View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Elsevier - Publisher Connector

JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 15, 183-199 (1966)

The Formation and Decay of Hydraulic Shock Waves[1]

R. R. BURNSIDE

AND

A. G. MACKIE

University of Toronto, Toronto, Ontario, Canada
Victoria University of Wellington, New Zealand

Submitted by F. V. Atkinson

Two standard problems in the propagation of plane shock waves are those of the formation of a shock due to the uniform acceleration of a piston moving into a gas at rest and of the decay of a shock when it interacts with a simple wave. This latter problem can arise when a piston moving with uniform velocity into a gas at rest is suddenly stopped and thereby sends out a rarefaction wave to interact with the shock wave which is sent ahead of the piston. In gas dynamics these problems are highly complicated because of entropy variations in the flow. This paper is concerned with the analogous situations in shallow water theory. In the hydraulic analogy there is no counterpart to entropy variations, and hence the flow, even behind a nonuniform shock (bore), can be investigated by the classical method of Riemann invariants. Integral equations for the shock path in the x, t-plane are set up and solved. In the final section an alternative method is given for determining the early stages of decay of a shock. The results are compared with those obtained by the simple wave approximation as described by Friedrichs.

1. INTRODUCTION

In the theory of one-dimensional, unsteady flow of a gas there are two well-known problems connected with shock formation and decay. The first problem is to investigate the growth of a shock wave formed when a piston is pushed with uniform acceleration into a gas at rest. The second is to

[1] One of the authors (R.R.B.) wishes to make acknowledgement to the Department of Scientific and Industrial Research in the United Kingdom for a grant which was held during the course of this work.

examine the rate of decay of a shock wave advancing into a region at rest when it interacts with a point-centered simple wave which overtakes it. Now in either problem the shock wave is "bounded" (in the x, t-plane) on one side by a uniform region at rest, and thus any variation in the strength of the shock must imply entropy variations which propagate along the particle paths behind the shock. These variations make any analytic approach extremely difficult. Fortunately, however, a considerable simplification is effected if the shock strength is not too great. For the entropy variation occurs to the third power of the shock strength and if this can be neglected then the solution can be found approximately by the methods of homentropic flow. Moreover it can be shown that to the same order of approximation the appropriate Riemann invariant through the shock is constant and the problems of formation and decay both reduce to finding the equation of a shock wave bordering a simple wave with the Rankine-Hugoniot equations acting as boundary conditions. A detailed account is given by Friedrichs [1]. Higher order approximations have been obtained by Lighthill [2] and Pillow [3] who give a first order estimate of the perturbation on the simple wave.

This paper is concerned with the analogous problems regarding the behavior of bores in shallow water. Although the approximate shallow water theory runs along lines parallel to those of the theory of gas dynamics there is one difference which is crucial in what follows. This is that there is nothing in shallow water theory which corresponds to the entropy in gas dynamics, and hence, *even behind a bore of variable strength*, the flow can be characterized by the Riemann invariants as in the classical theory of the unsteady, homentropic motion of a gas. The approximate simple wave theory is applicable as before, but in seeking a more accurate representation we need only the mathematics used to describe the homentropic motion of a gas with adiabatic index $\gamma = 2$. Because the language used throughout is that of gas dynamics we find it more suitable to refer to bores as shocks in what follows.

It is often, of course, very difficult to obtain an explicit representation of solutions in gas dynamics, even for homentropic flow, and indeed the complicated nature of the boundary conditions here precludes a complete or exact solution. Nevertheless a considerable improvement can be effected over the simple wave theory. The main problem to solve arises as follows. When a shock advances into a region of water at rest the image of the back of the shock is a fixed curve in the r, s-plane. This is the plane of the Riemann invariants in which the time t satisfies the linear Euler-Poisson equation. The resulting boundary value problem is not a classical one in which two data are given on the non-characteristic curve but a slightly different one in which one datum is given on this curve and one on a characteristic bounding the region in which the solution is sought. The application of Riemann's method then leads to an integral equation for t as a function of the shock strength on the shock. This can be solved to any desired degree of accuracy for the shock formation problem. It can also be solved to describe the complete history of the decay of the shock when we neglect terms in $\epsilon_M{}^4$, ϵ_M being a parameter denoting the initial strength of the shock. At first sight this would appear only one degree better than Friedrich's approximation which neglects $\epsilon_M{}^3$, but, in fact, it is two degrees better, since the cumulative effect of the error for large values of t is not taken into account in the simple wave theory as it is here.

