Finite pseudocomplemented lattices and ‘permutoedre’

C. Chameni Nembua
Université de Yaoundé, 54 bd Raspail, F-75270 Paris Cedex 06, France

B. Monjardet
Université Paris 5 et CAMS, 54 bd Raspail, F-75270 Paris Cedex 06, France

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Abstract

We study finite pseudocomplemented lattices and especially those that are also complemented. With regard to the classical results on arbitrary or distributive pseudocomplemented lattices (Glivenko, Stone, Birkhoff, Frink, Grätzer, Balbes, Horn, Varlet, ...), the finiteness property allows one to bring significant, more precise, details on the structural properties of such lattices. These results can especially be applied to the lattices defined by the ‘weak Bruhat order’ on a Coxeter group (and, for instance, to the lattice of permutations, called, in french, ‘le treillis permutoédre’) and to the lattice of binary bracketings.

1. Introduction

Let \( L \) be a lattice with a least element denoted as 0; \( g(t) \in L \) is a meet pseudocomplement of \( t \in L \) if \( x \land t = 0 \) if and only if \( x \leq g(t) \); \( L \) is meet-pseudocomplemented if every element of \( L \) has a meet pseudocomplement. One defines dually the notion of a join pseudocomplement \( f(t) \) of \( t \) and of a join-pseudocomplemented lattice. A lattice is pseudocomplemented if it is meet- and join-pseudocomplemented (take care: usually ‘pseudocomplemented’ means only ‘meet-pseudocomplemented’ and a join-pseudocomplemented lattice is sometimes called a ‘dual pseudocomplemented’ lattice). Two classes of meet-pseudocomplemented lattices have been intensively studied. First, the Brouwerian (called also Heyting or implicative) lattices, that are the ‘relatively meet-pseudocomplemented’ lattices, implying that they are distributive [3,9]. Second, the Stone lattices, that are distributive meet-pseudocomplemented
lattices satisfying an additional condition [1, 14, 15]. References [10, 15, 16] provide excellent accounts of the known results on arbitrary meet-pseudocomplemented lattices or the above special classes. Note that in most of the studied cases the considered lattices are infinite. We are interested here in the specific properties of the class of finite (meet) pseudocomplemented lattices. Indeed, the lattice of permutations (called, in French, le 'treillis permutoèdre' [9]) and, more generally, the lattices defined by the 'weak Bruhat order' on a Coxeter group (see [4]) are (meet- and join-) pseudocomplemented lattices [2]; it is also the case of the lattice of the binary bracketings (see [11, 12]); moreover, all these lattices are also complemented.

In this paper we give a summary of our results on the structure of finite meet-pseudocomplemented, (meet- and join-)pseudocomplemented, and pseudocomplemented and complemented lattices; for a detailed account of the proofs of these results, see [6]. The specific theory of finite meet-pseudocomplemented lattices begins with the easy but crucial observation that a (finite) lattice is meet-pseudocomplemented if and only if each of its atoms has a meet pseudocomplement; so, in such lattices, the meet-pseudocomplements can be expressed by means of the meet pseudocomplements of the atoms (Theorem 2.1). Then one shows (Theorem 2.2) that the joins of atoms define a boolean lattice isomorphic with the lattice of the meet pseudocomplements, thus reobtaining the classical result [7] that this last lattice

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{The Glivenko classes of the lattice of permutations on 4 elements (treillis permutoèdre).}
\end{figure}
is boolean. Propositions 2.4 and 2.5 study the properties of an element in a meet-pseudocomplemented lattice and especially when it admits a complement. Theorem 2.6 characterizes the meet-pseudocomplemented lattices that are complemented, such lattices being the same as the complemented join-pseudocomplemented lattices and, thus, that the complemented (and) pseudocomplemented lattices (Theorem 2.7). Theorem 2.8 gives other characterizations of such complemented pseudocomplemented lattices, whereas the last theorem summarizes all our results on the structure of such lattices; for instance, we show that the Glivenko congruence, ‘to have the same meet pseudocomplement’, is the same as ‘to have the same join pseudocomplement’ or that ‘to have the same complements’ or, etc. (see Theorem 2.9(4)); the classes of this congruence are the $2^n$ intervals $[\bigvee A(x), \bigwedge C(x)]$ with $A(x) = \{\text{atoms } a : a \leq x\}$, $C(x) = \{\text{coatoms } c : c \geq x\}$, and $n$ the number of atoms – or coatoms – of $L$. These results can be applied to the ‘concrete’ lattices mentioned above; for instance, Fig. 1 shows the lattice of permutations on 4 elements with the $2^3$ classes of the Glivenko congruence.

