T(C), \delta_{f,\infty} > 0 \), both independent of \( \varepsilon \), \( \delta_{f,\infty} \) also independent of \( C \), such that

\[
\|u_0 \pi_{\varepsilon,\infty} t\|_{\infty} \leq \|u_0\|_{\infty} + \delta_{f,\infty}, \quad t \geq 0,
\]

\[
\|u_0 \pi_{\varepsilon,\infty} t\|_{\infty} \leq \delta_{f,\infty}, \quad t \geq T, \quad \|u_0\|_{\infty} \leq C.
\]

Proof. We shall bound \( \dot{g}_{\varepsilon,p} \), \( g_{\varepsilon,p} \) as in Lemma 4.1, independently of \( \varepsilon \) and \( p \geq p_0 \), which allows us to get a bound for \( \|u_0 \pi_{\varepsilon,\infty} t\|_{\infty} \) and the said absorbing set.

We already mentioned that in this situation inequality (3.4) holds too, with \( p_0 \) replaced by \( p \).

Set \( u_\varepsilon(t) = u_0 \pi_{\varepsilon,\infty} t \). By Lemma 4.1, (3.4) and condition H2

\[
\dot{g}_{\varepsilon,p}(t) \leq -p \left( 1 - \frac{2}{p} \right) C \|u_\varepsilon(t)\|_{L^p}^p + \int_{\{x,y\in\Omega:|u_\varepsilon(t)|\leq C_{f,2}\}} |u_\varepsilon(t)|^{p-1}|\tilde{f}_\varepsilon(u_\varepsilon(t))| \, dx \, dy \]

\[
\leq p(C_1 C_{f,2}^p - C_2 g_{\varepsilon,p}(t)),
\]

where \( C_1, C_2 > 0 \) are independent of \( \varepsilon \) and \( p \). Thus

\[
g_{\varepsilon,p}(t) \leq g_{\varepsilon,p}(0)e^{-pC_1 t} + \frac{C_1}{C_2} C_{f,2}^p
\]

and

\[
\|u_\varepsilon(t)\|_{L^p} \leq \|u_0\|_{\infty}|\Omega|^{1/p} e^{-C_1 t} + \left( \frac{C_1}{C_2} \right)^{1/p} C_{f,2}, \quad p \geq p_0, t \geq 0.
\]

(4.2)

It is well known that \( \|u\|_{\infty} \leq C \) iff \( \|u\|_{L^p} \leq |\Omega|^{1/p} C \) for all \( p \) big enough (see e.g. Problem 7.1 [10]). Hence (4.2) implies the boundedness of \( u_\varepsilon(t) \) in \( \| \cdot \|_{\infty} \), thus proving \( \pi_{\varepsilon,\infty} \) to be global.

Set \( \delta_{f,\infty} := 2C_{f,2} \) and the conclusion of the lemma holds.

Note for latter use that as in the Proposition 3.1 here also \( \delta_{f,\infty} = \delta_{\tilde{f},\infty} \), if \( \tilde{f}(x,u) = f(z,u)\chi(u) \) and the \( C^\infty \) cut-off-function \( \chi \) satisfies \( \chi(u) \equiv 1 \) for \( |u| \leq C_{f,2} \). □
Lemma 4.2 allows us to restrict \( \pi_0 \) to \( L^\infty_s \):

**Proposition 4.1.** Let \( f \) satisfy condition H2.

Then the solution \( u(t) \) of (4.1) for \( \varepsilon = 0 \) and with initial condition \( u_0 \in L^\infty_s \) exists and \( u_0 \pi_{0,\infty} t := u(t) \) defines a global semi-flow on \( L^\infty_s \) with respect to \( \| \cdot \|_{L^p_\infty} \).

**Proof.** Let \( u_0 \in L^\infty_s \). We want to apply results of Section three, so take a \( C^\infty \) cut-off-function \( \chi \) such that \( \chi(u) \equiv 1 \) if \( |u| \leq \|u_0\|_\infty + \delta_{f,\infty} \), \( \delta_{f,\infty} \) as in Lemma 4.2. Set \( \tilde{f}(z, u) = f(z, u) \chi(u) \). \( \tilde{f} \) satisfies conditions H1 and H2.