Finally we find an alternative method, based on Meyer's focusing equations [4], for describing the *early* stages of decay of a shock of arbitrary strength. A comparison is made with the early stages of decay of the relatively weak shocks considered earlier. This comparison acts as a guide in assessing the range of shock strength over which the description of the complete history of decay might be considered valid.

2. Formulation of the Problem

If $h(x, t)$, $u(x, t)$ are respectively the height of the water surface above the (flat) bottom and the velocity then

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0,$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0.$$

If we introduce the variable c, defined by $c^2 = gh$, these equations become

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + \frac{c}{2}\frac{\partial u}{\partial x} = 0,$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + 2c\frac{\partial c}{\partial x} = 0.$$

The Riemann invariants are defined by

$$r = \tfrac{1}{2}u + c, \qquad s = -\tfrac{1}{2}u + c. \tag{2.1}$$

As in gas dynamics we have $r =$ constant on the characteristics defined by $dx/dt = u + c$ and $s =$ constant on the characteristics defined by $dx/dt = u - c$.

The relations connecting the variables on the two sides of a shock which advances with speed v are

$$c_1^2(v - u_1) = c_2^2(v - u_2),$$
$$c_1^2(v - u_1)^2 + \tfrac{1}{2} c_1^4 = c_2^2(v - u_2)^2 + \tfrac{1}{2} c_2^4.$$

If a shock wave is advancing with speed v into a region in which $u_1 = 0$, $c_1 = 1$, then the values of u and c behind the shock are given by

$$c^2(v - u) = v$$
$$c^2(v - u)^2 + \tfrac{1}{2} c^4 = v^2 + \tfrac{1}{2}.$$

If now v is eliminated from these equations and the new variables r, s, defined by (2.1), are introduced then the image of the back of the shock wave in the r, s-plane is the curve

$$P(r, s) \equiv 4\sqrt{2}\,(r^2 - s^2) - \{(r + s)^2 + 4)^{1/2}\}\{(r + s)^2 - 4\} = 0.$$

The time t, regarded as a function of r and s, must satisfy a certain relation on the curve. For $dx/dt = v(r, s)$ on $P(r, s) = 0$, and we have the further equations

$$\frac{\partial x}{\partial r} = (u - c)\frac{\partial t}{\partial r}, \qquad \frac{\partial x}{\partial s} = (u + c)\frac{\partial t}{\partial s} \tag{2.2}$$

arising from the fact that r, s are characteristic variables. After some algebra we find that

$$\frac{\partial t}{\partial s} = G(r, s)\frac{\partial t}{\partial r} \qquad \text{on} \qquad P(r, s) = 0, \tag{2.3}$$

where

$$G(r, s) = \left(\frac{r - 3s - 2}{3r - s - 2}\right)\frac{\partial P/\partial s}{\partial P/\partial r} \tag{2.4}$$

and

$$v = \frac{r + s}{4\sqrt{2}}\{(r + s)^2 + 4\}^{1/2}.$$

From (2.2) the differential equation for t is seen to be

$$(r + s)\frac{\partial^2 t}{\partial r\,\partial s} + \frac{3}{2}\left(\frac{\partial t}{\partial r} + \frac{\partial t}{\partial s}\right) = 0. \tag{2.5}$$

