J. Differential Equations 248 (2010) 1879–1897

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Minimal sets in monotone and sublinear skew-product semiflows I: The general case ☆

Carmen Núñez [a], Rafael Obaya [a,*], Ana M. Sanz [b]

[a] Departamento de Matemática Aplicada, E.T.S. de Ingenieros Industriales, Universidad de Valladolid, 47011 Valladolid, Spain
[b] Departamento de Matemática Aplicada, E.U. Ingenierías Agrarias, Universidad de Valladolid, 42004 Soria, Spain

ARTICLE INFO

Article history:
Received 18 June 2009
Revised 2 December 2009
Available online 27 January 2010

MSC:
37B55
37C65
37C70
37C75

Keywords:
Topological dynamics
Skew-product semiflow
Monotonicity
Sublinearity
Omega-limit sets
Minimal sets
Attractivity properties

ABSTRACT

The dynamics of a general monotone and sublinear skew-product semiflow is analyzed, paying special attention to the long-term behavior of the strongly positive semiorbits and to the minimal sets. Four possibilities arise depending on the existence or absence of strongly positive minimal sets and bounded semiorbits, as well as on the coexistence or not of bounded and unbounded strongly positive semiorbits. Previous results are unified and extended, and scenarios which are impossible in the autonomous or periodic cases are described.

1. Introduction

Dynamical methods for sublinear, concave or convex and monotone semiflows have been extensively studied in different contexts during the last decades. The obvious motivations are the high interest of the mathematical problems involved as well as the conclusions provided by the analysis of

☆ The authors were partly supported by Junta de Castilla y León under project VA060A09, and Ministerio de Ciencia e Innovación under project MTM2008-00700/MTM.

* Corresponding author.
E-mail addresses: carnun@wmatem.eis.uva.es (C. Núñez), rafoba@wmatem.eis.uva.es (R. Obaya), anasan@wmatem.eis.uva.es (A.M. Sanz).

0022-0396/$ – see front matter
doi:10.1016/j.jde.2009.12.007

these problems, fundamental in the description of different models appearing in engineering, biology, economics and other applied sciences.

In this paper we work with a real continuous skew-product semiflow on a product space $\Omega \times X_+$: the base component is given by a minimal flow on a compact metric space Ω; and the fiber or state component of the semiflow is defined on the normal positive cone X_+ of a strongly ordered Banach space X. We assume standard monotonicity and sublinearity properties referred to the state component. We also assume that every bounded semiorbit is a relatively compact set, and hence it gives rise to a compact omega-limit set. In many of the applications, this type of setting arises from the analysis of a nonautonomous differential equation for which the time dependence presents recurrence properties. The space Ω is then the hull of the coefficients, which reduces to a point in the autonomous case and to a circle in the periodic one. Although the presentation made in this paper is more general, by analogy with this particular case, we talk about the autonomous or periodic case when Ω is a point or a circle, respectively.

In our abstract and quite general framework, we analyze the long-term behavior of the strongly positive semiorbits and describe the global dynamics, paying special attention to the minimal sets of the semiflow, which frequently agree with the omega-limit sets of the bounded semiorbits and always provide useful information about them. The so-called copies of the base, which play the equivalent role to the equilibrium points in the autonomous setting or to the T-periodic solutions in the T-periodic one, are crucial in this description. The conclusions of the paper unify and extend several previous results of different authors, showing in addition the occurrence of dynamical scenarios which cannot appear in the autonomous or periodic cases. The arguments here used can also be adapted to provide a version of the results for the case of a not real but discrete skew-product semiflow, although this point is not included in the present paper.

Let us briefly summarize several interesting references closely related to this work, which is highly motivated by them. Pioneer results of the theory are due to Krasnoselskii [18,19]. After this, a first set of works refers to the existence of a constant or a periodic solution in the autonomous or periodic cases which is globally asymptotically stable. This question is analyzed via dynamical arguments in the classical papers of Hirsch [9,10], Selgrade [36] and Smith [39], among others. Their results require some of the properties of monotonicity and sublinearity, convexity or concavity to be strong. Much more recently but in the same line, Zhao [44] and Novo et al. [25] obtain more general versions of these results, valid for recurrent nonautonomous differential equations; and Núñez et al. [28] obtain the analogous conclusions in the case of recurrent families of functional differential equations with infinite delay, by analyzing the omega-limit sets for the compact-open topology.

Other series of well-known references are devoted to describe the different limiting sets attracting asymptotically the semiorbits. In [40], Takáç studies the asymptotic behavior of discrete time semigroups of sublinear strongly increasing maps, proving that the set of strongly positive equilibria constitutes an arc: an initial one and its multiples, the multiplying parameter varying in an interval. In [41], Takáç extends these results to ray contractive maps defined on abstract Hausdorff topological cones. Novo and Obaya [24] analyze strongly monotone and convex skew-product semiflows generated by recurrent families of functional differential equations in order to prove that the minimal sets are multiples of an initial one, providing in this way a nonautonomous version of Takáç' results. And Wang and Zhao [43] and Jiang and Zhao [12], among other authors, weaken the conditions of monotonicity ensuring the convergence of different semiorbits to different minimal sets whose structure they describe precisely.

A third type of works study the global dynamics of a monotone and sublinear semiflow in a general situation. Frequently a "limit set trichotomy" statement is obtained: it describes the unique three possibilities for the asymptotic behavior of the semiorbits. This theory was introduced by Krause and Ranft [21] for monotone and ray concave transformations and extended later by Krause and Nussbaum [20] for more general non-expansive continuous maps. The version of Nesemann [22] can be applied to certain classes of nonautonomous difference equations. And Freedman and Zhao [7] obtain results of this type for quasimonotone systems of periodic reaction–diffusion equations. We finally point out that the basis for an alternative general theory for random dynamical systems including the three aspects mentioned has been established by Arnold and Chueshov [2,3] and Chueshov [5].

Let us now sum up the results we obtain in the very general setting described above. The monotonicity and sublinearity properties ensure that a semiorbit is either eventually strongly positive or contained in the border of the phase space, given by those points with non-strongly positive state component. Roughly speaking, we show that one of the following four dynamical situations occurs.

Case A. Every semiorbit is bounded, the omega-limit set of every strongly positive state is a strongly positive minimal set, and every non-strongly positive omega-limit set is contained in the border of the phase space. There can be a unique strongly positive minimal set, in which case it is a copy of the base (the nonautonomous analogue of an equilibrium) with strong attractiveness properties (Case A1); or infinitely many, in which case either there exists a lowest strongly positive minimal set given by a copy of the base which attracts asymptotically the eventually strongly positive semiorbits starting below it (Case A2), or the strongly positive minimal sets approach $\Omega \times (X_+ - \operatorname{Int} X_+)$ (Case A3). In Case A2 or A3 there may exist or not a top strongly positive minimal set. If fact, its existence is equivalent to the boundedness on $\Omega \times X_+$ of the union of all the strongly positive minimal sets. The top minimal set is also a copy of the base and attracts asymptotically the semiorbits starting above it. In addition, except for Case A1, the existence of a strongly positive copy of the base ensures the existence of a continuous one-parameter family of copies of the base. The key point in the achievement of these results is the construction of a labeling map, closely related to the part metric.

We also prove that the dynamics fits Case A1 if the semiflow is eventually strongly sublinear at least on the semiorbits with initial data projecting on a fixed base point. And, assuming a strong separation property for the semiorbits whose initial data project on a dense subset of Ω, we deduce that in Cases A2 and A3 each minimal set is a copy of the base and all of them are multiples of a fixed one, the multiplying parameter varying in a bounded or unbounded interval. This unifies and extends the nonautonomous results of [44,24,25] cited before.

Case B. Every semiorbit is bounded, and every minimal set is contained in the border of the phase space. Under the strong separation property mentioned in Case A, the inferior limit as $t \to \infty$ of the norm of the state component of every semiorbit is zero. It is possible for an omega-limit set to contain also strongly positive states. These situations allow the occurrence of a highly complicated dynamics: compact pinched sets which are not copies of the base, sensitive dependence with respect to initial data, several ergodic measures, etc. The examples of monotone and sublinear discrete skew-product semiflows of Keller [17] and Jäger [11], samples of this dynamical situation, present in addition positive Lyapunov exponents and strange non-chaotic attractors.

Case C. There coexist strongly positive initial states giving rise to bounded and unbounded semiorbits. As before, although every minimal set is contained in the border of the phase space, strongly positive states and non-strongly positive ones can coexist in an omega-limit set. Assuming again the strong separation property, we show the existence of a residual set $\Omega_0 \subsetneq \Omega$ giving rise to strongly positive semiorbits whose states oscillate (in norm) from 0 to ∞ as $t \to \infty$. This is an infinite-dimensional version of the classical oscillation result of Johnson [13,15] for scalar linear nonautonomous equations. In Case C there can again occur the complicated dynamics associated to compact pinched sets.

Case D. Every strongly positive initial state gives rise to an unbounded semiorbit. In this situation, an oscillatory behavior of the strongly positive semiorbits similar to the one described in Case C can occur, and there can also exist strictly positive bounded semiorbits.

Note that in Cases B and C we talk about the omega-limit sets of bounded semiorbits as well as about the minimal sets they contain. Note also that the term "trichotomy" is not suitable in this deterministic nonautonomous setting, since four quite different dynamical situations arise.

It is an interesting open question to determine whether all the minimal sets are or not copies of the base in Cases A–D when no additional monotonicity conditions hold.

We finally want to remark again that the hypotheses assumed on this paper are very general: neither a lattice structure nor a product structure with componentwise separation property for each factor are required on the space X_+. Clearly, these additional conditions make it possible to describe more accurately the minimal sets and the global dynamics. This is the main objective of the paper Núñez et al. [29], natural continuation of this one, where we study in detail the dynamics generated by sublinear and monotone families of two-dimensional nonautonomous differential equations, of ordinary, finite-delay and reaction–diffusion type.

Section 2 contains a summary of the basic concepts we need, and Section 3 contains the dynamical description we obtain.

2. Basic notions and framework of the problems

Let us divide this preliminary section in two parts: we begin by recalling some basic concepts and properties on topological dynamics, which can be found in Ellis [6], Sacker and Sell [34], Shen and Yi [38] and references therein; and after this we describe more precisely the framework of our work.

A (*real and continuous*) *global flow* on a complete metric space Ω is a continuous map $\sigma : \mathbb{R} \times \Omega \to \Omega$, $(t, \omega) \mapsto \sigma(t, \omega)$ satisfying $\sigma_0 = \mathrm{Id}$ and $\sigma_{s+t} = \sigma_t \circ \sigma_s$ for each $s, t \in \mathbb{R}$, where $\sigma_t(\omega) = \sigma(t, \omega)$. By replacing $\mathbb{R}$ by $\mathbb{R}_+ = \{t \in \mathbb{R} \mid t \geqslant 0\}$, we obtain the definition of a (*real and continuous*) *global semiflow* on Ω.

Let $(\Omega, \sigma, \mathbb{R})$ be a global flow. The *orbit* of a point $\omega \in \Omega$ is the set $\{\sigma_t(\omega) \mid t \in \mathbb{R}\}$. A subset $\Omega_1 \subset \Omega$ is *σ-invariant* if $\sigma_t(\Omega_1) = \Omega_1$ for every $t \in \mathbb{R}$. A σ-invariant subset $\Omega_1 \subset \Omega$ is *minimal* if it is compact and does not contain properly any other compact σ-invariant set; or, equivalently, if the orbit of any one of its elements is dense in it. The continuous flow $(\Omega, \sigma, \mathbb{R})$ is *recurrent* or *minimal* if Ω itself is minimal.

In the case of a global semiflow $(\Omega, \sigma, \mathbb{R}_+)$, the (*positive*) *semiorbit* of a point ω is the set $\{\sigma_t(\omega) \mid t \geqslant 0\}$. A subset Ω_1 of Ω is *positively σ-invariant* if $\sigma_t(\Omega_1) \subset \Omega_1$ for all $t \geqslant 0$. A positively σ-invariant subset $K \subset \Omega$ is *minimal* for the semiflow if it is compact and it does not contain properly any closed, positively σ-invariant subset. And $(\Omega, \sigma, \mathbb{R}_+)$ is a *minimal* semiflow if Ω itself is minimal.

A *flow extension* of the semiflow $(\Omega, \sigma, \mathbb{R}_+)$ is a continuous flow $(\Omega, \tilde{\sigma}, \mathbb{R})$ such that $\tilde{\sigma}(t, \omega) = \sigma(t, \omega)$ for each $\omega \in \Omega$ and $t \geqslant 0$. A compact positively σ-invariant subset *admits a flow extension* if the restricted semiflow does. It is proved by Shen and Yi [38] that a positively σ-invariant compact set K admits a flow extension if every point in K admits a unique backward orbit which remains inside the set K. A *backward orbit* of a point $\omega \in \Omega$ is a continuous map $\psi : \mathbb{R}_- \to \Omega$ such that $\psi(0) = \omega$ and for each $s \leqslant 0$ it is $\sigma(t, \psi(s)) = \psi(s + t)$ whenever $0 \leqslant t \leqslant -s$.

