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The kernel and cokernel of a differential operator in several variables. II

by Arno van den Essen

Mathematisch Instituut, Budapestlaan 6, 3508 TA Utrecht, the Netherlands

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We fix the following notations. Let $n \in \bar{\mathbb{N}} := \mathbb{N} \cup \{0\}$ and let k be a field of characteristic zero. Put $\mathcal{O}_n = k$ if $n=0$ and $\mathcal{O}_n$ is the ring of formal power series in n variables $x_1, \ldots, x_n$ over k if $n \in \mathbb{N}$. Instead of $\mathcal{O}_n$ we write A. Furthermore $B := A[[t]]$ is the ring of formal power series in t with coefficients in A. Instead of the derivation $\partial/\partial t$ we write ∂. If R is a ring, $\underline{\underline{M}}(R)$ denotes the category of left R-modules of finite type.

Let $P: B \to B$ be a differential operator of the form

$$P = \sum_{i=0}^{r} p_i \partial^i, \text{ with } r \in \bar{\mathbb{N}},\ p_i \in B.$$

In [2] we showed that if p_r is t-regular i.e. $p_r(0, \ldots, 0, t) \neq 0$, both ker P and coker P $(=B/PB)$ belong to $\underline{\underline{M}}(A)$.

In this paper we give a much simpler proof of this result, based on work of Malgrange in [3]. Moreover, the theorems we obtain below generalize the previous results of [2] and are complete in the sense that they clarify under what conditions on P the kernel and cokernel are finitely generated A-modules.

THEOREM 1. Put $P(0) := \sum_{i=0}^{r} p_i(0, \ldots, 0, t)\partial^i$. Then coker $P \in \underline{\underline{M}}(A)$ iff $P(0) \neq 0$.

REMARK 1. This result can be viewed as a generalization of Weierstrass' division theorem, by taking $r=0$.

PROOF. i) Assume coker $P \in \underline{\underline{M}}(A)$. Since A is noetherian the A-submodule $\sum A\bar{t}^i$ of coker P is in $\underline{\underline{M}}(A)$ where $\bar{t}^i := t^i + PB$. Hence

$$t^N + a_{N-1}t^{N-1} + \ldots + a_0 = Pb \text{ for some } N \in \mathbb{N},\ a_i \in A,\ b \in B.$$

Now substitute $x_1 = \ldots = x_n = 0$ and we get $P(0) \neq 0$.

ii) Conversely, let $P(0) \neq 0$. We use induction on n. The case $n = 0$ is proved by Malgrange in [3]. So let $n \in \mathbb{N}$. Put

$$B_0 = \mathcal{O}_{n-1}[[t]],\ P_0 = \sum p_i(x_1, \ldots, x_{n-1}, 0, t)\partial^i.$$

We have the following isomorphism of B_0-modules

$$B/PB + x_nB = B/x_nB + P_0B \simeq B_0/P_0B_0.$$

By the induction hypothesis $B_0/P_0B_0 \in \underline{\underline{M}}(\mathcal{O}_{n-1})$, so we get

$$B/PB + x_nB \in \underline{\underline{M}}(\mathcal{O}_{n-1}).$$

Hence there exist $s \in \mathbb{N}$, $e_1, \ldots, e_s \in B$ satisfying

$$B \subset \sum \mathcal{O}_{n-1}e_i + PB + x_nB.$$

Let $b \in B$. Then

(1) $\qquad b = \sum c_ie_i + x_ng + Ph$, some $c_i \in \mathcal{O}_{n-1}, g, h \in B$.

Similarly

(2) $\qquad g = \sum c_i'e_i + x_ng' + Ph'$, some $c_i' \in \mathcal{O}_{n-1}, g', h' \in B$.

Substituting (2) in (1) gives

(3) $\qquad b = \sum (c_i + c_i'x_n)e_i + x_n^2g' + P(h + x_nh')$.

Again we can substitute a formula for g' in (3) and so on.

Since A and B are complete local rings the process above gives

$$b \in \sum Ae_i + PB, \text{ all } b \in B.$$

Hence $B \subset \sum Ae_i + PB$, which implies coker $P \in \underline{\underline{M}}(A)$, as desired.

From the proof above we immediately conclude:

COROLLARY 2. Assume $P(0) \neq 0$. If coker $P(0)$ $(= k[[t]]/P(0)k[[t]])$ is generated as a k-module by the residue classes of some elements $e_1, \ldots, e_d \in k[[t]]$, then coker P is generated as an A-module by the residue classes of $e_1, \ldots, e_d$.

In general coker P is not a free A-module, as can be seen as follows.

Let $n = 1$. So $A = k[[x]]$, $B = k[[x, t]]$. Put $P := t\partial + (x - 1)$. So $P(0) = t\partial - 1$. It is easily seen that coker $P(0) \simeq k\bar{t}$, where the residue class $\bar{t} := t + P(0)\ k[[t]]$ is non-zero. By Cor. 2 we get coker $P \simeq A\bar{\bar{t}}$, with $\bar{\bar{t}} := t + PB \neq 0$. Finally, since $xt = Pt$ we get $x\bar{\bar{t}} = 0$. So coker P is not a free A-module. Observe that ker $P(0) \neq 0$, since $P(0)t = 0$. However, the following general result completely clarifies the situation when ker $P(0) = 0$.