For both the shock formation and decay problems the domain in which this equation has to be solved is bordered by the shock locus $P(r, s) = 0$ and an s-characteristic. The time t is known on the s-characteristic and we also have relation (2.3) holding on the shock. Thus the problem of the determination of the shock locus in the x, t-plane reduces to that of solving the Euler-Poisson equation (2.5) in a fixed domain of the r, s-plane with one datum given on a characteristic and one on a non-characteristic curve. We shall see how $t(r, s)$ may thus be determined on the shock, after which the equation of the shock in the x, t-plane can be easily obtained. For later convenience we now introduce a parameter defined by

$$\epsilon = \tfrac{2}{3}(u + c - 1). \tag{2.6}$$

This parameter will only be used with reference to points on the shock. Clearly as $\epsilon \to 0$ the shock becomes vanishinsly weak. Below we list the expansions of certain quantities on the shock expressed as power series in ϵ. These are required in what follows.

$$\left.\begin{aligned} r &= 1 + \epsilon - \frac{1}{64}\epsilon^3 + \frac{1}{128}\epsilon^4 - \frac{11}{12288}\epsilon^5 + \cdots \\ s &= 1 - \frac{3}{64}\epsilon^3 + \frac{3}{128}\epsilon^4 - \frac{11}{4096}\epsilon^5 + \cdots \\ v &= 1 + \frac{3}{4}\epsilon + \frac{5}{32}\epsilon^2 - \frac{5}{128}\epsilon^3 + \frac{3}{2048}\epsilon^4 + \frac{51}{8192}\epsilon^5 + \cdots . \end{aligned}\right\} \tag{2.7}$$

3. Formation of a Shock

When a piston is pushed with constant acceleration and with initial velocity of zero into a column of water at rest, a forward-facing simple compression wave is set up. We let $x = \frac{1}{2}at^2$ be the piston path and let the simple wave be characterized by a parameter τ, the time on the piston path. The equation of the r-characteristic through the point $(\frac{1}{2}a\tau^2, \tau)$ is

$$x - \frac{1}{2}a\tau^2 = \left(1 + \frac{3a\tau}{2}\right)(t - \tau)$$

when $c = 1$ in the region ahead of the advancing wave. These characteristics intersect to form an evelope, the first point of which is the point N with coordinates $(2/3a, 2/3a)$ on the leading characteristic $x = t$ (Fig. 1). We try to anticipate the formation of this envelope by inserting a shock wave NX beginning at this point. The shape of this shock wave is to be determined. The simple wave approximation is to assume that $s = 1$ behind the shock as it is in front. From (2.7), $s = 1 + 0(\epsilon^3)$ so that this is a reasonably good approximation in the very early stages. However the r-characteristics meeting the shock do in fact send signals back into the water and these signals interact with the oncoming characteristics and so affect the subsequent growth of the shock. Anything like a complete description of the whole history of the shock is out of the question because of the later reflection of waves off the piston itself and the probable formation of secondary shocks. However the boundary value problem, when correctly formulated in the r, s-plane leads to a description of the early stages of the shock which is much more accurate than that given by the simple wave approximation.

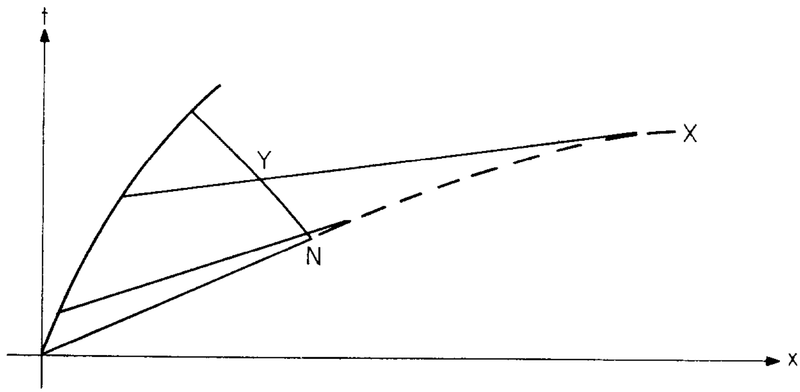

FIG. 1

The configuration in the r, s-plane is shown in Fig. 2. NX is the curve $P(r, s) = 0$, and NY is the characteristic $s = 1$. From the theory of the simple wave we know that on NY