Proposition 3.1 assures the local existence of the solution \( u(t) \), \( 0 \leq t < T = T(u_0) \) of (4.1) with \( \varepsilon = 0 \), initial condition \( u_0 \) and \( f \) replaced by \( \tilde{f} \).

Let \( \varepsilon_n \downarrow 0 \) and \( u_n \in C(\bar{\Omega}) \) with \( u_n \to u_0 \) in \( \| \cdot \|_{L^p_\infty} \), \( \| u_n \|_{L^\infty} \leq 2 \| u_0 \|_{L^\infty} \).

We write \( \pi_{n,\infty} \), \( A_n \), etc. for \( \pi_{\varepsilon_n,\infty} \), \( A_{\varepsilon_n} \), etc.

By Lemma 4.2 exist the global semi-flows \( u_n \pi_{n,\infty} t \) and satisfy

\[
\| u_n \pi_{n,\infty} t \|_\infty \leq 2 \| u_0 \|_\infty + \delta_{f,2}, \quad \quad t \geq 0, \quad \varepsilon > 0.
\]

Proposition 3.2 shows the convergence of \( u_n \pi_{n,\infty} t \) to \( u(t) \) in \( \| \cdot \|_{\varepsilon,d} \), \( 0 < d < 1 \), \( 0 < t < T \).

In particular we have pointwise convergence \( u_n \pi_{n,\infty} t(x, y) \to u(t)(x, y) \), for every \( 0 < t < T \) and a.a. \((x, y) \in \Omega \). Thus \( \| u(t) \|_\infty \leq 2 \| u_0 \|_\infty + \delta_{f,\infty}, \quad 0 < t < T \), i.e. \( u(t) \) is bounded and has to exist for all \( t \geq 0 \).

Since \( u(t) \) is bounded by \( 2 \| u_0 \|_\infty + \delta_{f,\infty} \) it is the solution of (4.1) with the original \( f \). The conclusion of the proposition follows immediately. \( \Box \)

By Lemma 4.2 there are absorbing sets for \( \pi_{\varepsilon,\infty} \). With an argument similar to that in the proof of Proposition 3.3 these semi-flows have attractors, say \( A_{\varepsilon,\infty} \). Let us assume for a moment that \( f \) satisfies H1. Then trivially \( A_{\varepsilon,\infty} \subset A_\varepsilon \). This means the family \( \{ A_{\varepsilon,\infty} : \varepsilon > 0 \} \cup \{ A_0 \} \) is by Theorem 1.1 upper-semi-continuous at \( \varepsilon = 0 \). The problem here is that \( A_0 \) might contain solutions which are not in \( L^\infty \). We shall see in the next proposition that we can replace \( A_0 \) by an appropriate \( \omega \)-limit set which is bounded in \( \| \cdot \|_\infty \). Thus we can still use the argument above even if \( f \) does not satisfy H1.

**Proposition 4.2.** Let \( f \) satisfy condition H2. Then for \( \varepsilon > 0 \) the semi-flows \( \pi_{\varepsilon,\infty} \) have global attractors \( A_{\varepsilon,\infty} \) which attract bounded sets of \( C(\bar{\Omega}) \). \( A_{\varepsilon,\infty} \) is compact and connected in \( C(\bar{\Omega}) \). Moreover, \( A_{\varepsilon,\infty} \) is the \( \omega \)-limit set with respect to \( \pi_{\varepsilon,\infty} \)

\[
A_{\varepsilon,\infty} = \omega([u \in C(\bar{\Omega}) : \| u \|_\infty \leq \delta_{f,\infty}] )
\]

\[
\subset \{ u \in C(\bar{\Omega}) \cap H^1 : \| u \|_\infty \leq \delta_{f,\infty}, \| u \|_{\varepsilon,1} \leq \tilde{\delta}_f \},
\]