PROPOSITION 3. If $\ker P(0)=0$, then $\operatorname{coker} P$ is a free A-module of rank d, where $d:=\dim_k \operatorname{coker} P(0)$. More precisely, for every d-tuple $e_1,\dots,e_d$ in $k[[t]]$ such that $(\bar{e}_1,\dots,\bar{e}_d)$ is a k-basis of $\operatorname{coker} P(0)$, the elements e_i+PB, $1\le i\le d$ form an A-basis of $\operatorname{coker} P$.

To prove this result we also generalise [2], Th. II.1). Observe that if $P=0$, then $\ker P=B\notin \underline{\underline{M}}(A)$. In fact this is the only case where $\ker P\notin \underline{\underline{M}}(A)$. This follows from

THEOREM II. $\ker P\in \underline{\underline{M}}(A)$ iff $P\neq 0$.

PROOF. i) Write $P=\sum P_j x_n^j$, $P_j\in B_0[\eth]$. Assume $P\neq 0$. Then there exists some minimal $j_0\in \mathbb{N}$ with $P_{j_0}\neq 0$. Put $P=x_n^{j_0}Q$. Obviously $\ker P=\ker Q$ and $Q_0=P_{j_0}\neq 0$. So we may assume $P_0\neq 0$.

ii) Let $b\in\ker P$. Write $b=\sum b_j x_n^j$, $b_j\in B_0$. Then $b_0\in\ker P_0$. So we get a map

$$\phi:\ker P\ni b\to b_0\in\ker P_0.$$

If $b_0=0$, then $b=x_n b'$, some $b'\in B$. Since $0=Pb=x_nPb'$ and B has no zero-divisors, $b'\in\ker P$ follows. Hence $\ker\phi=x_n\ker P$. So ϕ induces an injective $\mathcal{O}_{n-1}$-module homomorphism

$$\bar{\phi}:\ker P/x_n\ker P\to\ker P_0.$$

iii) Now we prove our theorem by induction on n. The case $n=0$ is proved by Malgrange in [3]. So let $n\in\mathbb{N}$. Then $\ker P_0\in\underline{\underline{M}}(\mathcal{O}_{n-1})$. So by ii) $\ker P/x_n \ker P\in\underline{\underline{M}}(\mathcal{O}_{n-1})$, say

$$\ker P/x_n\ker P=\sum_{i=1}^{q}\mathcal{O}_{n-1}(e_i+x_n\ker P),\ \text{some } e_i\in\ker P.$$

Then, arguing as in the second part of the proof of TH. I we find $\ker P=\sum Ae_i\in\underline{\underline{M}}(A)$, which concludes the proof.

COROLLARY 4. If $\ker P(0)=0$, then $\ker P=0$.

PROOF. Induction on n. So we may assume $\ker P_0=0$. Then by ii) above $\ker P/x_n\ker P=0$, whence $\ker P=0$ by the result of iii) above.

PROOF OF PROPOSITION 3. Let $e_1,\dots,e_d\in k[[t]]$ be such that $(\bar{e}_1,\dots,\bar{e}_d)$ is a k-basis of $\operatorname{coker} P(0)$.

Using Cor. 2 it remains to verify that the elements e_i+PB, $1\le i\le d$ are A-linearly independent. So let $\sum a_ie_i=Pb$, some $a_i\in A$, $b\in B$. We use induction on n. Therefore we write as before

$$a_i=\sum a_{ij}x_n^j,\ b=\sum b_jx_n^j,\ P=\sum P_jx_n^j.$$

We get

$$(*) \qquad \sum a_{ij} e_i = P_j b_0 + P_{j-1} b_1 + \ldots + P_0 b_j, \text{ all } j \in \mathbb{N}.$$

By induction on j we shall prove: $a_{ij}=0$, all i, $b_0 = \ldots = b_j = 0$. From this we deduce $a_i = 0$, all i as desired.

The case $j=0$. Put $j=0$ in (*). This gives

$$\sum a_{i0} e_i = P_0 b_0.$$

So by our induction on n we get $a_{i0}=0$, all i. Whence $P_0 b_0 = 0$ implying $b_0 = 0$ (Cor. 4). Now let $j \in \mathbb{N}$ and assume $b_0 = \ldots = b_{j-1} = 0$.

Then (*) gives

$$\sum a_{ij} e_i = P_0 b_j.$$

Whence, by induction on n, $a_{ij}=0$, all i, implying $b_j = 0$ by Cor. 4 as desired.

FINAL REMARK. An application of TH. I is given in [1] where it is shown that the cokernel of a holonomic $\mathscr{D}_n$-module under the operator $\partial/\partial x_n$ is a holonomic $\mathscr{D}_{n-1}$-module if M satisfies some generic condition. The importance of this last result for the theory of $\mathscr{D}$-modules will be pointed out in a forthcoming paper.

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