$$t = \frac{2}{15a}\{16\sqrt{2}\,(r+1)^{-3/2} + 3(2r-3)\} \tag{3.1}$$

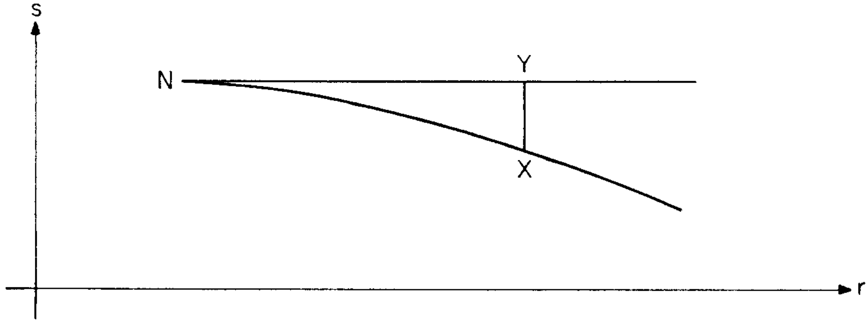

FIG. 2

By the application of Riemann's method we can now write down

$$t_Y = \frac{1}{2}(Wt)_N + \frac{1}{2}(Wt)_X + \int_N^X \left\{\frac{3Wt}{2(r+s)} + \frac{1}{2}\left(W\frac{\partial t}{\partial r} - t\frac{\partial W}{\partial r}\right)\right\} dr$$
$$- \int_N^X \left\{\frac{3Wt}{2(r+s)} + \frac{1}{2}\left(W\frac{\partial t}{\partial s} - t\frac{\partial W}{\partial s}\right)\right\} ds, \qquad (3.2)$$

where W is the Riemann function and the curvilinear integrals are taken along the curve $P(r, s) = 0$. In this case we do not know all of the quantities t, $\partial t/\partial r$ and $\partial t/\partial s$ on the shock. We merely have the relation (2.3) connecting them. Thus instead of regarding (3.2) as an equation for t_Y we can, after some manipulation, rewrite it as an integral equation for t at the point X on the shock.

It seems desirable first to make some comment on the notation used. A Riemann function is a function of four variables, the "current" coordinates r, s and the "field" co-ordinates r_0, s_0. In the present application the field point Y will always be on $s = 1$ and we take its coordinates to be $(r_\theta, 1)$. Thus $(r_\theta$ is the r-coordinate of the point X on the shock (Fig. 2), and we shall use s_θ as the corresponding s-coordinate so that $P(r_\theta, s_\theta) = 0$. Furthermore we find it convenient to use a function w which is $(r + s)^{-1}$ times the classical Riemann function and is defined by

$$w(r, s; r_\theta, 1) = \frac{(r+1)^{1/2}(r_\theta + s)^{1/2}}{(r_\theta + 1)^2} F\left(-\frac{1}{2}, -\frac{1}{2}; 1; p\right),$$

where

$$p = \frac{(r - r_\theta)(s-1)}{(r+1)(r_\theta + s)},$$

and the standard notation for hypergeometric functions is used.