where \( \delta_{f,\infty} \) and \( \tilde{\delta}_f \) are as in Lemma 4.2 and Proposition 3.3.
Let \( \mathcal{A}_{0,\infty} \) be the \( \omega \)-limit set with respect to \( \pi_{0,\infty} \)

\[
\mathcal{A}_{0,\infty} := \omega(\{ u \in L_s^\infty : \| u \|_\infty \leq \delta_{f,\infty} \}) 
\]

\[
\subset \{ u \in L_s^\infty \cap H_s^1 : \| u \|_\infty \leq \delta_{f,\infty}, \| u \|_{H^1} \leq \tilde{\delta}_f \},
\]

then the family \( \mathcal{A}_{\varepsilon,\infty} \) is upper-semi-continuous at \( \varepsilon = 0 \), i.e. for \( 0 \leq \varepsilon < 1 \)

\[
\lim_{\varepsilon \downarrow 0} \sup_{u \in \mathcal{A}_{\varepsilon,\infty}} \inf_{v \in \mathcal{A}_{0,\infty}} \| u - v \|_{\varepsilon,d} = 0.
\]

**Proof.** We shall first prove the proposition if \( f \) satisfies H1 too.

Let \( C > 0 \) be a fixed number to be specified latter and define \( \tilde{f}(z,u) = f(z,u)\chi(u) \), where \( \chi \) is a cut-off-function with \( \chi(u) \equiv 0 \) for \( |u| \leq C \). \( \tilde{f} \) satisfies conditions H1 and H2 and we can apply the results of the previous section.

Denote all entities which we get using \( \tilde{f} \) by a tilde.

We have already mentioned how to prove the existence of the attractors \( \tilde{A}_{\varepsilon,\infty} \) and the characterization of the \( \omega \)-limit sets follows directly from Lemma 4.2 and Proposition 3.3.

That \( \tilde{A}_{0,\infty} \) is bounded by \( \delta_{\tilde{f},\infty} \) in \( \| \cdot \|_\infty \) follows from Lemma 4.2 and the convergence of \( \tilde{\pi}_\varepsilon \) to \( \tilde{\pi}_0 \) (Proposition 3.2).

Assume \( \tilde{A}_{\varepsilon,\infty} \) is not upper-semi-continuous at \( \varepsilon = 0 \). Then for some \( 0 < \varepsilon < 1 \) there is a \( \delta > 0 \) and sequences \( \varepsilon_n \downarrow 0 \), \( u_n \in \tilde{A}_{\varepsilon_n,\infty} \) such that

\[
\inf_{v \in \tilde{A}_{0,\infty}} \| u_n - v \|_{\varepsilon_n,d} > \delta, \quad \forall n.
\]

As in the proof of Theorem 1.1 there are full solutions \( \tilde{\sigma}_n \) to \( \tilde{\pi}_{\varepsilon_n,\infty} \) and \( \tilde{\sigma}_0 \) to \( \tilde{\pi}_0 \) with \( \tilde{\sigma}_n(0) = u_n, \| \tilde{\sigma}_n(t) - \tilde{\sigma}_0(t) \|_{\varepsilon_n,d} \rightarrow 0 \). Thus \( \| \tilde{\sigma}_n(t) \|_{\varepsilon_n,\infty} \leq \delta_{\tilde{f},\infty} \) implies \( \| \tilde{\sigma}_0(t) \|_{\varepsilon_n,\infty} \leq \delta_{\tilde{f},\infty} \) and \( \tilde{\sigma}_0 \in \tilde{A}_0 \cap \{ u \in L_s^\infty : \| u \|_\infty \leq \delta_{\tilde{f},\infty} \} \). That is \( \tilde{\sigma}_0(0) \in \tilde{A}_{0,\infty} \) and \( \| u_n - \tilde{\sigma}_0(0) \|_{\varepsilon_n,d} \rightarrow 0 \).

This proves the proposition if the nonlinearity is \( \tilde{f} \).