Finally, the *omega-limit set* of a point $\omega_0 \in \Omega$ (or of its semiorbit) whose semiorbit for a global flow or semiflow σ is relatively compact, $\mathcal{O}(\omega_0)$, is given by those points $\omega \in \Omega$ such that $\omega = \lim_{n\to\infty} \sigma(t_n, \omega_0)$ for some sequence $(t_n) \uparrow \infty$. The omega-limit set is nonempty, compact, connected and positively σ-invariant, and each one of its points admits a backward orbit inside this set. Clearly, a minimal set is the omega-limit set of any of the semiorbits contained in it.

As said in the Introduction, we will work with global skew-product semiflows defined on a trivial bundle whose base is a compact metric space and whose fiber is the normal positive cone of a strongly ordered Banach space. Some monotonicity and sublinearity properties will be also imposed. Let us briefly describe these concepts, which can also be found in Amann [1], Vulikh [42], and references therein.

In what follows, $(\Omega, \sigma, \mathbb{R})$ represents a real continuous minimal global flow on a compact metric space, and we denote $\sigma(t, \omega) = \omega \cdot t$. The minimality of the base plays a fundamental role in the description of the dynamics we are going to obtain. Given a Banach space X, a real global *skew-product semiflow* $(\Omega \times X, \tau, \mathbb{R}_+)$ projecting onto $(\Omega, \sigma, \mathbb{R})$ is a real continuous global semiflow on $\Omega \times X$ of the form

$$\tau : \mathbb{R}_+ \times \Omega \times X \to \Omega \times X, \qquad (t, \omega, x) \mapsto \big(\omega \cdot t, u(t, \omega, x)\big).$$

That is, τ reproduces the flow on the base space Ω and is given on the fiber by a map $u:\mathbb{R}_+ \times \Omega \times X \to X$ which satisfies the cocycle property

$$u(s+t,\omega,x) = u\big(t,\omega\cdot s,u(s,\omega,x)\big) \quad \text{for } s,t \geqslant 0 \text{ and } (\omega,x)\in\Omega\times X. \tag{2.1}$$

Let K be a minimal subset of $\Omega \times X$. It is immediate to deduce from the minimality of the base that the section $K_\omega = \{x \in X \mid (\omega,x) \in K\}$ contains at least one element for each $\omega\in\Omega$. In the case that K_ω contains a unique element for every $\omega\in\Omega$, K is a *copy of the base*. Clearly, a copy of the base agrees with the graph of a continuous map $c:\Omega\to X$ which is a solution of the invariance equation $c(\omega\cdot t) = u(t,\omega,c(\omega))$, $\omega\in\Omega$ and $t\geqslant 0$. Such a map c is also referred to as a *continuous equilibrium*. By minimality and uniqueness, the restriction of the semiflow to this set admits a flow extension which reproduces the flow on the base.

The Banach space X is *strongly ordered* if there is a closed convex solid cone (i.e., a nonempty closed subset $X_+ \subset X$ satisfying $X_+ + X_+ \subset X_+$, $\mathbb{R}_+X_+ \subset X_+$ and $X_+ \cap (-X_+) = \{0\}$) with nonempty interior. The (partial) *strong order relation* in X is defined by

$$\begin{aligned} x \leqslant y \quad &\Leftrightarrow \quad y - x \in X_+; \\ x < y \quad &\Leftrightarrow \quad y - x \in X_+ \quad \text{and} \quad x \neq y; \\ x \ll y \quad &\Leftrightarrow \quad y - x \in \operatorname{Int} X_+. \end{aligned}$$

The relations $\geqslant$, $>$ and $\gg$ are defined in the obvious way. We will always assume that the positive cone X_+ is *normal*, that is, that the norm of the Banach space X is *semimonotone*: there is a positive constant $l > 0$ such that $\|x\| \leqslant l\|y\|$ whenever $0 \leqslant x \leqslant y$. The norm of X is *monotone* if $l = 1$. In a normal cone, any vector $e \gg 0$ makes it possible to define a monotone norm by $\|x\|_e = \inf\{r > 0 \mid -re \leqslant x \leqslant re\}$, which is equivalent to the one defining the topology of X (see [1] for details).

The skew-product semiflow $(\Omega\times X,\tau,\mathbb{R}_+)$ is *monotone* if

$$u(t,\omega,x) \leqslant u(t,\omega,y) \quad \text{for } t\geqslant 0, \omega\in\Omega \text{ and } x,y\in X \text{ with } x\leqslant y. \tag{2.2}$$

Note that monotone semiflows are forward dynamical systems on ordered metric spaces which preserve the order of initial states along the semiorbits. Given two minimal sets K_1, K_2 in a monotone skew-product semiflow, we say that $K_1 \leqslant K_2$ if for any $(\omega,x_1)\in K_1$ there exists $(\omega,x_2)\in K_2$ with $x_1 \leqslant x_2$. We remark that, thanks to minimality, it suffices for $K_1 \leqslant K_2$ that there exist $(\omega_0,x_1)\in K_1$ and $(\omega_0,x_1)\in K_2$ with $x_1\leqslant x_2$. We write $K_1 < K_2$ if $K_1 \leqslant K_2$ and $K_1 \neq K_2$.

A skew-product semiflow $(\Omega\times X,\tau,\mathbb{R}_+)$ is said to have the *positivity property* if $u(t,\omega,x)\geqslant 0$ for $t\geqslant 0$, $\omega\in\Omega$ and $x\geqslant 0$. In other words, if the positive cone is τ-invariant. In this case we can consider the restricted semiflow $(\Omega\times X_+,\tau,\mathbb{R}_+)$. Note that, if the semiflow is monotone and $u(t,\omega,0)\geqslant 0$ for any $t\geqslant 0$ and $\omega\in\Omega$, the semiflow has the positivity property. In this setting, we write $K\geqslant 0$ for a compact set $K\subset\Omega\times X_+$; $K>0$ if $K\geqslant 0$ and there exists $(\omega,x)\in K$ with $x>0$; and $K\gg 0$ if K is *strongly positive*, i.e., if $K\subset\Omega\times\operatorname{Int}X_+$; or equivalently, as easily deduced from the compactness of K, if there exists $e\gg 0$ such that $e\leqslant x$ for every $(\omega,x)\in K$.

A skew-product semiflow with the positivity property is *sublinear* if

$$u(t,\omega,\lambda x) \geqslant \lambda u(t,\omega,x) \quad \text{for } t\geqslant 0, \omega\in\Omega, x\in X_+ \text{ and } \lambda\in[0,1]; \tag{2.3}$$

or equivalently, if

$$u(t,\omega,\mu x) \leqslant \mu u(t,\omega,x) \quad \text{for } t\geqslant 0, \omega\in\Omega, x\in X_+ \text{ and } \mu\geqslant 1.$$

We now recall the standard definitions of uniform stability. A positive semiorbit $\{\tau(t,\omega,x) \mid t \geqslant 0\}$ of the skew-product semiflow is said to be *uniformly stable* if for every $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that, if $s \geqslant 0$ and $\|u(s,\omega,x) - y\| \leqslant \delta(\varepsilon)$ for certain $y \in X$, then for $t \geqslant 0$,

$$\|u(s+t,\omega,x) - u(t,\omega\cdot s,y)\| = \|u\big(t,\omega\cdot s,u(s,\omega,x)\big) - u(t,\omega\cdot s,y)\| \leqslant \varepsilon.$$

A positively invariant compact set $M \subseteq \Omega \times X$ is *uniformly stable* if given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $(\omega,y) \in M$ and $(\omega,x) \in \Omega \times X$ satisfy $\|y - x\| \leqslant \delta(\varepsilon)$, then $\|u(t,\omega,y) - u(t,\omega,x)\| \leqslant \varepsilon$ for any $t \geqslant 0$. It is well known that, if the semiorbit of certain (ω,x) is relatively compact and uniformly stable, then the omega-limit set $\mathcal{O}(\omega,x)$ is a uniformly stable positively invariant compact set (see [37]).

We finally recall the concept of distallity in the fiber. A pair of points sharing base component, (ω,x) and (ω,y), are (*positively*) *distal* if $\inf_{t\geqslant 0}\|u(t,\omega,x) - u(t,\omega,y)\| > 0$; i.e., if the semiorbits keep at a positive distance. And a positively invariant compact set M is (*positively*) *fiber-distal* if any two distinct points in M form a distal pair.

3. Long-term behavior and minimal sets

Throughout the rest of the paper we consider a minimal global flow $(\Omega,\sigma,\mathbb{R})$, a strongly ordered Banach space X with normal positive cone X_+, and a real continuous skew-product global semiflow $(\Omega \times X_+,\tau,\mathbb{R}_+)$. As said in the Introduction, the aim of this section is to describe the semiflow dynamics, with special interest in the minimal sets and the long-term behavior of the semiorbits corresponding to strongly positive initial states, which we do assuming that

(h1) the semiflow is monotone and sublinear, and
(h2) any bounded semiorbit is a relatively compact set.

It is clear that, under condition (h2), any bounded semiorbit provides a well-defined omega-limit set, which in turn contains (possibly properly) a minimal set. We will show that under conditions (h1) and (h2) the dynamics fits one of the situations A1, A2, A3, B, C or D roughly described in the Introduction. Note that, due to monotonicity, the existence of a bounded semiorbit (and hence of a minimal set, projecting on the whole base Ω) is equivalent to the boundedness of the semiorbit starting at $(\omega,0)$ for any $\omega \in \Omega$.

We begin by deducing some easy but crucial consequences of the monotonicity and sublinearity assumptions made on the semiflow. In particular, we show that any semiorbit is either eventually strongly positive or contained in the border of the phase space (which, a priori, has no consequences on its omega-limit set). And after this, in Theorem 3.2, we show that in a minimal set there cannot coexist strongly positive states and non-strongly positive ones.

Proposition 3.1. *Assume that the skew-product semiflow on $\Omega \times X_+$ satisfies hypothesis* (h1)*, and fix $e \gg 0$. Then*:

(i) *$u(t,\omega,e) \gg 0$ for every $t \geqslant 0$ and $\omega \in \Omega$. In particular,* $\inf_{t\geqslant 0}\|u(t,\omega,e)\| = 0$ *if and only if* $\liminf_{t\to\infty}\|u(t,\omega,e)\| = 0$.
(ii) *Let us fix $\omega \in \Omega$. Then* $\sup_{t\geqslant 0}\|u(t,\omega,e)\| < \infty$ (*resp.* $\inf_{t\geqslant 0}\|u(t,\omega,e)\| = 0$) *if and only if* $\sup_{t\geqslant 0}\|u(t,\omega,x)\| < \infty$ (*resp.* $\inf_{t\geqslant 0}\|u(t,\omega,x)\| = 0$) *for every $x \geqslant 0$.*
(iii) *The sets $\Omega_b = \{\omega \in \Omega \mid \sup_{t\geqslant 0}\|u(t,\omega,e)\| < \infty\}$ and $\Omega_u = \Omega - \Omega_b$ are invariant for the base flow.*
(iv) *The set $\Omega_c = \{\omega \in \Omega \mid \inf_{t\geqslant 0}\|u(t,\omega,e)\| = 0\}$ is either empty or residual in Ω and invariant for the base flow.*

Proof. (i) Since $u(0,\omega,e) = e \gg 0$ for every $\omega \in \Omega$, the continuity of the semiflow and the compactness of Ω ensure the existence of $t_0 > 0$ such that $u(t,\omega,e) \gg 0$ for every $t \in [0,t_0]$ and

$\omega \in \Omega$. Consequently, given $x \gg 0$, we take $\lambda \in (0,1)$ with $x \geqslant \lambda e$ to conclude from the monotonicity and sublinearity properties (2.2) and (2.3) that $u(t,\omega,x) \geqslant u(t,\omega,\lambda e) \geqslant \lambda u(t,\omega,e) \gg 0$ for every $t \in [0,t_0]$ and $\omega \in \Omega$. Now, we reason by induction assuming that $u(t,\omega,e) \gg 0$ for $t \in [0,nt_0]$ and $\omega \in \Omega$, where $n \in \mathbb{N}$, and take $t \in [0,t_0]$. Then, by the cocycle property (2.1), $u(nt_0+t,\omega,e) = u(t,\omega \cdot nt_0, u(nt_0,\omega,e)) \gg 0$, which proves the first assertion in (i). The second one is an immediate consequence.