We note also that Eq. (3.2) includes terms in $\partial t/\partial r$ and $\partial t/\partial s$. When use is made of (2.3), however, these terms can be written as multiples of the total derivative dt/dr along the curve, and this derivative is then eliminated by integrating by parts. The result of this operation leads to the equation

$$\left[\left(\frac{3}{2}r - \frac{1}{2}s - v\right)wt\right]_X$$
$$= \left[\left(\frac{1}{2}r - \frac{3}{2}s - v\right)wt\right]_N + t_Y - \int_1^{r_\theta} t\left[\left(1 - \frac{ds}{dr}\right)w - \frac{d}{dr}\{(r - s - v)w\}\right.$$
$$\left. - \frac{r+s}{2}\left(\frac{\partial w}{\partial r} - \frac{\partial w}{\partial s}\frac{ds}{dr}\right)\right] dr. \qquad (3.3)$$

If we now make use of (3.1) and perform the necessary simplification we obtain

$$\left\{\frac{3}{2} r_\theta - \frac{1}{2} s_\theta - v(r_\theta, s_\theta)\right\} (r_\theta + s_\theta)^{1/2} (r_\theta + 1)^{-3/2} t(r_\theta, s_\theta)$$

$$= \frac{2}{5a}\{2\sqrt{2}(r_\theta + 1)^{-3/2} + (2r_\theta - 3)\} - \int_1^{r_\theta} t(r, s)$$

$$\times \left[\left(1 - \frac{ds}{dr}\right) w - \frac{d}{dr}\{(r - s - v)\, w\} - \frac{r + s}{2}\left(\frac{\partial w}{\partial r} - \frac{\partial w}{\partial s}\frac{ds}{dr}\right)\right] dr. \tag{3.4}$$

We have now obtained a nonhomogeneous Volterra integral equation of the second type for $t(r_\theta, s_\theta)$. In the above equation everything (except t) outside the integral is known as a function of r_θ, while the square bracket inside the integral is a known function of r and r_θ, any s which occurs being understood as a function of r through the relation $P(r, s) = 0$.

It is not possible to obtain a general solution of this equation, but t may be found as a series in powers of ϵ to any required number of terms. We have, in fact, found t to the fourth power in ϵ by substituting an expansion for t in (3.4) and equating powers of ϵ after expanding the integrand also. The resulting expression is

$$t = \frac{2}{3a}\left(1 + \frac{4}{3}\epsilon + \frac{5}{72}\epsilon^2 + \frac{89}{120}\epsilon^3 + \frac{250379}{110592}\epsilon^4 + \cdots\right). \tag{3.5}$$

In order to find the actual equation of the shock path in the x, t-plane we use the relation (2.6)

$$\frac{dx}{dt} = v = 1 + \frac{3}{4}\epsilon + \frac{5}{32}\epsilon^2 - \frac{5}{128}\epsilon^3 + \cdots,$$

which gives x as a function of ϵ; namely

$$x = \frac{2}{3a}\left(1 + \frac{4}{3}\epsilon + \frac{41}{72}\epsilon^2 + \frac{203}{240}\epsilon^3 + \frac{327343}{221184}\epsilon^4 + \cdots\right) \tag{3.6}$$

(3.5) and (3.6) then define the equation of the shock in the x, t-plane parametrically.

The simple wave theory as described by Friedrichs gives the first three terms of the expansion in (3.5) or (3.6), that is up to the term in ϵ^2. The additional terms obtained in (3.5) show that the rate at which the shock wave grows in intensity is thus slightly overestimated by the simple wave approximation.

An analogous result was obtained by Pillow [3] for the gas dynamical problem. In fact the method we have used is very similar to that of Pillow, although, because of the absence of entropy variations here, the integral equation is now exact whereas in Pillow's work the equivalent equation was derived as an approximation valid to order ϵ^3. Thus Pillow does not continue the expansion beyond the term in ϵ^3, since further terms would not be significant. As here the coefficient of ϵ^3 in the expression for t is positive, indicating that the shock grows somewhat less rapidly than the simple wave approximation suggests. Pillow appears to have misinterpreted his results, stating that the shock grows more rapidly when determined from the higher approximation,

4. The Shock Decay Problem

The second problem which we consider is that of a uniform shock which decays as the result of interaction with a point-centered simple wave. The position in the x, t-plane is shown in Fig. 3. A piston is moved along PO, the path $x = u_M t$ sending in front of it a shock PM at speed v_M. At O the piston is suddenly stopped and a rarefaction wave, point-centered at O, overtakes the shock and modifies its shape. As in the previous problem a simple wave theory can be used for weak shocks. However, as before the signals sent back along the s-characteristics from the shock will interact with the oncoming r-characteristics and so affect the rate of decay of the shock. It is our purpose to take into account the effect of this interaction.