Choose \( C > \delta_{\tilde{f}} + \tilde{C}_{f,2} \). It has been commented before in the proofs of Proposition 3.1 and Lemma 4.2 that then \( \delta_{f} = \delta_{\tilde{f}} \) and \( \delta_{f,\infty} = \delta_{\tilde{f},\infty} \).

By Lemma 4.2 \( \| u_0 \pi_{\varepsilon,\infty} t \|_{\varepsilon,\infty} \leq C \) if \( \| u_0 \|_{\varepsilon,\infty} \leq \delta_f \), for all \( t \geq 0 \), \( \varepsilon > 0 \), i.e. \( \pi_{\varepsilon,\infty} = \tilde{\pi}_{\varepsilon,\infty} \) on \{ \( u \in C(\tilde{Q}) : \| u \|_{\varepsilon,\infty} \leq \delta_f \) \} and \( \tilde{A}_{\varepsilon,\infty} = \tilde{A}_{\varepsilon,\infty} \) is an attractor for \( \pi_{\varepsilon,\infty} \) as stated above.

By Proposition 3.2 \( \| u_0 \tilde{\pi}_{0,\infty} t \|_{\infty,\infty} \leq C \) if \( \| u_0 \|_{\infty,\infty} \leq \delta_f \) too. Again \( \tilde{\pi}_{0,\infty} = \pi_{0,\infty} \) and \( \tilde{A}_{0,\infty} = \tilde{A}_{0,\infty} \).

This proves the proposition in the general case. \( \square \)
As a last point we want to show how it is possible to get semi-flows on $L^p_0$ which—after a finite time independent of $\varepsilon$—become semi-flows on $C(\bar{\Omega})$ (for $\varepsilon > 0$). That is for these flows we can apply the results above.

**Corollary 4.1.** Let $f$ satisfy conditions $H1$ and $H3$.

Then there is a $T > 0$, independent of $\varepsilon \geq 0$, such that for any $u \in L^p_0$, $u_0 \in L^p_0$ we have $u\pi_t \in C(\bar{\Omega})$ and $u_0 \pi_0 \in L^\infty_s$ for $t \geq T$. In particular $\pi_\varepsilon = \pi_{\varepsilon,\infty}$ for $t \geq T$ and $A_\varepsilon = A_{\varepsilon,\infty}$, for $\varepsilon \geq 0$.

**Proof.** In a way similar to the proof of Lemma 4.2 we shall use the functions $g_{\varepsilon,p}$ in differential inequalities. Thus we shall get a bound on $g_{\varepsilon,p}(T)$ which is independent of $\|g_{\varepsilon,p}(0)\|_{L^p}$ and $\varepsilon \geq 0$, implying $\|u_0 \pi_\varepsilon T\|_{L^\infty}$ is bounded. By an abstract theorem we get even $u_0 \pi_\varepsilon t \in C(\bar{\Omega})$.

For the moment let $u \in C(\bar{\Omega})$, then $u_\varepsilon(t) = u\pi_\varepsilon t = u\pi_{\varepsilon,\infty} t$ and $g_{\varepsilon,p}$ is defined for all $p \geq p_0$.

As in the proof of Lemma 4.2

$$g'_{\varepsilon,p}(t) \leq p \left( \int_{\{z \in \Omega: |u_\varepsilon| \leq C_{f,2}\}} |u_\varepsilon|^{p-1} |f(x, \varepsilon y, u_\varepsilon)| \, dx \, dy 
+ \int_{\{z \in \Omega: |u_\varepsilon| \geq C_{f,2}\}} |u_\varepsilon|^{p-2} u_\varepsilon f(x, \varepsilon y, u_\varepsilon) \, dx \, dy \right)$$

$$\leq p(C_1 C_{f,2}^{p-1} - C_{f,3} \int_{\{z \in \Omega: |u_\varepsilon| \geq C_{f,2}\}} |u_\varepsilon|^{p-2+\beta_f}),$$

where $C_1 > 0$ is independent of $\varepsilon$, $p$ and $u_0$.