(ii) Assume that $\sup_{t\geqslant 0} \|u(t,\omega,e)\| < \infty$ (resp. $\inf_{t\geqslant 0} \|u(t,\omega,e)\| = 0$). Given $x \geqslant 0$ we look for $\mu > 1$ such that $x \leqslant \mu e$. Monotonicity and sublinearity ensure that $0 \leqslant u(t,\omega,x) \leqslant \mu u(t,\omega,e)$ for $t \geqslant 0$. Consequently, the semimonotonicity of the norm shows that $\sup_{t\geqslant 0} \|u(t,\omega,x)\| < \infty$ (resp. $\inf_{t\geqslant 0} \|u(t,\omega,x)\| = 0$). The converse assertions are obvious.

(iii) Take $\omega \in \Omega_b$ and $s > 0$. Using again (2.1), $u(t,\omega\cdot s, u(s,\omega,e)) = u(s+t,\omega,e)$, so that (i) and (ii) imply that $\omega \cdot s \in \Omega_b$. Now take $s < 0$. Then, for $t \geqslant -s$, $u(t,\omega\cdot s,e) = u(s+t,\omega,u(-s,\omega\cdot s,e))$, and again the result follows from (i) and (ii). This shows the invariance of Ω_b and, consequently, also that of Ω_u.

(iv) Since the function $i_e : \Omega \to \mathbb{R}_+$, $\omega \mapsto \inf_{t\geqslant 0} \|u(t,\omega,e)\|$ is upper semicontinuous, it has a residual set of continuity points (see e.g. Choquet [4]), which we call Ω_c. Let us check that if i_e vanishes at some point of Ω, then it vanishes in an invariant set which agrees with Ω_c. Assume first that $i_e(\omega_0) = 0$. Then, for $s > 0$, $\inf_{t\geqslant 0} \|u(t,\omega_0\cdot s, u(s,\omega_0,e))\| = \inf_{t\geqslant 0} \|u(s+t,\omega_0,e)\| = 0$, and for $s < 0$, $\inf_{t\geqslant 0} \|u(t,\omega_0\cdot s,e)\| = \inf_{t\geqslant 0} \|u(s+t,\omega_0,u(-s,\omega_0\cdot s,e))\|$. In both cases, (i) and (ii) prove that $i_e(\omega_0 \cdot s) = 0$, and hence the invariance of the vanishing set of i_e. Since any continuity point $\omega \in \Omega_c$ can be written as $\lim_{n\to\infty} \omega_0 \cdot t_n$ for a suitable sequence $(t_n) \uparrow \infty$, we get $i_e(\omega) = \lim_{n\to\infty} i_e(\omega_0 \cdot t_n) = 0$. Conversely, assume that $i_e(\omega_0) = 0$ and take a sequence (ω_n) with limit ω_0. The upper semicontinuity of i_e means that $0 = i_e(\omega_0) \geqslant \limsup_{n\to\infty} i_e(\omega_n) \geqslant 0$, so that the inequalities must be equalities and hence $\omega_0 \in \Omega_c$, as asserted. □

Theorem 3.2. *Assume that the skew-product semiflow on $\Omega \times X_+$ satisfies hypothesis* (h1) *and it admits a minimal set K. Then, either $K \subset \Omega \times \operatorname{Int} X_+$ or $K \subset \Omega \times (X_+ - \operatorname{Int} X_+)$.*

Proof. Given $(\omega, y) \in K$, we can choose a backward extension of (ω, y) in K (existing but not necessarily unique), $\alpha(\omega,y) = \{(\omega\cdot s, y_s) \mid s \leqslant 0\} \subset K$, and define $\mathcal{A}(\omega,y)$ as the set of points of the form $\lim_{n\to\infty}(\omega\cdot s_n, y_{s_n})$ for any subsequence $(s_n) \downarrow -\infty$; i.e., as the alpha-limit set of the backward semiorbit. It is easy to check that $\mathcal{A}(\omega,y)$ is a closed and positively invariant subset of K, and hence, by minimality $K = \mathcal{A}(\omega,y)$.

Now assume the existence of $(\omega,y) \in K \cap (\Omega \times (X - \operatorname{Int} X_+))$. Proposition 3.1(i) shows that $\alpha(\omega,y)$ is contained in $\Omega \times (X - \operatorname{Int} X_+)$, which is closed. So that $K \subset \Omega \times (X - \operatorname{Int} X_+)$, which proves the result. □

The rest of the section is divided in two parts. In the first one we will work assuming that

(h3) there exists a strongly positive minimal set,

while the second one corresponds to the opposite situation, in which

(nh3) there is no strongly positive minimal set.

It is clear from the description of Cases A, B, C and D made in the Introduction that Case A holds if and only if (h3) is satisfied. In addition, the dynamics fits Case A or B if and only if $\Omega = \Omega_b$, and Case D if and only if $\Omega = \Omega_u$. Examples fitting each situation will be provided.

In each subsection, a more accurate description will be obtained under the more restrictive monotonicity hypothesis

(h4) there are a dense set $\widetilde{\Omega} \subseteq \Omega$ and $\tilde{t} > 0$ such that if $(\omega,x), (\omega,y) \in \widetilde{\Omega} \times X_+$ admit backward extensions $(\omega\cdot s, x_s)$ and $(\omega\cdot s, y_s)$ for $s \leqslant 0$, and for a $\lambda \in [0,1]$ it is $x > \lambda y$ and $x_s \geqslant \lambda y_s$ for any $s < 0$, then $u(t,\omega,x) \gg \lambda u(t,\omega,y)$ for any $t \geqslant \tilde{t}$.

Note that this condition is weaker than eventual strong monotonicity (see [24]). In applications, the dense set will be often a semiorbit in Ω.

3.1. Dynamics under the presence of a strongly positive minimal set

There are different dynamical situations occurring under hypotheses (h1)–(h3), comprised in the so-called Case A, which are described in Proposition 3.4 and Theorem 3.8. Their proofs require some previous analysis of the behavior of the semiorbits and the minimal sets, which we do in what follows.

Note that condition (h3), assumed in this subsection, is equivalent to the existence of a positively invariant strongly positive compact set. In fact minimality plays no role in the first result, in which we prove several basic properties which will be continuously used in what follows, sometimes with no explicit reference.

Proposition 3.3. *Assume that the skew-product semiflow on $\Omega \times X_+$ satisfies hypotheses* (h1)–(h3)*. Then*:

(i) *Any semiorbit is bounded and hence relatively compact.*
(ii) *For any $\omega \in \Omega$ and $x \gg 0$ the semiorbit of (ω, x) is uniformly stable.*
(iii) *The omega-limit set $M = \mathcal{O}(\omega, x)$ is strongly positive whenever $x \gg 0$.*
(iv) *If an omega-limit set M is strongly positive, then it is a uniformly stable and fiber-distal minimal set. Besides, it admits a flow extension and the section map $\Omega \to \mathcal{P}_c(M_X)$, $\omega \mapsto M_\omega = \{x \in X \mid (\omega, x) \in M\}$ is continuous, considering the set $\mathcal{P}_c(M_X)$ of closed subsets of the projection of M over X endowed with the Hausdorff metric.*
(v) *If $M = \mathcal{O}(\omega, x)$ is strongly positive and such that $x \leqslant z$ for any $(\omega, z) \in M$, or such that $x \geqslant z$ for any $(\omega, z) \in M$, then M is a copy of the base.*

Proof. Hypothesis (h3) provides a strongly positive minimal set K, for which we fix $e_1, e_2 \in X$ such that $0 \ll e_1 \leqslant y \leqslant e_2$ for any $(\omega, y) \in K$.

(i) For every $\omega \in \Omega$ there exists $(\omega, y) \in K$, which hence has a bounded semiorbit. Assertion (i) follows from Proposition 3.1(ii) and (h2).

(ii) Let us fix $(\omega, x) \in \Omega \times \operatorname{Int} X_+$ and take $(\omega, y) \in K$ and $\lambda \in (0,1)$ such that $\lambda y \leqslant x \leqslant (1/\lambda) y$. Properties (2.2) and (2.3) guaranteed by (h1) lead us to

$$0 \ll \lambda e_1 \leqslant \lambda u(t, \omega, y) \leqslant u(t, \omega, x) \leqslant (1/\lambda) u(t, \omega, y) \leqslant (1/\lambda) e_2 \tag{3.1}$$

for any $t \geqslant 0$. Let us now fix $\alpha \in (0,1)$. It is not difficult to deduce the existence of $\delta = \delta(\alpha) > 0$ such that, if $z \in X_+$ satisfies $\|u(s, \omega, x) - z\| < \delta$ for certain $s \geqslant 0$, then $\alpha u(s, \omega, x) \leqslant z \leqslant (1/\alpha) u(s, \omega, x)$, and hence, again by the monotonicity and sublinearity properties, $\alpha u(s+t, \omega, x) \leqslant u(t, \omega \cdot s, z) \leqslant (1/\alpha) u(s+t, \omega, x)$ for any $t \geqslant 0$. Therefore

$$\begin{aligned}(1-1/\alpha)\lambda e_1 &\leqslant (1-1/\alpha) u(s+t, \omega, x) \leqslant u(s+t, \omega, x) - u(t, \omega \cdot s, z)\\ &\leqslant (1-\alpha) u(s+t, \omega, x) \leqslant (1-\alpha)(1/\lambda) e_2.\end{aligned}$$

The uniform stability follows easily.

(iii) It follows immediately from the lower bound for $u(t, \omega, x)$ in (3.1).

(iv) Note that M is the omega-limit set of a pair (ω, x) with $x \gg 0$. Once we know that the semiorbit of this point is uniformly stable, its omega-limit set is uniformly stable too (see [37] and [26]). We can apply Theorem 3.4 and Proposition 3.6(ii) in [26] to deduce all the other statements in (iv).

(v) The proof is inspired in that of Proposition 3.4 (part II) in [38]. Assume that $x \leqslant z$ for any $(\omega, z) \in M = \mathcal{O}(\omega, x)$. Suppose that for an $\widetilde{\omega} \in \Omega$ there exist $(\widetilde{\omega}, x_1)$, $(\widetilde{\omega}, x_2) \in M$. Let us take $(t_n) \uparrow \infty$ such that $(\widetilde{\omega}, x_1) = \lim_{n\to\infty}(\omega \cdot t_n, u(t_n, \omega, x))$. Since, by (iv), $M_{\omega \cdot t_n} \to M_{\widetilde{\omega}}$ and $x_2 \in M_{\widetilde{\omega}}$, there are $(\omega \cdot t_n, y_n) \in M$ such that $y_n \to x_2$, and since M admits a flow extension, $y_n = u(t_n, \omega, z_n)$ for certain

$(\omega, z_n) \in M$. By hypothesis $x \leqslant z_n$, and the monotonicity of the semiflow ensures that $u(t_n, \omega, x) \leqslant y_n$. Taking limits, $x_1 \leqslant x_2$. Interchanging the roles of x_1 and x_2, we get $x_1 = x_2$. The proof in the symmetric situation is analogous. □

In particular, this result shows that the limiting behavior of the eventually strongly positive semiorbits is determined by the shape of the strongly positive minimal sets and the dynamics on them. As we will see in Theorem 3.8, there may appear very different dynamical situations. Even the existence of a *lowest strongly positive minimal set* K^- and/or of a *top strongly positive minimal set* K^+, in the sense that any other minimal set $M \gg 0$ satisfies $M > K^-$ and/or $M < K^+$, does not provide a more accurate description in the general case we are dealing with. However, the existence of these sets can be characterized in terms of the boundedness of the union of all the strongly positive minimal sets, and it has important dynamical consequences. To explain these points is the purpose of the next result.

Proposition 3.4. *Assume that the skew-product semiflow on $\Omega \times X_+$ satisfies hypotheses* (h1)–(h3)*, and define $\mathcal{M} = \{K \subset \Omega \times \operatorname{Int} X_+ \mid K \text{ is minimal}\}$. Then:*

(i) *The existence of a lowest strongly positive minimal set K^- is equivalent to the existence of $e_1 \gg 0$ such that $e_1 \leqslant x$ for every $(\omega, x) \in \bigcup_{K \in \mathcal{M}} K$. Besides, the lowest strongly positive minimal set is unique, it is a copy of the base, say $K^- = \{(\omega, c^-(\omega)) \mid \omega \in \Omega\}$, and $\lim_{t\to\infty} \|c^-(\omega\cdot t) - u(t, \omega, x)\| = 0$ whenever $0 \ll x \leqslant c^-(\omega)$.*

(ii) *A strongly positive minimal set K is the lowest one if and only if, whenever $0 \ll x \leqslant y$ for $(\omega, y) \in K$, it is $\lim_{t\to\infty} \|u(t, \omega, x) - u(t, \omega, y)\| = 0$.*

(iii) *The existence of a top strongly positive minimal set K^+ is equivalent to the existence of $e_2 \gg 0$ such that $x \leqslant e_2$ for every $(\omega, x) \in \bigcup_{K \in \mathcal{M}} K$. Besides, the top strongly positive minimal set is unique, it is a copy of the base, say $K^+ = \{(\omega, c^+(\omega)) \mid \omega \in \Omega\}$, and $\lim_{t\to\infty} \|c^+(\omega\cdot t) - u(t, \omega, x)\| = 0$ whenever $x \geqslant c^+(\omega)$.*

(iv) *A strongly positive minimal set K is the top one if and only if, whenever $x \geqslant y$ for $(\omega, y) \in K$, it is $\lim_{t\to\infty} \|u(t, \omega, x) - u(t, \omega, y)\| = 0$.*

Proof. (i) If there exists a lowest strongly positive minimal set K^-, there is $e_1 \gg 0$ such that $e_1 \leqslant y$ for every $(\omega, y) \in K^-$. If $M \gg 0$ is minimal, for any $(\omega, x) \in M$ there exists $(\omega, y) \in K^-$ with $y \leqslant x$, and hence $e_1 \leqslant x$.