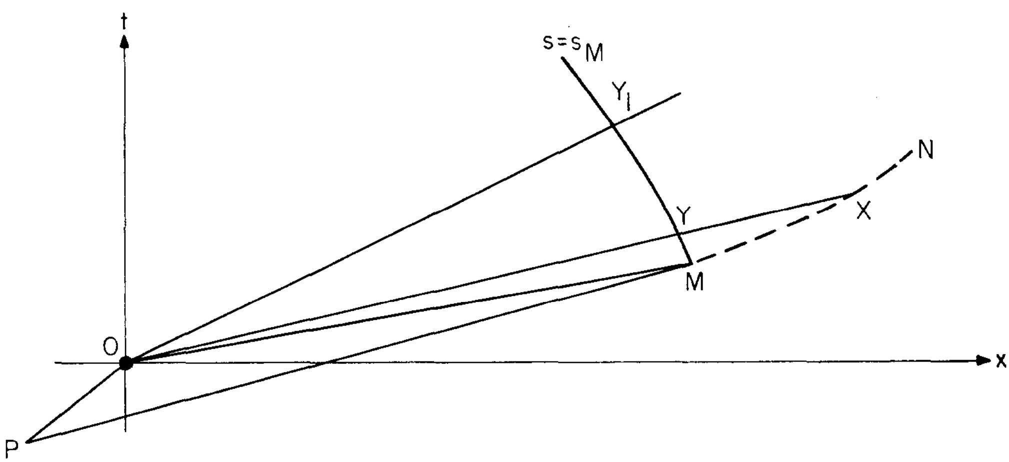

Fig. 3

If we let the suffix M denote conditions at the point M (just behind the shock) we will therefore have to consider the curvilinear triangle NMY_1 in the r, s-plane (Fig. 4) bounded by the curve $P(r, s) = 0$ and the charac-

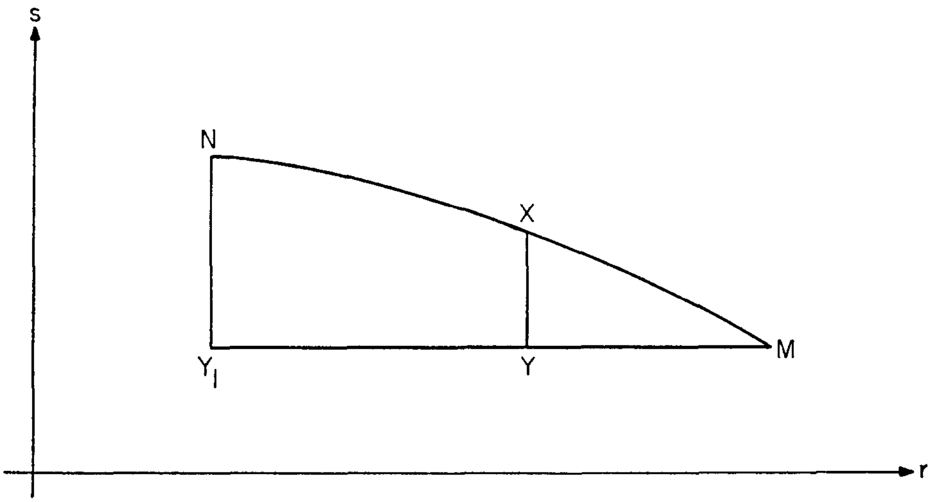

FIG. 4

teristics $s = s_M$ and $r = 1$. The time t is now known along the characteristic $s = s_M$, since this characteristic MYY_1 in the physical plane (Fig. 3) is the last s-characteristic of the simple wave unaffected by the modification to the shock wave. In fact on $s = s_M$ it is easily shown that