A simple estimate proves: if $C > 0$, $\tilde{p} \geq 2$ and $\|u\|_{L^{\tilde{p}}(\Omega)} \geq 2|\Omega|^{1/p}$, then

$$\int_{\{(x,y) \in \Omega: |u(x,y)| \geq C\}} |u(x,y)|^{\tilde{p}} \, dx \, dy \geq \frac{1}{2} \|u\|_{L^{\tilde{p}}}^{\tilde{p}}.$$

Apply this, setting

$$C_2 := \sup_{p \geq p_0} \left( C_{f,2}^{p-1} (2|\Omega|)^{-1/p-2+\beta_f}, (4C_1 C_{f,3}^{p-1} C_{f,2}^{-2+\beta_f})^{-1/p-2+\beta_f} \right) < \infty$$

to get for $\|u_\varepsilon\|_{L^{p-2+\beta_f}} \geq C_2$

$$g'_{\varepsilon,p}(t) \leq -p \frac{C_{f,3}}{4} \|u_\varepsilon\|_{L^{p-2+\beta_f}}^{p-2+\beta_f} \leq - p C_3 g_{\varepsilon,p}(t) \frac{p^{p-2+\beta_f}}{p},$$
where $C_3 > 0$ is independent of $\varepsilon$, $p$ and $u_0$. Hence as long as $\|u_\varepsilon(t)\|_{L^{p-2+\beta_f}} \geq C_2$, we have

$$g_{\varepsilon,p}(t) \leq \left( (\beta_f - 2)C_3 t + |\Omega|^{-\frac{\beta_f - 2}{p}} \|u_0\|_\infty^{-(\beta_f - 2)} \right)^{-\frac{p}{\beta_f - 2}} 0 \leq t. \quad (4.3)$$

For $t \geq T := (C_3(\beta_f - 2))^{-1}$ the right-hand side of (4.3) is less than 1, yielding

$$g_{\varepsilon,p}(t) \leq \max(1, |\Omega|^{\frac{\beta_f - 2}{p-2+\beta_f}} C_2^p), \quad t \geq T.$$

This in turn shows

$$\|u_\varepsilon(t)\|_\infty \leq \max(1, C_2) =: C_4, \quad t \geq T.$$ 

Now let $u_0 \in L^{p_0}$. Then there is a sequence $u_n \in C(\bar{\Omega})$, $u_n \to u_0$ in $\|\cdot\|_{L^{p_0}}$. Since $\|u_n\pi_\varepsilon t - u_0\pi_\varepsilon t\|_{L^{p_0}} \to 0$, we get $\|u_0\pi_\varepsilon t\|_{\infty} \leq C_4$ and thus by Proposition 3.2 for $v_0 \in L^{p_0}$. Also $\|v_0\pi_0 t\|_{\infty} \leq C_4$, i.e. $v_0\pi_0 t \in L^{p_0}_\infty$ for $t \geq T$.

We still have to show $u_0\pi_\varepsilon t \in C(\bar{\Omega})$ for $t \geq T$ and $\varepsilon > 0$. To do this let

$$\hat{D}_\varepsilon := \left\{ u \in \bigcap_{p \geq 1} W^{2,p}_{\text{loc}} : u, A_\varepsilon u \in L^\infty, B_\varepsilon u|_{\partial\Omega} = 0 \right\}.$$

Then $A_\varepsilon|_{\hat{D}_\varepsilon} : \hat{D}_\varepsilon \to L^\infty$ is sectorial and $\hat{D}_\varepsilon \subset C^1(\bar{\Omega})$ (see [13] Corollary 3.1.24). There is a mild solution of (4.1) with initial value $u_0$ and for $t > 0$ this mild solution has values in $\hat{D}_\varepsilon$ (see [13] Theorem 7.1.2 and Proposition 7.1.10).