Conversely, if an $e_1 \gg 0$ satisfying the condition in (i) exists, then by Proposition 3.3(iii)–(v) the set $K^- = \mathcal{O}(\omega, e_1) \gg 0$ is a copy of the base. It is immediate to deduce from the monotonicity that $K^- = \{(\omega, c^-(\omega)) \mid \omega \in \Omega\}$ is the lowest strongly positive minimal set, and that it agrees with the omega-limit set of any (ω, x) with $0 \ll x \leqslant c^-(\omega)$, from where the remaining assertions in (i) follow.

(ii) The condition in (ii) implies that K is the omega-limit set of any (ω, x) with $0 \ll x \leqslant y$ for $(\omega, y) \in K$. Let M be another strongly positive minimal set, and take $x \gg 0$ such that $x \leqslant y$ for an $(\omega, y) \in K$ and $x \leqslant z$ for an $(\omega, z) \in M$. Then, by monotonicity, $K = \mathcal{O}(\omega, x) \leqslant \mathcal{O}(\omega, z) = M$, so that K is the lowest minimal set. The converse assertion follows immediately from (i).

(iii)&(iv) The proofs of the last assertions are identical to the previous ones with the obvious changes. □

Before stating Theorem 3.8, we give some more preliminary results which will be used in its proof. The first one reflects a fundamental property of fiber-distal minimal sets in monotone (not necessarily sublinear) skew-product semiflows (see Theorem 3.1 (part II) in [38] for an inspiring result).

Proposition 3.5. *Assume that the skew-product semiflow on $\Omega \times X$ is monotone. A positively fiber-distal minimal set K does not contain any ordered pair, that is, any two points $(\omega, y_1), (\omega, y_2) \in K$ with $y_1 < y_2$.*

Proof. Assume by contradiction the existence of $y_1, y_2 \in K_\omega$ with $y_1 < y_2$. The first step of the proof shows the existence of a maximal element in K_ω: a point $y_* \in K_\omega$ which is not less than any other element in the same fiber.

We define $\mathcal{L} = \{J \subset K_\omega \mid J \supseteq \{y_1, y_2\}$ and is a totally ordered set$\}$. (This means that any two elements of J are ordered.) Note that $\{y_1, y_2\} \in \mathcal{L}$. We define a partial order in $\mathcal{L}$ by inclusion: $J_1 \leqslant J_2$ if $J_1 \subset J_2$. Then, any totally ordered set of elements of $\mathcal{L}$ has an upper bound, given by the union of the sets of the chain. Zorn's Lemma ensures the existence of a maximal element in $\mathcal{L}$, which we call J_*. It is easy to check that the closure of this set (in the topology of K_ω, inherited from that of X) also belongs to $\mathcal{L}$, which means that J_* is closed and hence a compact set.

Let us fix $e \gg 0$. By compactness, for each $n \in \mathbb{N}$ there are points $y_1^n, \dots, y_{j_n}^n \in J_*$ such that $J_* \subset \bigcup_{l=1}^{j_n} \{y \in X \mid \|y - y_l^n\|_e < 1/n\}$. We call $y_*^n = \max(y_1^n, \dots, y_{j_n}^n)$ and conclude that $y \leqslant y_*^n + (1/n)e$ for any $y \in J_*$ and any $n \in \mathbb{N}$. Then, if y_* is the limit of a convergent subsequence of (y_*^n), we have $y \leqslant y_*$ for any $y \in J_*$. It follows easily from the maximality of J_* that y_* is a maximal element of K_ω.

Now, we consider the elements $y_1 < y_*$ in K_ω and look for $(t_n) \uparrow \infty$ with $\lim_{n\to\infty}(\omega \cdot t_n, u(t_n, \omega, y_1)) = (\omega, y_*)$ such that there exists $\lim_{n\to\infty} u(t_n, \omega, y_*) = z_*$. By monotonicity, $z_* \geqslant y_*$, and the fiber-distallity shows that $z_* \neq y_*$. Consequently, $z_* > y_*$, contradicting the maximality of y_*. □

The second preliminary result refers to the properties of a function λ^* which will be fundamental in the proof of the main theorems. Let $K \subset \Omega \times X_+$ be a strongly positive minimal set. Fixed $(\widetilde{\omega}, \tilde{y}) \in K$, we consider a *strongly increasing path* $\gamma : [0, 1] \to X$ *connecting* 0 *and* $\tilde{y}$; that is, γ is continuous and, if $\tilde{x}_s = \gamma(s)$, then $\tilde{x}_0 = 0$, $\tilde{x}_1 = \tilde{y}$, and $\tilde{x}_{s_1} \ll \tilde{x}_{s_2}$ for $s_1 < s_2$. (Note that such a path always exists: for instance, $\gamma_0(s) = s\tilde{y}$.) For $s \in [0, 1]$ and $t \geqslant 0$, we choose $\lambda_s(t) = \lambda(s, t, \widetilde{\omega}, \tilde{y}, \gamma)$ as the maximum point in $[0, 1]$ such that

$$u(t, \widetilde{\omega}, \tilde{x}_s) \geqslant \lambda_s(t) u(t, \widetilde{\omega}, \tilde{y}). \tag{3.2}$$

The cocycle property (2.1), the monotonicity and sublinearity imply that, if $l > 0$,

$$u(t + l, \widetilde{\omega}, \tilde{x}_s) \geqslant \lambda_s(t) u(t + l, \widetilde{\omega}, \tilde{y}),$$

which means that $\lambda_s(t + l) \geqslant \lambda_s(t)$. In other words, $\lambda_s : [0, \infty) \to [0, 1]$ increases with t for each fixed $s \in [0, 1]$. Note also that $\lambda_s(0) > 0$ for any $s > 0$, since $\tilde{x}_s \gg 0$. In addition, $\lambda_s(t)$ increases in s for any fixed $t > 0$, as easily deduced from the inequality $u(t, \widetilde{\omega}, \tilde{x}_{s_1}) \leqslant u(t, \widetilde{\omega}, \tilde{x}_{s_2})$ for $s_1 \leqslant s_2$. And finally, $\lambda_1(t) = 1$ for every $t \geqslant 0$. These properties guarantee that

$$\lambda^* : (0, 1] \to (0, 1], \qquad s \mapsto \lambda_s^* = \lim_{t\to\infty} \lambda_s(t) \tag{3.3}$$

is well defined and nondecreasing, with $\lambda_1^* = 1$. Besides, λ^* is continuous, as proved in the following result, which will be repeatedly used from now on.

Proposition 3.6. *Assume that the skew-product semiflow on $\Omega \times X_+$ satisfies hypotheses* (h1)–(h3)*. Let K be a strongly positive minimal set and let γ be a strongly increasing path connecting* 0 *with $\tilde{y}$ for a fixed pair $(\widetilde{\omega}, \tilde{y}) \in K$. Then*:

(i) *The map $\lambda^* : (0, 1] \to (0, 1]$, $s \mapsto \lambda_s^*$ given by* (3.3) *is nondecreasing, continuous and $\lambda^*((0, 1])$ is a* (*possibly degenerate*) *interval containing* 1.
(ii) *$K_s = \mathcal{O}(\widetilde{\omega}, \gamma(s))$ is a uniformly stable fiber-distal strongly positive minimal set for every $s \in (0, 1]$. Besides, if $K_{s_1} = K_{s_2}$ then $\lambda_{s_1}^* = \lambda_{s_2}^*$, and if $\lambda_s^* = 1$ then $K_s = K$.*
(iii) *If $\lambda^* \not\equiv 1$, then $K_s \neq K$ for every $s \in (0, 1)$.*
(iv) *The map $(0, 1] \to \mathcal{P}_c(\Omega \times X_+)$, $s \mapsto K_s$ is monotone and continuous for the Hausdorff topology of the set $\mathcal{P}_c(\Omega \times X_+)$ of closed parts of $\Omega \times X_+$, and the set $\bigcup_{s\in I} K_s \subset \Omega \times X_+$ is connected for any interval $I \subset (0, 1]$.*

Proof. We represent $\gamma(s)=\tilde{x}_s$ for $s\in[0,1]$.

(i) Let us fix $s_1\in(0,1]$ and $\varepsilon>0$. As seen in (3.1), $u(t,\widetilde{\omega},\tilde{x}_{s_1})\geqslant e$ for any $t\geqslant 0$ for an $e\gg 0$. The uniform stability of the semiorbit of $(\widetilde{\omega},\tilde{x}_{s_1})$ guaranteed by Proposition 3.3(ii) and the continuity of the path γ show the existence of $\rho=\rho(\varepsilon)>0$ such that if $s_2\in(0,1]$ and $|s_1-s_2|<\rho$, then $\|u(t,\widetilde{\omega},\tilde{x}_{s_1})-u(t,\widetilde{\omega},\tilde{x}_{s_2})\|_e\leqslant\varepsilon$ for any $t\geqslant 0$. Hence, for any $t\geqslant 0$,

$$-\varepsilon u(t,\widetilde{\omega},\tilde{x}_{s_1})\leqslant -\varepsilon e\leqslant u(t,\widetilde{\omega},\tilde{x}_{s_2})-u(t,\widetilde{\omega},\tilde{x}_{s_1})\leqslant \varepsilon e\leqslant \varepsilon u(t,\widetilde{\omega},\tilde{x}_{s_1}),$$

which implies $(1-\varepsilon)u(t,\widetilde{\omega},\tilde{x}_{s_1})\leqslant u(t,\widetilde{\omega},\tilde{x}_{s_2})\leqslant(1+\varepsilon)u(t,\widetilde{\omega},\tilde{x}_{s_1})$. It follows easily from the maximal character of $\lambda_s(t)$ that $(1-\varepsilon)\lambda_{s_1}(t)\leqslant\lambda_{s_2}(t)\leqslant(1+\varepsilon)\lambda_{s_1}(t)$ for any $t\geqslant 0$. Taking limits as $t\to\infty$ we get $(1-\varepsilon)\lambda^*_{s_1}\leqslant\lambda^*_{s_2}\leqslant(1+\varepsilon)\lambda^*_{s_1}$, or in other words, $|\lambda^*_{s_1}-\lambda^*_{s_2}|\leqslant\varepsilon\lambda^*_{s_1}\leqslant\varepsilon$. This proves the continuity of the map.

(ii) Proposition 3.3(iii) and (iv) guarantees the first part. Now assume that $K_{s_1}=K_{s_2}$ for $s_1<s_2$ and take $(t_n)\uparrow\infty$ with $(\widetilde{\omega},\tilde{y})=\lim_{n\to\infty}(\widetilde{\omega}\cdot t_n,u(t_n,\widetilde{\omega},\tilde{y}))$ such that there exists

$$\lim_{n\to\infty}\big(\widetilde{\omega}\cdot t_n,u(t_n,\widetilde{\omega},\tilde{x}_{s_i})\big)=(\widetilde{\omega},\tilde{y}_{s_i})\in K_{s_1}=K_{s_2}$$

for $i=1,2$. Since $\tilde{y}_{s_1}\leqslant\tilde{y}_{s_2}$, Proposition 3.5 ensures that $\tilde{y}_{s_1}=\tilde{y}_{s_2}$. Taking limits in (3.2) for $\tilde{x}_{s_2}$ and (t_n) we conclude that $\lambda^*_{s_2}\tilde{y}\leqslant\tilde{y}_{s_2}=\tilde{y}_{s_1}$, which together with the definition of $\lambda_{s_1}(t_n)$ provides an easy proof of $\lambda^*_{s_2}\leqslant\lambda^*_{s_1}$; that is, they agree.

Now assume that $\lambda^*_s=1$. Then, if $(\widetilde{\omega},\tilde{y})=\lim_{n\to\infty}(\widetilde{\omega}\cdot t_n,u(t_n,\widetilde{\omega},\tilde{y}))$ and $(\widetilde{\omega},\tilde{y}_s)=\lim_{n\to\infty}(\widetilde{\omega}\cdot t_n,u(t_n,\widetilde{\omega},\tilde{x}_s))$, we deduce from (3.2) that $\tilde{y}=\lambda^*_s\tilde{y}\leqslant\tilde{y}_s\leqslant\tilde{y}$, and hence $K_s=K$.