2. Results

In this paper $L$ denotes a finite lattice; $0$ ($1$) denotes the least (the greatest) element of $L$; $\leq, \wedge, \text{ and } \vee$ denotes the order relation (the covering relation) the infimum, – or meet – operation, the supremum – or join – operation) defined on $L$.

An element $t_*$ ($t^*$) of $L$ is a meet-pseudocomplement (a join-pseudocomplement) of an element $x$ of $L$ if $x \wedge t = 0 \iff x \leq t_*$ ($x \vee t = 1 \iff x \geq t^*$); in other words, the set of all elements $x$ such that $x \wedge t = 0$ has a greatest element $t_*$. A lattice $L$ is meet-pseudocomplemented (join-pseudocomplemented) if each element of $L$ has a meet pseudocomplement (a join pseudocomplement); then we denote by $g$ ($f$) the map $t \mapsto g t = t_*$ ($t \mapsto f t = t^*$); $L$ is pseudocomplemented if it is both meet- and join-pseudocomplemented (take care: usually ‘pseudocomplemented’ means only ‘meet-pseudocomplemented’).

An obvious – but significant – observation made by Birkhoff [3] is that in a meet-pseudocomplemented lattice $L$ the map $g$ of pseudocomplementation is a ‘symmetric Galois connection’ (for the definitions of a Galois connection and of other notions of lattice theory not defined here, see [3] or [10]; then $g^3 = g$, $g^2 = \varphi$ is a closure operator on $L$ and the set $G = g(L)$ of all meet pseudocomplements is a lattice sub-meet-semilattice of $L$. We recall also the following classical results:

$$g(x \vee y) = gx \wedge gy, \quad g(x \wedge y) = g(\varphi x \wedge \varphi y)$$

$$\varphi(x \vee y) = g[gx \wedge gy], \quad \varphi(x \wedge y) = \varphi x \wedge \varphi y.$$

An atom (a coatom) of $L$ is an element covering $0$ (covered by $1$); we denote by $A$ ($C$) the set of all atoms (coatoms) of $L$; for $x$ in $L$, $A(x) = \{a \in A : a \leq x\}$, $C(x) = \{c \in C : c \geq x\}$; $A'(x) = A - A(x)$; $C'(x) = C - C(x)$. 


Theorem 2.1. Let $L$ be a (finite) lattice; the two following conditions are equivalent:
1. $L$ is a meet-pseudocomplemented lattice,
2. each atom of $L$ has a meet-pseudocomplement (denoted by $g_a$)
These conditions imply that, for all $x, y$ in $L$,
3. $g_a x = \bigwedge g[A(x)] = g[\bigvee A(x)]$,
4. $\varphi x = \varphi[\bigvee A(x)] = \bigwedge (g[A(gx)])$,
5. $A(x) = A(\varphi x)$,
6. $g_a x \rightarrow g_a y \Leftrightarrow \varphi x \rightarrow \varphi y \Leftrightarrow A(x) = A(y)$.

We denote by $\Pi$ the equivalence defined on $L$ by the conditions in (6) above; it is easy to see that $\Pi$ is a congruence on $L$ called the Glivenko congruence; then $L/\Pi$ is isomorphic to the lattice $G$ of the meet pseudocomplements; $\Pi x$ will denote the congruence class of $x$.

Let $\mathcal{A}^\vee = \{ \bigvee X, X \subseteq A \}$ be the set of all the join of sets of atoms of $L$; $\mathcal{A}^\vee$ is a lattice for the order defined on $L$ (note that $\bigvee \emptyset = 0$).

Theorem 2.2. Let $L$ be a meet-pseudocomplemented lattice:
1. The lattice $\mathcal{A}^\vee$ of the join of atoms of $L$ is a boolean lattice, sub-join-semilattice of $L$ and with same 0.
2. $g(\varphi)$ induces an anti-isomorphism (an isomorphism) between $\mathcal{A}^\vee$ and the lattice $G$ of the meet pseudocomplements of $L$:
   \[ A(g_a x) = A'(x), \quad g(x) = \varphi[\bigvee A'(x)]. \]
3. The classes of the Glivenko congruence $\Pi$ are the $2^{|A|}$ intervals $[\bigvee X, \varphi(\bigvee X)]$, where $X \subseteq A$:
   \[ \Pi x = [\bigvee A(x), \varphi(\bigvee A(x))], \quad \Pi(g_a x) = [\bigvee A'(x), \varphi(\bigvee A'(x))]. \]

Corollary 2.3. The lattice $G$ of all meet pseudocomplements of a (finite) lattice $L$ is a boolean lattice, sub-meet-semilattice of $L$, with same 0 and 1.