Thus for $t > T$ $u_0\pi_\varepsilon t = (u_0\pi_\varepsilon T)\pi_\varepsilon(t - T) \in \hat{D}_\varepsilon \subset C^1(\bar{\Omega})$, which proves the corollary. \[\Box\]

5. Comments

We would like to make some comments on the differences and difficulties which arise because the operators $A_\varepsilon$ are not self-adjoint.

We already mentioned the two most important differences in the linear case: in the non-self-adjoint case it is unclear if there is complete system of eigenfunctions of $A_\varepsilon$, but this difficulty can be overcome by the technique introduced by Antoci and Prizzi [2] using the convergence of the resolvents. The other point is the slightly weaker convergence of the linear semigroups in the norm $\|\cdot\|_{s,d}$, $d < 1$. The reason for this is that—unlike in the self-adjoint case—if $u_1 \in H^1 \setminus H^1_\varepsilon$, $u_2 \in H^1_\varepsilon$, then $a_\varepsilon(u_1, u_2)$ contains a term $\frac{1}{\varepsilon}\nabla_\varepsilon u_1$, which ultimately yields $a_\varepsilon(u_2, u_2) \nrown a_0(u_2, u_2)$.
In the nonlinear case the difficulties are greater.

If $A_\varepsilon$ is self-adjoint, then $D(A_\varepsilon^{\frac{1}{2}}) = H^1$ (see e.g. [14]) and one can work with fractional power spaces. Above fact is based upon the equality $a_\varepsilon(u, u) = (A_\varepsilon u, u)_{L^2} = (A_\varepsilon^{\frac{1}{2}} u, A_\varepsilon^{\frac{1}{2}} u)_{L^2}$ for $u \in D(A_\varepsilon)$. In our non-self-adjoint case the analogous equality is

$$a_\varepsilon(u, u) = (A_\varepsilon u, u)_{L^2} = (A_\varepsilon^{\frac{1}{2}} u, A_\varepsilon^{\frac{1}{2}} u)_{L^2}$$

($A_\varepsilon^{*}$ denotes the adjoint of $A_\varepsilon$) and one can’t bound $\|A_\varepsilon^{\frac{1}{2}} u\|_{L^2}$ without knowing something about $\|A_\varepsilon^{\frac{1}{2}} u\|_{L^2}$.

Not knowing $D(A_\varepsilon^{\frac{1}{2}})$ we defined semi-flows $\pi_\varepsilon$ on $L^{p_0}$, $p_0$ as in condition H1.

The semi-flows are rather on $L^{p_0}$ than on $H^1$ because there is still another difficulty: The natural Lyapunov-function (with respect to $\|\cdot\|_{\varepsilon, 1}$, see e.g. [2,7,14]) involves a term $a_\varepsilon(u, u)$. Now if $u(t)$ is a solution of Eq. (3.1) then $t \mapsto a_\varepsilon(u(t), u(t))$ is still differentiable, with derivative $a_\varepsilon(u_t(t), u_t(t)) + a_\varepsilon(u(t), u_t(t))$. In the self-adjoint case both these terms are equal to $(A_\varepsilon u(t), u_t(t))_{L^2}$ which is easy to bound. In the non-self-adjoint case we have (assuming sufficient regularity and $\beta_1 = 0$) $a_\varepsilon(u_t, u) = (A_\varepsilon u, u_t)_{L^2} + r_\varepsilon(u, u_t)$, where $r_\varepsilon$ is a boundary term of the same order as $a_\varepsilon(u, u_t)$ and if $v_1 \in H^2 \cap H^{\frac{1}{2}}_1$, $v_2 \in H^1$, then in general $r_\varepsilon(v_1, v_2) \to \infty$, as $\varepsilon \to 0$.

That the semi-flow on $H^1$ still has an attractor follows via the absorbing set and attractor of the semi-flow on $L^{p_0}$. This indicates that $L^{p_0}$ might be the more natural space for the semi-flows in this situation.

Acknowledgments

T.E. expresses his gratitude to Professor Dr. K.P. Rybakowski and Dr. M. Prizzi for their helpful advices and valuable suggestions.

References