(iii) The monotonicity of the semiflow and Proposition 3.5 ensure that the set $I=\{s\in(0,1]\mid K_s=K\}$ is an interval, which contains 1. As by hypothesis there exists $s_0\in(0,1)$ with $\lambda^*_{s_0}<1$, by (ii) $K_{s_0}\neq K$, and therefore $I\neq(0,1]$. We define $s_1=\inf I$ and assume by contradiction that $s_1<1$. Note that $s_1\geqslant s_0>0$. Let us choose $\delta>0$ and $e\gg 0$ such that $\tilde{x}_{s_1+\delta}+e\leqslant\tilde{y}$. The uniform stability of the semiorbit of $(\widetilde{\omega},\tilde{x}_{s_1})$, ensured by Proposition 3.3(ii), and the continuity of the path γ allow us to take $0<\varepsilon<\delta$ small enough to guarantee that $\|u(t,\widetilde{\omega},\tilde{x}_{s_1+\varepsilon})-u(t,\widetilde{\omega},\tilde{x}_{s_1-\varepsilon})\|_e\leqslant 1/2$ for any $t>0$. The definition of s_1 allows us to choose $(t_n)\uparrow\infty$ such that $(\widetilde{\omega},\tilde{y})=\lim_{n\to\infty}(\widetilde{\omega}\cdot t_n,u(t_n,\widetilde{\omega},\tilde{x}_{s_1+\varepsilon}))$ and such that there exists $\tilde{y}_{s_1-\varepsilon}=\lim_{n\to\infty}u(t_n,\widetilde{\omega},\tilde{x}_{s_1-\varepsilon})$. Consequently, $\|\tilde{y}-\tilde{y}_{s_1-\varepsilon}\|_e\leqslant 1/2$, and hence $\tilde{x}_{s_1+\delta}\leqslant\tilde{y}_{s_1-\varepsilon}<\tilde{y}$. This, together with the monotonicity and Proposition 3.5, leads to a contradiction with the definition of s_1, and we are done.

(iv) The monotonicity is immediate from the monotonicity of the semiflow. The continuity follows easily from the continuity of the path and the uniform stability of the semiorbit of $(\widetilde{\omega},\tilde{x}_s)$ for any $s\in(0,1]$. Finally, to see that $\bigcup_{s\in I}K_s\subset\Omega\times X_+$ is connected for any interval $I\subset(0,1]$, assume that it is contained in the disjoint union of two open sets A and B of $\Omega\times X_+$. As omega-limit sets are well known to be connected, each K_s must be contained either in A or in B. Call $I_A=\{s\in I\mid K_s\subset A\}$ and $I_B=\{s\in I\mid K_s\subset B\}$, with $I=I_A\cup I_B$. Again the uniform stability of the semiorbits implies that both I_A and I_B are open, so that I must be contained in one of them, and from this fact the proof is easily concluded. □

Remark 3.7. If $\gamma:[0,\infty)\to X_+$ is a continuous and increasing map with $\gamma(s)\gg 0$ for every $s>0$, the conclusions of Proposition 3.6(iv) also hold for the map $(0,\infty)\to\mathcal{P}_c(\Omega\times X_+)$, $s\mapsto K_s$, where $K_s=\mathcal{O}(\omega,\gamma(s))$. The proof is identical.

We are finally in a position to state the main result of this section, which describes the three possible scenarios for the dynamics in Case A. The information it provides must be completed with Proposition 3.4, which describes the asymptotic properties of the lowest and the top minimal sets, provided that they exist.

Theorem 3.8. *Assume that the skew-product semiflow on $\Omega\times X_+$ satisfies hypotheses* (h1)–(h3)*. Then, the dynamics is described by one of the following situations*:

Case A1. *There is a unique strongly positive minimal set K. In this case K is an asymptotically stable copy of the base, i.e., $K=\{(\omega,c(\omega))\mid \omega\in\Omega\}$ for a continuous map $c:\Omega\to X$, and $\lim_{t\to\infty}\|c(\omega\cdot t)-u(t,\omega,x)\|=0$ for any (ω,x) with $x\gg 0$.*

Case A2. *There are more than one strongly positive minimal sets and, among them all, there exists one, K^-, which is the lowest one. It can either exist a top strongly positive minimal set K^+ (Case* A2.1*) or not (Case* A2.2*).*

In addition, given any minimal set $K_1\gg 0$ with $K_1\neq K^-$, there exists a continuous and connected family of (infinitely many) strongly positive minimal sets, $(K_s)_{s\in[s_-,\infty)}$, for an $s_-\in(0,1)$, such that $K_{s_1}\leqslant K_{s_2}$ for $s_1<s_2$, with $K^-=K_{s_-}<K_s<K_1$ for any $s_-<s<1$ and either $K_s=K^+$ for every $s\geqslant s_+\geqslant 1$ in Case A2.1 *or $\bigcup_{s\geqslant 1}K_s$ unbounded in Case* A2.2.

Case A3. *Given any minimal set $K\gg 0$ there exists another one $M\gg 0$ such that $M<K$. It can either exist a top strongly positive minimal set K^+ (Case* A3.1*) or not (Case* A3.2*).*

In addition, given any minimal set $K_1\gg 0$ there exists a continuous and connected family of (infinitely many) strongly positive minimal sets, $(K_s)_{s\in(0,\infty)}$, such that $K_{s_1}\leqslant K_{s_2}$ for $s_1<s_2$, with $0\ll K_s<K_1$ for any $0<s<1$ and either $K_s=K^+$ for every $s\geqslant s_+\geqslant 1$ in Case A3.1 *or $\bigcup_{s\geqslant 1}K_s$ unbounded in Case* A3.2. *Furthermore,*

(i) *if $(\omega,x)\in\Omega\times X_+$ is such that for any $s\in(0,1]$ there exists $(\omega,y_s)\in K_s$ with $x\leqslant y_s$, then $x\in X_+-\operatorname{Int}X_+$;*
(ii) *for any strongly positive minimal set M there is $s\in(0,1)$ such that $K_s<M$.*

Proof. *Case* A1: Since K is at the same time the top and lowest strongly positive minimal set, Proposition 3.4 and an easy application of the monotonicity prove the assertion.

Case A2: Assume that there exist the lowest strongly positive minimal set, $K^-=\{(\omega,c^-(\omega))\mid \omega\in\Omega\}$, and other minimal set $K_1>K^-$. We fix $(\widetilde{\omega},\tilde{y})\in K_1$ and build a strongly increasing continuous path $\gamma:(0,\infty)\to X_+$ such that $\gamma(0)=0$, $\gamma(1)=\tilde{y}$, and $\gamma(s)=s\tilde{y}$ for $s>1$. The family $(K_s)_{s\in(0,\infty)}$, with $K_s=\mathcal{O}(\widetilde{\omega},\gamma(s))\gg 0$, is monotone, continuous and connected in the sense of Proposition 3.6(iv) and Remark 3.7. Let us check that this family satisfies the assertions. Proposition 3.4(i) ensures that $K_s=K^-$ for every $s\in(0,1)$ with $\gamma(s)\leqslant c^-(\widetilde{\omega})$. Hence $s_-=\max\{s\in(0,1)\mid K_s=K^-\}$ is well defined, belongs to $(0,1)$, and satisfies $\lambda_s^*=\lambda_{s_-}^*<1$ for every $s\in(0,s_-]$, using Proposition 3.6(ii). Note that the fact that $\lambda^*((0,1])=[\lambda_{s_-}^*,1]$ and Proposition 3.6(ii) ensure that there are infinitely many different elements in the family, and that Proposition 3.6(iii) ensures that $K_s<K_1$ for every $s<1$. Now assume that the top minimal set $K^+=\{(\omega,c^+(\omega))\mid\omega\in\Omega\}$ exists. Then, for any s such that $s\tilde{y}\geqslant c^+(\widetilde{\omega})$ (so that $s\geqslant 1$), $K_s=K^+$, and the remaining conclusion holds for $s_+=\min\{s\geqslant 1\mid K_s=K^+\}$. If, on the contrary, K^+ does not exist, the family is unbounded: if this is not the case, it is easy to deduce from the expression of γ that Proposition 3.4(iii) holds, impossible.

Case A3: Assume that there is not a lowest strongly positive minimal set. In this situation, Proposition 3.4 shows that for any strongly positive minimal set K there exists $x\gg 0$ such that $\mathcal{O}(\omega,x)<K$. Hence the only dynamical alternative to Cases A1 and A2 is the one described in Case A3.

Now we fix a strongly positive minimal set K_1 and consider the monotone, continuous and connected family $(K_s)_{s\in(0,\infty)}$, associated to a suitable path γ constructed as in Case A2. We take a minimal set M with $0\ll M<K_1$, fix $(\widetilde{\omega},\tilde{x})\in M$ and take $s_0\in(0,1)$ with $\tilde{x}_{s_0}<\tilde{x}$. Then $K_{s_0}\leqslant M<K_1$. This implies that $\lambda^*\not\equiv 1$ and consequently, by Proposition 3.6(iii), $0\ll K_{s_1}\leqslant K_{s_2}<K_1$ for any $0<s_1<s_2<1$. The assertions concerning the existence or absence of top minimal set are proved as in Case A2. Finally, we prove (i) and (ii).

(i) Assume for contradiction that there are $\omega\in\Omega$ and $x\gg 0$ such that for any $s\in(0,1]$ there exists $(\omega,y_s)\in K_s$ with $x\leqslant y_s$. Then, by monotonicity, $0\ll\mathcal{O}(\omega,x)\leqslant K_s$ for any $s\in(0,1)$. It follows from the continuity of γ that $\mathcal{O}(\omega,x)$ is the lowest strongly positive minimal set, which is impossible in Case A3.

(ii) Given a strongly positive minimal set M we take $x \gg 0$ with $\mathcal{O}(\omega, x) < M$ and $s_0 \in (0,1)$ with $\tilde{x}_{s_0} \leqslant x$. Then $0 \ll K_{s_0} \leqslant \mathcal{O}(\omega, x) < M$, as asserted. □

Remark 3.9. Using results by Novo et al. [26], it can be proved that, when conditions (A1)–(A4) in the work of Jiang and Zhao [12] are fulfilled, as happens in many applications, then all the minimal sets in Theorem 3.8 are copies of the base.

The classification provided in Theorem 3.8 is optimal, in the sense that there exist monotone and sublinear semiflows in each one of the cases described. Very simple examples of Cases A1 and A3 are given by the positive semiflow induced on $\mathbb{R}_+$ by the solutions of the autonomous scalar equations $x' = x(1-x)$ and $x' = 0$, respectively. (Here the set Ω reduces to a point and hence plays no role.) The equation $x' = xf(x)$, where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is C^1 and decreasing, with $f(x) > 0$ for $x \in [0,1)$ and $f(x) = 0$ for $x \in [1,\infty)$, provides an example of Case A2. Easy modifications provide examples of Cases A2 and A3 admitting a top minimal set.

Remark 3.10. We point out that the families of minimal sets occurring in Cases A2 and A3 are not necessarily unique, even when a top strongly positive minimal set exists. Examples of this lack of uniqueness are described in [29]. However, the existence of a strongly positive copy of the base $K = \{(\omega, c(\omega)) \mid \omega \in \Omega\}$ ensures the existence of a continuous family of strongly positive copies of the base satisfying the properties described in Theorem 3.8. To check this assertion we fix $\widetilde{\omega} \in \Omega$, denote $K_s = \mathcal{O}(\widetilde{\omega}, sc(\widetilde{\omega})) \gg 0$ for $s > 0$ and deduce from the sublinearity and the continuity of c that, for any $(\widetilde{\omega}, \tilde{y}_s) \in K_s$, it is $\tilde{y}_s \geqslant sc(\widetilde{\omega})$ if $s \in (0,1)$ and $\tilde{y}_s \leqslant sc(\widetilde{\omega})$ if $s > 1$. These inequalities and Proposition 3.3(v) prove that K_s is a copy of the base for every $s > 0$. The assertion follows from the fact that this family is associated to the path $\gamma(s) = sc(\widetilde{\omega})$.

The next result offers a precise characterization of each of the three Cases A, in terms of the behavior of the strongly increasing paths joining 0 with points in the fibers of strongly positive minimal sets.