Remark. Indeed this last result is true for an arbitrary (meet) pseudocomplemented meet-semilattice [7]. The Frink proof uses a concise axiomatic of a boolean lattice, whereas Grätzer [10] gives a direct proof of the distributivity of $G$.

Obviously, there are dual results for the join-pseudocomplemented lattices, with the coatoms of $L$ playing the role of atoms. We now give two propositions on the (meet- and join-)pseudocomplemented lattices, preparing our results on the complemented pseudocomplemented lattices.

Proposition 2.4. Let $x$ be an element of the pseudocomplemented lattice $L$, $|A| = n$,
\[ |C| = p, f^\prime x = \bigvee A'(x), g'x = \bigwedge C'(x); \]
1. $x \bigvee y = 1 \Rightarrow y \gg f^\prime x$. 

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(2) \( x \wedge y = 0 \Rightarrow y \leq g'x \),
(3) \( f'x \leq gx \wedge fx, \quad g'x \geq gx \vee fx, \)
(4) \( C'(x) = g[A(x)] \iff gx = g'x \Rightarrow C(gx) = C(g'x) \Leftrightarrow C(gx) = C(fx) \Rightarrow |C(x)| + |C(gx)| = p. \)

Proposition 2.5. Let \( x \) be an element of the pseudocomplemented lattice \( L \); the following conditions (1) or (2) imply the equivalent conditions (3) and (4):

1. \( gx = g'x \),
2. \( fx = f'x \),
3. \( fx \leq gx \),
4. \( x \) has a complement in \( L \).

Examples of pseudocomplemented lattices are the distributive lattices, since \( gx = \bigvee \{ t \in L : x \wedge t = 0 \} \), and \( fx = \bigwedge \{ t \in L : x \vee t = 1 \} \). On the contrary (nondistributive), upper locally distributive lattices (see [13] for a presentation of these lattices first studied by Dilworth) are meet-pseudocomplemented not pseudocomplemented lattices. As said in the introduction, the lattices defined by the 'weak Bruhat order' on a (finite) Coxeter group and the lattice of binary bracketings are pseudocomplemented and complemented lattices. So, we now come to our results on complemented pseudocomplemented lattices. First we give 6 characterizations of such lattices (see also Theorem 2.7).

Theorem 2.6. Let \( L \) be a meet-pseudocomplemented lattice; the following conditions are equivalent:

1. \( L \) is complemented,
2. \( \Pi \Pi \{1\} \),
3. \( A(x) = A \iff x = 1 \),
4. \( \bigvee A = 1 \),
5. \( g \) induces a bijection between \( A \) and \( C \),
6. \( \varphi \) is the identity map on \( C \),
7. \( L \) is strictly meet-semicomplemented (i.e. \( x \neq 1 \) implies that there exists \( y \neq 0 \), with \( x \wedge y = 0 \)).

Remark. In Theorem 2.6 condition (7) is due to Varlet [15].

Indeed, the significant following result shows that the complemented meet-pseudocomplemented lattices are the complemented pseudocomplemented lattices (and also the complemented join-pseudocomplemented lattices).
Theorem 2.7. For a complemented lattice \( L \), the three following conditions are equivalent:

1. \( L \) is meet-pseudocomplemented.
2. \( L \) is join-pseudocomplemented.
3. \( L \) is pseudocomplemented.

We now give other characterizations of pseudocomplemented lattices \( L \) that are also complemented; in the following results, \( \Pi^* \) denotes the congruence on \( L \) defined by \( x \Pi^* y \iff f_x = f_y \iff C(x) = C(y) \).

Theorem 2.8. Let \( L \) be a pseudocomplemented lattice having \( n \) atoms and \( p \) coatoms; the following conditions are equivalent:

1. \( L \) is complemented,
2. For every \( x \in L \), \( g_x = g'_x \) (or \( |C(x)| + |C(g_x)| = p \)),
3. For every \( x \in L \), \( f_x = f'_x \) (or \( |A(x)| + |A(f_x)| = n \)),
4. For every \( x \in L \), \( f_x \leq g_x \),
5. For every \( x \in L \), \( \Pi(g_x) = \Pi^*(f_x) \),
6. \( \Pi = \Pi^* \),
7. \( \Pi 0 = \Pi^* 0 \),
8. \( \Pi 1 = \Pi^* 1 \).

The following theorem summarizes all our results on the structure of complemented pseudocomplemented lattices; there \( \Psi = f^2 \) is the anticlosure operator defined on a (join) pseudocomplemented lattice \( L \), \( F = f(L) = \Psi L \), and \( C \) is the lattice formed by all the meet of coatoms.