Theorem 3.11. *Assume that the skew-product semiflow on $\Omega \times X_+$ satisfies hypotheses* (h1)–(h3)*. With reference to Theorem* 3.8:

Case A1 *is equivalent to the fact that for any strongly positive minimal set K, any $(\widetilde{\omega}, \tilde{y}) \in K$ and any strongly increasing path connecting* 0 *with $\tilde{y}$, it is $\lambda^* \equiv 1$.*

Case A2 *is equivalent to the existence of a strongly positive minimal set K such that for a* (*actually for every*) *strongly increasing path γ joining* 0 *with $\tilde{y}$, for a pair $(\widetilde{\omega}, \tilde{y}) \in K$, $\lim_{s\to 0^+} \lambda^*_s = \lambda^*_-$ with $0 < \lambda^*_- < 1$. In this case necessarily $\lambda^*((0,1]) = [\lambda^*_-, 1]$. In fact, in Case* A2 *every strongly positive minimal set, other than the lowest one* (*for which $\lambda^* \equiv 1$ for any path*)*, fulfils this condition.*

Case A3 *is equivalent to the existence of a strongly positive minimal set K such that for a* (*actually for every*) *strongly increasing path joining* 0 *with $\tilde{y}$, for a pair $(\widetilde{\omega}, \tilde{y}) \in K$, $\lim_{s\to 0^+} \lambda^*_s = 0$, that is, $\lambda^*((0,1]) = (0,1]$. In fact, in Case* A3*, this behavior is common to every strongly positive minimal set.*

Proof. *Case* A1: If Case A1 holds, the assertion follows from Proposition 3.6(ii). Conversely, if all the maps λ^* associated to a minimal set $K \gg 0$ satisfy this property, it follows easily from Proposition 3.6(ii) that $\mathcal{O}(\widetilde{\omega}, \tilde{x}) = K$ whenever $0 \ll x \ll y$ for an $(\omega, y) \in K$. It is immediate to deduce that if K_1 and K_2 are two strongly positive minimal sets, they agree.

Case A2: Assume the existence of a minimal set $K \gg 0$, a point $(\widetilde{\omega}, \tilde{y}) \in K$ and a strongly increasing path γ connecting 0 with $\tilde{y}$, with $\lim_{s\to 0^+} \lambda^*_s = \lambda^*_- \in (0,1)$. We fix $e \gg 0$ with $e \leqslant \lambda^*_- y$ for every $(\omega, y) \in K$. It follows from (3.2) and from $\lambda^*_s \geqslant \lambda^*_-$ that $x \geqslant e$ for every $(\omega, x) \in \bigcup_{s\in(0,1]} K_s$, where $K_s = \mathcal{O}(\widetilde{\omega}, \gamma(s))$. From here it is immediate to check that e is also a lower bound for any state in a strongly positive minimal set, so that Proposition 3.4(i) ensures the existence of the lowest strongly positive minimal set. The previous paragraph shows that we are not in Case A1, so that the dynamics corresponds to Case A2. The proof of the converse property and the remaining assertions are included in the proof of Theorem 3.8.

Case A3: The above properties show that $\lim_{s\to 0^+}\lambda_s^*=0$ in Case A3, so that $\lambda^*((0,1])=(0,1]$. Cases A1 and A2 are precluded if $\lambda^*((0,1])=(0,1]$. □

There are some particular situations in which the description is more accurate. To describe them is the last purpose of this subsection. We first assume the additional sublinearity condition of existence of $\widetilde{\omega}\in\Omega$ and $\tilde{t}>0$ with

$$u(\tilde{t},\widetilde{\omega},\lambda x)\gg\lambda u(\tilde{t},\widetilde{\omega},x)\quad\text{for any } x\gg 0 \text{ and } \lambda\in(0,1),\tag{3.4}$$

in order to show that the dynamics fits Case A1. Once fixed $x\gg 0$ and $\lambda\in(0,1)$ we can take $\mu\in(\lambda,1)$ with $u(\tilde{t},\widetilde{\omega},\lambda x)\geqslant\mu u(\tilde{t},\widetilde{\omega},x)$, so that sublinearity, Proposition 3.1(i) and the cocycle equality (2.1) ensure that $u(t,\widetilde{\omega},\lambda x)\gg\lambda u(t,\widetilde{\omega},x)$ for every $t\geqslant\tilde{t}$. The same dynamics occurs when (3.4) is substituted by the less restrictive condition of existence of $\widetilde{\omega}\in\Omega$ and $\tilde{t}>0$ with

$$u(\tilde{t},\widetilde{\omega},\lambda x)>\lambda u(\tilde{t},\widetilde{\omega},x)\quad\text{for any } x\gg 0 \text{ and } \lambda\in(0,1)\tag{3.5}$$

if the additional monotonicity condition (h4) holds. We also analyze the dynamics assuming (h4) without imposing (3.5), showing that in particular the lack of uniqueness mentioned in Remark 3.10 in Case A2 or A3 is not possible.

Theorem 3.12. *Assume that the skew-product semiflow on $\Omega\times X_+$ satisfies hypotheses* (h1)–(h3) *and* (3.4)*. Then, the dynamics fits Case* A1 *in Theorem* 3.8.

Proof. Let $\widetilde{\omega}$ and $\tilde{t}$ be the ones appearing in (3.4). According with Theorem 3.11, it suffices to check that for any strongly positive minimal set K, any $(\widetilde{\omega},\tilde{y})\in K$ and any strongly increasing path connecting 0 with $\tilde{y}$, the map λ^* defined in (3.3) satisfies $\lambda^*\equiv 1$. So, let us fix such K and γ and denote $\tilde{x}_s=\gamma(s)$ for $s\in[0,1]$.

Assume by contradiction that $\lambda_{s_0}^*\in(0,1)$ for an $s_0\in(0,1)$. We choose a sequence $(t_n)\uparrow\infty$ such that $(\widetilde{\omega},\tilde{y})=\lim_{n\to\infty}(\widetilde{\omega}\cdot t_n,u(t_n,\widetilde{\omega},\tilde{y}))$ and there exists $\lim_{n\to\infty}u(t_n,\widetilde{\omega},\tilde{x}_{s_0})=\tilde{z}$. It follows from (3.2) that $\tilde{z}\geqslant\lambda_{s_0}^*\tilde{y}$, and by monotonicity and (3.4), $u(\tilde{t},\widetilde{\omega},\tilde{z})\gg\lambda_{s_0}^*u(\tilde{t},\widetilde{\omega},\tilde{y})$. In other words, by the cocycle property (2.1) and the continuity,

$$\lim_{n\to\infty}u(\tilde{t}+t_n,\widetilde{\omega},\tilde{x}_{s_0})\gg\lambda_{s_0}^*\lim_{n\to\infty}u(\tilde{t}+t_n,\widetilde{\omega},\tilde{y}).\tag{3.6}$$

This contradicts the definition of $\lambda_{s_0}^*$, and hence the theorem is proved. □

Theorem 3.13. *Assume that the skew-product semiflow on $\Omega\times X_+$ satisfies hypotheses* (h1)–(h4)*, and let $\widetilde{\Omega}$ be the dense set in* (h4)*. Then*:

- (i) *Any minimal set $K>0$ is strongly positive.*
- (ii) *Any strongly positive minimal set is a copy of the base, and all of them are multiples of a reference one $K=\{(\omega,c(\omega))\mid\omega\in\Omega\}$; that is, all of them are the graphs K_s of the maps sc, for s varying in a right-closed* (*maybe unbounded*) *interval $J\subset(0,\infty)$, which degenerates to a point if and only if the dynamics fits Case* A1*, with* $\inf J\in J$ *in Case* A2 *and* $\inf J=0$ *in Case* A3*. In particular,* 0 *is a continuous equilibrium in Case* A3.
- (iii) *If $x\gg 0$, then there exists $s\in J$ with $\lim_{t\to\infty}\|u(t,\omega,x)-sc(\omega\cdot t)\|=0$. The same happens if (ω,x) with $\omega\in\widetilde{\Omega}$ and $x>0$ admits a backward extension.*
- (iv) *If $s_-=\inf J>0$, $K^-=\{(\omega,s_-c(\omega))\mid\omega\in\Omega\}$ is the lowest strongly positive minimal set.*
- (v) *If $s_+=\sup J<\infty$, $K^+=\{(\omega,s_+c(\omega))\mid\omega\in\Omega\}$ is the top strongly positive minimal set.*

Furthermore, if in addition the semiflow satisfies (3.5)*, then the dynamics fits Case* A1 *in Theorem* 3.8.

Proof. Clearly, Theorem 3.8 applies here, so that one of the three Cases A holds.

(i) Let us check that for any $K>0$ there exists $(\widetilde{\omega},\tilde{x})\in K$ with $\widetilde{\omega}\in\widetilde{\Omega}$ and $\tilde{x}>0$. Assuming by contradiction that $\widetilde{\Omega}\times\{0\}\subseteq K$, it follows from the density of $\widetilde{\Omega}$ that $\Omega\times\{0\}\subseteq K$. Given $t>0$ and $\omega\in\Omega$, let $\{(\omega\cdot(t+s),y_s)\mid s\leqslant 0\}$ be a backward orbit of $(\omega\cdot t,0)$ in K. Then $0\leqslant u(t,\omega,0)\leqslant u(t,\omega,y_{-t})=0$. This shows that $\Omega\times\{0\}$ is positively invariant, and hence, by minimality, it agrees with K, impossible. Now, given such a point $(\widetilde{\omega},\tilde{x})\in K$, property (h4) applied to $x_1=x_2=\tilde{x}$ and $\lambda=0$ implies that $u(\tilde{t},\widetilde{\omega},\tilde{x})\gg 0$, and hence, by Proposition 3.3(iii) and (iv), $K=\mathcal{O}(\widetilde{\omega}\cdot\tilde{t},u(\tilde{t},\widetilde{\omega},\tilde{x}))\gg 0$.

(ii) Let us prove that any minimal set $K\gg 0$ is a copy of the base. Because K is fiber-distal as stated in Proposition 3.3(iv), it suffices to see that fixed $\widetilde{\omega}\in\widetilde{\Omega}$ there is a unique pair $(\widetilde{\omega},\tilde{y})\in K$. Suppose for contradiction that there exist $(\widetilde{\omega},\tilde{y}_1),(\widetilde{\omega},\tilde{y}_2)\in K$ with $\tilde{y}_1\neq\tilde{y}_2$. As done in (3.2), for each $t\geqslant 0$ let $\lambda(t)$ be the maximum value in $[0,1]$ with $u(t,\widetilde{\omega},\tilde{y}_1)\geqslant\lambda(t)u(t,\widetilde{\omega},\tilde{y}_2)$ and take $\lambda=\lim_{t\to\infty}\lambda(t)\in[0,1]$. Then, take $(t_n)\uparrow\infty$ such that there exist $\lim_{n\to\infty}(\widetilde{\omega}\cdot t_n,u(t_n,\widetilde{\omega},\tilde{y}_i))=(\widetilde{\omega},\tilde{z}_i)\in K$ for $i=1,2$. Note that $\tilde{z}_1\neq\tilde{z}_2$, since K is fiber-distal, and recall that Proposition 3.3(iv) also shows that K has a flow extension. Then, for each $s\leqslant 0$, taking limits in the inequality for $s+t_n$, we get that $u(s,\widetilde{\omega},\tilde{z}_1)\geqslant\lambda u(s,\widetilde{\omega},\tilde{z}_2)$. It cannot be $\tilde{z}_1>\lambda\tilde{z}_2$, because hypothesis (h4) would imply that $u(\tilde{t},\widetilde{\omega},\tilde{z}_1)\gg\lambda u(\tilde{t},\widetilde{\omega},\tilde{z}_2)$ and, arguing as in (3.6), we would get a contradiction with the definition of λ. Thus, $\tilde{z}_1=\lambda\tilde{z}_2\leqslant\tilde{z}_2$, and Proposition 3.5 gives us the searched contradiction.

We choose a strongly positive minimal set $K=\{(\omega,c(\omega))\mid\omega\in\Omega\}$, consider a strongly increasing continuous path γ joining 0 with $c(\widetilde{\omega})$ for a fixed $\widetilde{\omega}\in\Omega$, denote $\tilde{x}_s=\gamma(s)$ for $s\in(0,1]$, and write $K_s=\mathcal{O}(\widetilde{\omega},\tilde{x}_s)=\{(\omega,c_s(\omega))\mid\omega\in\Omega\}$ for certain $c_s:\Omega\to X$ continuous. Let us fix $\omega\in\widetilde{\Omega}$ and $s\in(0,1]$. Taking limits in the inequality (3.2) for a suitable sequence $(t_n)\to\infty$ we get $c_s(\omega)\geqslant\lambda_s^*c(\omega)$. If it were $c_s(\omega)>\lambda_s^*c(\omega)$, hypothesis (h4) would imply $c_s(\omega\cdot\tilde{t})\gg\lambda_s^*c(\omega\cdot\tilde{t})$, which as in (3.6) would lead to a contradiction. So that $c_s(\omega)=\lambda_s^*c(\omega)$ for any $\omega\in\widetilde{\Omega}$, and by continuity and density, $c_s=\lambda_s^*c$. It follows easily that any minimal set is of the form $\{(\omega,\lambda c(\omega))\mid\omega\in\Omega\}$ for a certain $\lambda>0$: if $M=\{(\omega,d(\omega))\mid\omega\in\Omega\}$ is a strongly positive minimal set, we take $\tilde{x}\gg 0$ with $\tilde{x}\ll c(\widetilde{\omega})$ and $\tilde{x}\ll d(\widetilde{\omega})$ and choose suitable paths joining 0 with $c(\widetilde{\omega})$ and $d(\widetilde{\omega})$ and passing through $\tilde{x}$ to conclude that $\mathcal{O}(\widetilde{\omega},\tilde{x})=\{(\omega,\lambda_1c(\omega))\mid\omega\in\Omega\}=\{(\omega,\lambda_2d(\omega))\mid\omega\in\Omega\}$ for certain $\lambda_1,\lambda_2>0$, which implies the assertion.

Let us finally define $J=\{s>0\mid sc$ is a continuous equilibrium$\}$. Clearly J is a right-closed set, with $J=\{1\}$ in Case A1, $\inf J=s_-\in J$ in Case A2 (for which $K^-=\{(\omega,s_-c(\omega))\mid\omega\in\Omega\}$) and $\inf J=0$ in Case A3 (for which 0 is an equilibrium). By Remark 3.7 applied to the path $\gamma_0(s)=sc(\widetilde{\omega})$, J is an interval.

(iii) To check the first assertion, we recall that whenever $x\gg 0$, $\mathcal{O}(\omega,x)$ is a strongly positive minimal set, and apply (ii). The other property is proved as (i).

(iv)&(v) These properties are obvious once all the strongly positive minimal sets are known.

Finally, in order to check the last assertion, note that if there were two strongly positive minimal sets $K=\{(\omega,c(\omega))\mid\omega\in\Omega\}$ and $K_\lambda=\{(\omega,\lambda c(\omega))\mid\omega\in\Omega\}$ for a certain $\lambda\in(0,1)$, by (3.5) we would obtain $\lambda c(\widetilde{\omega}\cdot\tilde{t})=u(\tilde{t},\widetilde{\omega},\lambda c(\widetilde{\omega}))>\lambda u(\tilde{t},\widetilde{\omega},c(\widetilde{\omega}))=\lambda c(\widetilde{\omega}\cdot\tilde{t})$, a contradiction. □

We point out that, under hypotheses (h1)–(h4), the fact that $\Omega\times\{0\}$ is a minimal set, for sure in Case A3, is also possible in Cases A1 and A2, as the examples described after Theorem 3.8 show.

Assume finally that, in the conditions of the previous theorem, the dynamics fits Case A2 or A3 and $K=\{(\omega,c(\omega))\mid\omega\in\Omega\}$ is a minimal set. Assume also that u is differentiable with respect to the state variable and that the linearized skew-product semiflow on K admits a "continuous separation" in the terms of Poláčik and Tereščák [30,31] (see also Shen and Yi [38]). Then the dominant one-dimensional subspace generated by a strongly positive vector (which determines the long-term behavior of the linear semiflow) is given, for each $\omega\in\Omega$, by $\operatorname{span}\{c(\omega)\}$; i.e., it is determined by the family of minimal sets, where the semiflow u has a linear restriction.

3.2. Dynamics under the absence of strongly positive minimal sets

The three dynamical possibilities occurring under hypotheses (h1), (h2) and (nh3), called Cases B, C and D, are described in Theorem 3.15. Recall that Proposition 3.1 shows that the set Ω_b of base points ω such that the semiorbit of (ω,x) is bounded for every $x\in X_+$, and its complementary Ω_u, are invariant subsets of Ω, and that together with (h2) it also shows that the omega-limit set $\mathcal{O}(\omega, x)$ exists for any $(\omega, x) \in \Omega_b \times X_+$. Recall also that, as proved in Theorem 3.2, under hypothesis (nh3), either there are not any minimal sets or they are contained in the border of the phase space. Before stating Theorem 3.15, we check that this last property is stronger under the extra monotonicity condition (h4).

Theorem 3.14. *Assume that the skew-product semiflow on $\Omega \times X_+$ satisfies hypotheses* (h1), (h2), (nh3) *and* (h4). *Then either there are not bounded semiorbits or $\Omega \times \{0\}$ is the only minimal set. Consequently, $\Omega \times \{0\} \subseteq \mathcal{O}(\omega, x)$ and hence $\liminf_{t\to\infty} \|u(t, \omega, x)\| = 0$ for every (ω, x) with bounded semiorbit.*

Proof. Let us assume the existence of a bounded semiorbit and take a minimal set K. We fix a point $(\omega, x) \in K$ with $\omega \in \widetilde{\Omega}$ (the dense set appearing in (h4)), and assume by contradiction that $x > 0$. Property (h4) applied to $x_1 = x_2 = x$ and $\lambda = 0$ shows that $u(\tilde{t}, \omega, x) \gg 0$, which according to Theorem 3.2 means that $K \gg 0$ and contradicts (nh3). This means that $K \cap (\widetilde{\Omega} \times X_+) = \widetilde{\Omega} \times \{0\}$, and hence that $\Omega \times \{0\} \subset K$. Arguing as at the beginning of the proof of Theorem 3.13(i), we deduce that $K = \Omega \times \{0\}$, as asserted. Clearly, any omega-limit set contains this unique minimal set, from where the last property follows. $\square$

Theorem 3.15. *Assume that the skew-product semiflow on $\Omega \times X_+$ satisfies hypotheses* (h1), (h2) *and* (nh3). *Then, one of the following situations holds*:

Case B: $\Omega = \Omega_b$ *and every minimal set is contained in* $\Omega \times (X_+ - \operatorname{Int} X_+)$.
Case C: $\Omega \neq \Omega_b$ *and* $\Omega \neq \Omega_u$. *In this case, Ω_u is residual, and $\mathcal{O}(\omega, x) \cap (\Omega_u \times \operatorname{Int} X_+) = \emptyset$ for every $(\omega, x) \in \Omega_b \times X_+$. In addition, if hypothesis* (h4) *holds for a dense subset $\widetilde{\Omega} \subseteq \Omega$, then there exists a residual set $\Omega_o \subseteq \Omega$ such that $\liminf_{t\to\infty} \|u(t, \omega, x)\| = 0$ and $\limsup_{t\to\infty} \|u(t, \omega, x)\| = \infty$ for every $(\omega, x) \in \Omega_o \times \operatorname{Int} X_+$. Furthermore, if $\widetilde{\Omega} = \Omega$, then $\mathcal{O}(\omega, x) \cap (\Omega_u \times X_+) = \Omega_u \times \{0\}$ for every $(\omega, x) \in \Omega_b \times X_+$.*
Case D: $\Omega = \Omega_u$.

Proof. It is obvious that Cases B, C and D exhaust the possibilities under hypothesis (nh3). We will prove the remaining assertions in Case C, so that we assume $\Omega \neq \Omega_b$ and $\Omega \neq \Omega_u$. Let us fix $e \gg 0$. According to Proposition 3.1 we can write $\Omega_b = \bigcup_{m\in\mathbb{N}} A_m$ with $A_m = \bigcap_{r\in\mathbb{Q}_+} \{\omega \in \Omega \mid \|u(r, \omega, e)\| < m\}$. Let us check $\operatorname{Int} \overline{A}_m$ is empty for every $m \in \mathbb{N}$. It is easy to check that $\overline{A}_m \subset \Omega_b$, so that Ω_b contains the open subset $\operatorname{Int} \overline{A}_m$. Assuming by contradiction that it is nonempty, we conclude from the minimality of Ω and the invariance of Ω_b that $\Omega_b = \Omega$: for any $\omega \in \Omega$ there is $t \in \mathbb{R}$ with $\omega \cdot t \in \operatorname{Int} \overline{A}_m \subset \Omega_b$, and hence $\omega \in \Omega_b$. Consequently, as $\Omega_b \neq \Omega$, then it is a first Baire category set and hence its complementary Ω_u is a residual set (see [4]).

According to Proposition 3.1(ii), if $(\widetilde{\omega}, \tilde{x}) \in \Omega_u \times \operatorname{Int} X_+$, the semiorbit of $(\widetilde{\omega}, \tilde{x})$ is unbounded. This means that $(\widetilde{\omega}, \tilde{x}) \notin \mathcal{O}(\omega, x)$ for any $(\omega, x) \in \Omega_b \times X_+$, as asserted.

Assume now that (h4) holds and for an $e \gg 0$ define $i_e(\omega) = \inf_{t\geqslant 0} \|u(t, \omega, e)\|$. We take $\omega \in \Omega_b$, and observe that Theorem 3.14 ensures that $i_e(\omega) = 0$. Consequently, according to Proposition 3.1(iv), the nonempty vanishing set of i_e, Ω_c, is residual. Let us define $\Omega_o = \Omega_u \cap \Omega_c$, also residual, and take $\omega \in \Omega_o$. Since $\omega \in \Omega_c$, $i_e(\omega) = \liminf_{t\to\infty} \|u(t, \omega, e)\| = 0$, and since $\omega \in \Omega_u$, $\limsup_{t\to\infty} \|u(t, \omega, e)\| = \infty$. This and Proposition 3.1(ii) prove the same properties for every $(\omega, x) \in \Omega_o \times \operatorname{Int} X_+$.

Assume finally that condition (h4) holds for every $\omega \in \Omega$, take $(\widetilde{\omega}, \tilde{x}) \in \mathcal{O}(\omega, x) \cap (\Omega_u \times X_+)$ for a point $(\omega, x) \in \Omega_b \times X_+$, and suppose by contradiction that $\tilde{x} > 0$. Then, by (h4) and Proposition 3.1(iii), $(\widetilde{\omega} \cdot \tilde{t}, u(\tilde{t}, \widetilde{\omega}, \tilde{x})) \in \mathcal{O}(\omega, x) \cap (\Omega_u \times \operatorname{Int} X_+)$, which as seen above is impossible. The proof is complete. $\square$

Note that, when applied to semiflows coming from differential equations, the oscillation properties in Case C under hypothesis (h4) are an extension to the monotone and sublinear general dimension case (finite or infinite) of the well known oscillation properties occurring in the linear scalar case:

see Johnson [13,15] and references therein; see also Theorem A2 in Jorba et al. [16] for a simplified proof.

As in the previous subsection, the above classification is optimal. Examples as simple as the positive semiflows induced on $\mathbb{R}_+$ by the solutions of the autonomous linear scalar equations $x' = -x$ and $x' = x$ respectively fit Cases B and D. Note that, as pointed out in the Introduction, Case D does not mean the absence of strictly positive bounded semiorbits, and Cases B and D are compatible with highly complicated dynamics in the nonautonomous cases. In addition, to find examples of Case C is a bit more difficult, since this type of dynamics cannot occur in the autonomous case, understanding by *autonomous* the case in which Ω reduces to a point (see e.g. [21]). In order to complete this information, we describe in what follows a family of well-known examples of scalar almost periodic ordinary differential equations giving rise to very interesting different phenomena in Cases C and D.