Theorem 2.9. (1) \( f^3 = f = f^F = f^\Psi = f^\Psi g = f = f \leq g = g^3 = g \leq \varphi \); \( \Psi - g \leq \varphi = g = f^\Psi g = g \leq \varphi \); \( \Psi - g \leq \varphi = g = f^\Psi g \leq \varphi \); \( g = g'x = g[A(x)]; f(x) = f'[x] = f[\sqrt{A}(x)]; C'(x) = C(gx) = C(fx) - g[A(x)]; A'(x) - A(fx) = A(gx) = f[A(x)]; \)

(2) \( G = g(L) = \varphi L = C^\varphi \) is a boolean lattice, sub-meet-semilattice of \( L \) with same 0 and 1; \( F = f(L) = \Psi L = A^\Psi \) is a boolean lattice, sub-join-semilattice of \( L \), with same 0 and 1; \( f \) and \( g \) (\( \varphi \) and \( \Psi \)) induce two inverse antiisomorphisms (isomorphisms) between \( G \) and \( F \); \( g \) on \( G \) (\( f \) on \( F \)) is the complementation in this lattice:

\[ gA = C = \varphi C, \quad fC = A = \Psi A. \]

(3) \( g(f) \) is a morphism of \( L \) on the dual of \( G(F) \). \( \varphi(\Psi) \) is a morphism of \( L \) on \( G(F) \).

(4) For \( x, y \in L \), \( g(x) = g(y) \iff g(x) = g(y) \iff A(x) = A(y) \iff \sqrt{A}(x) = \sqrt{A}(y) \iff f_x = f_y \iff \Psi x = \Psi y \iff C(x) = C(y) \iff \sqrt{C}(x) = \sqrt{C}(y) \iff [f_x, g_x] = [f_y, g_y] \iff [\Psi x, \varphi x] = [\Psi y, \varphi y] \).

The relation \( \Pi \) defined on \( L \) by these equivalent equalities is a congruence; the congruence classes of \( \Pi \) are the \( 2^n \) intervals \([\sqrt{A}(x), \sqrt{C}(x)]\) (with \( n = |A| - |C| \)) and \( g \) and \( f(\varphi \) and
\( \Psi \) are isomorphisms (involutive antiisomorphisms) between \( L/II, F \) and \( G \); let \( (IIx)' \) be the class complement of the class \( IIx \) in the boolean lattice \( L/II \); then

\[
\begin{align*}
gx &= \max [(IIx)'], \\
fx &= \min [(IIx)'], \\
\phi x &= \max (IIx), \\
\Psi x &= \min (IIx),
\end{align*}
\]

\[
\{\text{complements of } x \text{ in } L\} = (IIx)' = [fx, gx].
\]

(5) \( II1 = \{1\} - \sqrt{A}, \quad II0 = \{0\} = \bigvee C. \)

Note added in proof. Several results of this paper were first published in the Chameni-Nembua thesis (1989, see reference below) and in a CAMS report (C. Chameni-Nembua and B. Monjardet, *Les treillis pseudocomplémentés finis*, Rapport CAMS P 061, Paris, 1990). Since 1991 we become aware of a M.K. Bennett and G. Birkhoff preprint (1991, to appear in *Algebra Universalis*) Two families of Newman lattices, of a series G. Markowsky reports on the permutation (or ‘permutoédre’) lattice, beginning in 1990, (the last one being *Permutation lattices revisited*, August 1992, Univ. of Maine), and of a annex (Retracts and Glivenko intervals) written with G. Markowsky in 1992 to a Birkhoff paper (to appear in the Proceedings of the 1991 Darmstadt Conference on lattice theory); the Bennett and G. Birkhoff paper studies a class of lattices containing both the permutation lattice and the binary bracketings lattice (especially it contains new results on this last one); the Markowsky reports contains old and new results on the permutation lattice (some of them have been independently get by V. Duquenne and A. Cherfouh, *On the permutation lattice*, Rapport CAMS P 077, Paris, 1991); these reports and especially the above quoted annex contain also several results that have been obtained but not published by C. Le Conte de Poly-Barbut (see reference below) and that are special cases of theorems of this paper; indeed the C. Le Conte de Poly-Barbut results on the permutation lattice were our main motivation to study the more general class of finite pseudocomplemented (and possibly complemented) lattices. The reader will be able to find in the Chameni-Nembua and Monjardet paper *Les treillis pseudocomplémentés finis* (reference below) a much more complete bibliography on the permutation lattice and related topics (we just add here that the order dimension of the ‘multinomial lattices’ and especially of the permutation lattice has been determined by S. Flath, preprint 1492, Darmstadt, 1992).

References