The base space Ω will be a compact metric space with a minimal and almost periodic flow. *Almost periodicity* means equicontinuity of the family of transformations of the base $(\sigma_t)_{t\in\mathbb{R}}$; that is, two orbits remain uniformly close at any time provided that the initial points are close enough. It is well known that such a flow admits a unique ergodic measure m. Given a continuous function $f:\Omega\to\mathbb{R}$, we consider the family of linear equations $x' = f(\omega\cdot t)x$ for $\omega\in\Omega$, whose solutions define a monotone and linear semiflow τ on $\Omega\times\mathbb{R}_+$: $\tau(t,\omega,x) = (\omega\cdot t, u(t,\omega,x)) = (\omega\cdot t, x\exp(\int_0^t f(\omega\cdot s)\,ds))$. We assume that the equations do not have an exponential dichotomy, or in other words, that $\int_\Omega f\,dm = 0$. According to the results of Selgrade [35] and Sacker and Sell [32,33], this means that some equation in the family admits a non-trivial (positive) bounded solution. Let us represent the base sets corresponding to bounded and unbounded positive semiorbits by Ω_b^f (nonempty) and Ω_u^f. Since they are invariant and m is ergodic, their measure is 0 or 1. Note that $\omega\in\Omega_b^f$ if and only if $\limsup_{t\to\infty}\int_0^t f(\omega\cdot s)\,ds<\infty$. Then:

1. In the case that f admits a continuous primitive, i.e., a continuous function $g:\Omega\to\mathbb{R}$ with $g(\omega\cdot t)-g(\omega)=\int_0^t f(\omega\cdot s)\,ds$ for every $\omega\in\Omega$ and every $t\in\mathbb{R}$, then the omega-limit set of (ω_0,x_0) is $\{(\omega, x_0\exp(g(\omega)-g(\omega_0)))\mid\omega\in\Omega\}$, and hence the dynamics fits the one described in Theorem 3.13, Case A3.
2. In the rest of the cases, the oscillation theorem before mentioned ([13,15,16]) shows that the points $\omega\in\Omega$ such that $\limsup_{t\to\infty}\int_0^t f(\omega\cdot s)\,ds=\infty$ and $\liminf_{t\to\infty}\int_0^t f(\omega\cdot s)\,ds=-\infty$ form a residual invariant subset $\Omega_o^f\subsetneq\Omega$. Since $\Omega_o^f\subseteq\Omega_u^f$ and Ω_b^f is nonempty, the dynamics fits Case C. Moreover, if $\omega_0\in\Omega_b^f$ the positively invariant compact set $K_0=\mathrm{Cls}_{\,\Omega\times\mathbb{R}_+}\{\tau(t,\omega_0,1)\mid t\geqslant 0\}$ is *pinched*. This means that the non-null semicontinuous function $s(\omega)=\sup\{x\in\mathbb{R}_+\mid(\omega,x)\in K_0\}$ vanishes at every $\omega\in\Omega_o^f$. See Núñez and Obaya [27] for details. It follows from $s(\omega\cdot t)=s(\omega)\exp\int_0^t f(\omega\cdot s)\,ds$ that $\omega\in\Omega_b^f$ if $s(\omega)\neq 0$.
3. In spite of being in Case C, the dynamics of the examples mentioned in point 2 can be highly different. There are functions f admitting measurable but not continuous primitive g (see e.g. Furstenberg [8] and Novo and Obaya [23]). If this happens, since $s(\omega)=\exp(g(\omega)-g(\omega_0))\neq 0$ almost everywhere, then $m(\Omega_b^f)=1$ and hence $m(\Omega_u^f)=0$. But there are also many functions f with $\int_\Omega f\,dm=0$ without measurable primitive (see Johnson [14]). In this case, $m(\Omega_b^f)=0$ and hence $m(\Omega_u^f)=1$. This conclusion follows from a contradiction argument: assuming $m(\Omega_b^f)=1$ one constructs a compact invariant pinched set for which the corresponding semicontinuous function s is different from 0 almost everywhere, and hence its logarithm provides a measurable primitive.

In addition, it is not hard to check that in the case that f admits a measurable but not a continuous primitive, then there exist omega-limit sets which have indeed nonempty intersection with $\Omega\times\{0\}$ and $\Omega\times\mathrm{Int}\,X_+$, as well as different ergodic measures for the semiflow τ.

Let us employ the previous family to construct a two-dimensional example of Case D. If the function f does not have a continuous primitive, then the dynamics of the (monotone and linear) semiflow $\tilde{\tau}$ defined on $\Omega \times \mathbb{R}_+^2$ by the solutions of the family of two-dimensional linear systems $x' = f(\omega \cdot t)x$, $y' = -f(\omega \cdot t)y$ for $\omega \in \Omega$ provides an interesting example of Case D. To check this assertion, note that $\tilde{\tau}(t, \omega, x, y) = (\omega \cdot t, x \exp(\int_0^t f(\omega \cdot s)\,ds), y \exp(-\int_0^t f(\omega \cdot s)\,ds))$, so that any $(\omega, x, y) \in \Omega_u^f \times \operatorname{Int}\mathbb{R}_+^2$ gives rise to a semiorbit with unbounded first component. This excludes Cases A and B. Assume now that $\omega \in \Omega_b^f$. Applying Theorem 3.14 to the scalar equation $x' = f(\omega \cdot t)x$, we have $\liminf_{t\to\infty} \exp(\int_0^t f(\omega \cdot s)\,ds) = 0$, so that the second component of the semiorbit for $\tilde{\tau}$ of any $(\omega, x, y) \in \Omega_b^f \times \operatorname{Int}\mathbb{R}_+^2$ is unbounded: we are in Case D, as asserted. What makes this example interesting is that the absence of strongly positive bounded semiorbits does not imply that the norm of the solutions goes to ∞ as time increases: they can be oscillating solutions. In fact, for every $\omega \in \Omega_o^f$ the first component of the semiorbit of $(\omega, 1, 1)$ approaches ∞ (resp. 0) along a sequence of times for which the second one approaches 0 (resp. ∞); and given any $\omega \in \Omega_o^f$ and $x > 0$ there exists a sequence $((\omega \cdot t_n, x, 1/x))$ (obviously norm-bounded) in the semiorbit of $(\omega, 1, 1)$ with $(t_n) \uparrow \infty$. Note also that in this case it makes sense to talk about the omega-limit set of the unbounded semiorbits contained at $\Omega_o^f \times \operatorname{Int}\mathbb{R}_+^2$. This omega-limit set makes also sense in the case that $\omega \in \Omega_b^f$: if $\lim_{n\to\infty}(\omega \cdot t_n, \exp(\int_0^{t_n} f(\omega \cdot s)\,ds)) = (\omega_1, x)$ with $x > 0$ for $(t_n) \uparrow \infty$ (so that $\omega_1 \in \Omega_b^f$), then $\tilde{\tau}(t_n, \omega, 1, 1)$ tends to $(\omega_1, x, 1/x)$. In fact this type of limit is the only possible, so that the corresponding omega-limit set (closed and positively invariant, but not compact) does not project on the whole base Ω.

We also point out that there are more sophisticated examples, like the non-linear ones due to Keller [17] and Jäger [11], fitting Case B, for which the dynamical scenery is even more complex than the one previously described for Case C: apart from the existence of omega-limit sets given by pinched sets and different ergodic measures, in those examples there appear non-null Lyapunov exponents as well as strange non-chaotic attractors.

We finish this paper by recalling that the dynamics can be more accurately described in the case of a semiflow determined by a recurrent two-dimensional system of differential equations. We refer the reader to [29] for a detailed description of the set of all the minimal sets in that case.

References

[1] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Rev. 18 (1976) 620–709.

[2] L. Arnold, I.D. Chueshov, Order-preserving random dynamical systems: equilibria, attractors, applications, Dyn. Syst. 13 (1998) 265–280.

[3] L. Arnold, I.D. Chueshov, A limit set trichotomy for order-preserving random systems, Positivity 5 (2) (2001) 95–114.

[4] G. Choquet, Lectures on Analysis. Integration and Topological Vector Spaces, vol. I, Math. Lect. Notes, Benjamin, Reading, MA, 1969.

[5] I.D. Chueshov, Monotone Random Systems. Theory and Applications, Lecture Notes in Math., vol. 1779, Springer-Verlag, Berlin, Heidelberg, 2002.

[6] R. Ellis, Lectures on Topological Dynamics, Benjamin, New York, 1969.

[7] H.I. Freedman, X.-Q. Zhao, Global asymptotics in some quasimonotone reaction–diffusion systems with delays, J. Differential Equations 137 (2) (1997) 340–362.

[8] H. Furstenberg, Strict ergodicity and transformations of the torus, Amer. J. Math. 85 (1961) 573–601.

[9] W.M. Hirsch, Systems of differential equations which are competitive or cooperative I: limit sets, SIAM J. Math. Anal. 13 (1982) 167–179.

[10] W.M. Hirsch, Systems of differential equations which are competitive or cooperative II: convergence almost everywhere, SIAM J. Math. Anal. 16 (1985) 423–439.

[11] T. Jäger, On the structure of strange nonchaotic attractors in pinched skew products, Ergodic Theory Dynam. Systems 27 (2) (2007) 493–510.

[12] J. Jiang, X.-Q. Zhao, Convergence in monotone and uniformly stable skew-product semiflows with applications, J. Reine Angew. Math. 589 (2005) 21–55.

[13] R. Johnson, Minimal functions with unbounded integral, Israel J. Math. 31 (2) (1978) 133–141.

[14] R. Johnson, Almost-periodic functions with unbounded integral, Pacific J. Math. 87 (2) (1980) 347–362.

[15] R. Johnson, On a Floquet theory for almost-periodic, two-dimensional linear systems, J. Differential Equations 37 (1980) 184–205.

[16] À. Jorba, C. Núñez, R. Obaya, J.C. Tatjer, Old and new results on strange nonchaotic attractors, Internat. J. Bifur. Chaos 17 (11) (2007) 3895–3928.

[17] G. Keller, A note on strange non-chaotic attractors, Fund. Math. 151 (1996) 139–148.

[18] M.A. Krasnoselskii, Positive Solutions of Operator Equations, Groningen, Noordhoff, 1964.
[19] M.A. Krasnoselskii, The Operator of Translation Along Trajectories of Differential Equations, Transl. Math. Monogr., vol. 19, Amer. Math. Soc., Providence, 1968.
[20] U. Krause, R.G. Nussbaum, A limit set trichotomy for self-mappings of normal cones in Banach spaces, Nonlinear Anal. 20 (1993) 855–870.
[21] U. Krause, P. Ranft, A limit set trichotomy for monotone nonlinear dynamical systems, Nonlinear Anal. 19 (1992) 375–392.
[22] R.G. Nesemann, A limit set trichotomy for positive nonautonomous discrete dynamical systems, J. Math. Anal. Appl. 237 (1999) 55–77.
[23] S. Novo, R. Obaya, An ergodic classification of bidimensional linear systems, J. Dynam. Differential Equations 8 (3) (1996) 373–406.
[24] S. Novo, R. Obaya, Strictly ordered minimal subsets of a class of convex monotone skew-product semiflows, J. Differential Equations 196 (2004) 249–288.
[25] S. Novo, R. Obaya, A.M. Sanz, Attractor minimal sets for cooperative and strongly convex delay differential equations, J. Differential Equations 208 (1) (2005) 86–123.
[26] S. Novo, R. Obaya, A.M. Sanz, Stability and extensibility results for abstract skew-product semiflows, J. Differential Equations 235 (2) (2007) 623–646.
[27] C. Núñez, R. Obaya, A non-autonomous bifurcation theory for deterministic scalar differential equations, Discrete Contin. Dyn. Syst. 9 (3–4) (2008) 701–730.
[28] C. Núñez, R. Obaya, A.M. Sanz, Global attractivity in concave or sublinear monotone infinite delay differential equations, J. Differential Equations 246 (2009) 3332–3360.
[29] C. Núñez, R. Obaya, A.M. Sanz, Minimal sets in monotone and sublinear skew-product semiflows II: Two-dimensional systems of differential equations, J. Differential Equations (2010), doi:10.1016/j.jde.2009.12.006, in press.
[30] P. Poláčik, I. Tereščák, Exponential separations and invariant bundles for maps in ordered Banach spaces with applications to parabolic equations, J. Dynam. Differential Equations 5 (1993) 279–304.
[31] P. Poláčik, I. Tereščák, Erratum to: Exponential separations and invariant bundles for maps in ordered Banach spaces with applications to parabolic equations, J. Dynam. Differential Equations 6 (1) (1994) 245–246.
[32] R.J. Sacker, G.R. Sell, Dichotomies and invariant splittings for linear differential systems I, J. Differential Equations 15 (1974) 429–458.
[33] R.J. Sacker, G.R. Sell, Dichotomies and invariant splittings for linear differential systems II, J. Differential Equations 22 (1976) 478–496.
[34] R.J. Sacker, G.R. Sell, Lifting properties in skew-products flows with applications to differential equations, Mem. Amer. Math. Soc. 190 (1977).
[35] J.F. Selgrade, Isolated invariant sets for flows on vector bundles, Trans. Amer. Math. Soc. 203 (1975) 359–390.
[36] J.F. Selgrade, Asymptotic behavior of solutions to single loop positive feedback systems, J. Differential Equations 38 (1980) 80–103.
[37] G.R. Sell, Topological Dynamics and Ordinary Differential Equations, Van Nostrand Reinhold, London, 1971.
[38] W. Shen, Y. Yi, Almost Automorphic and Almost Periodic Dynamics in Skew-Products Semiflows, Mem. Amer. Math. Soc., vol. 647, Amer. Math. Soc., Providence, 1998.
[39] H.L. Smith, Cooperative systems of differential equations with concave nonlinearities, Nonlinear Anal. 10 (1986) 1037–1052.
[40] P. Takáç, Asymptotic behavior of discrete-time semigroups of sublinear, strongly increasing mappings with applications to biology, Nonlinear Anal. 14 (1) (1990) 35–42.
[41] P. Takáç, Convergence in the part metric for discrete dynamical systems in ordered topological cones, Nonlinear Anal. 26 (11) (1996) 1753–1777.
[42] B.Z. Vulikh, Introduction to the Theory of Partially Ordered Spaces, Woltest-Nordhoff Scientific Publications LTD, Groningen, 1967.
[43] Y. Wang, X.-Q. Zhao, Convergence in monotone and subhomogeneous discrete dynamical systems on product Banach spaces, Bull. Lond. Math. Soc. 35 (2003) 681–688.
[44] X.-Q. Zhao, Global attractivity in monotone and subhomogeneous almost periodic systems, J. Differential Equations 187 (2003) 494–509.