$$t = t_M \left(\frac{r_M + s_M}{r + s_M}\right)^{3/2}. \tag{4.1}$$

The Riemann function is similar to that defined in the previous section. However a typical field point in the present application now lies on the characteristic $s = s_M$ instead of $s = 1$. Accordingly in the present section we use

$$w(r, s; r_\theta; s_M) = \frac{(r + s_M)^{1/2} (r_\theta + s)^{1/2}}{(r_\theta + s_M)^2} F\left(-\frac{1}{2}, -\frac{1}{2}; 1; p\right),$$

where

$$p = \frac{(r - r_\theta)(s - s_M)}{(r + s_M)(r_\theta + s)}.$$

This function is $(r + s)^{-1}$ times the classical Riemann function with field point (r_θ, s_M). As before s_θ is the s-coordinate of the point on the shock curve whose r-coordinate is r_θ. This is the point X in Fig. 4.

There is one main difference between the solution in this case and that of the previous section. This is because we now wish to consider the whole history of the shock and must therefore seek a formula which is valid for all $t \geqslant t_M$, since we would expect the shock to decay to zero strength only after infinite time. However the image of the whole of the shock is confined to the finite segment NM in the r, s-plane. We shall see that, as expected, t will become infinite as the point N is approached—the resulting integral equation now being singular.

This equation is derived in exactly the same way as (3.2) and is indeed identical with it except that N is now replaced by M and the lower limit of the integral is r_M instead of 1. When we make use of (2.3) and (2.4), we have the form corresponding to (3.4), which is

$$\left\{\frac{3}{2} r_\theta - \frac{1}{2} s_\theta - v(r_\theta , s_\theta)\right\} (r_\theta + s_M)^{-3/2}\, t(r_\theta , s_\theta)$$

$$= (r_M + s_M)^{1/2} (r_\theta + s_M)^{-3/2} \left(\frac{3}{2} r_M - \frac{1}{2} s_M - v_M\right) t_M - \int_{r_M}^{r_\theta} t(r, s)$$

$$\times \left[\left(1 - \frac{ds}{dr}\right) w - \frac{d}{dr}\{(r - s - v)\, w\} - \frac{r + s}{2} \left(\frac{\partial w}{\partial r} - \frac{\partial w}{\partial s}\frac{ds}{dr}\right)\right] dr. \tag{4.2}$$

Mathematically the most significant difference between this equation and (3.4) is that the term other than the integral on the right hand side does not vanish when $r_\theta = 1$ in (4.2) as it does in (3.4). Since the coefficient of t on the left hand side is zero when $r_\theta = 1$ in both cases, this explains why we get an infinite value of t when $r_\theta = 1$ in (4.2), although not in (3.4). We can emphasize this by rewriting (4.2) in terms of a new variable

$$T(\epsilon) = \left(\frac{3r_\theta - s_\theta - 2v(r_\theta , s_\theta)}{3r_M - s_M - 2v_M}\right) \frac{(r_\theta + s_\theta)^{1/2}}{(r_M + s_M)^{1/2} (r_\theta + s_M)^{3/2}} \frac{t(\epsilon)}{t_M}, \tag{4.3}$$

where ϵ is defined by (2.6) and r_θ, s_θ and v are known functions of ϵ through the Rankine-Hugoniot equations. They are given approximately by (2.7). The integral equation now becomes

$$T(\epsilon) = (r_\theta + s_M)^{-3/2} - \int_{\epsilon_M}^{\epsilon} T(\rho) \frac{2(r + s)^{-1/2} (r + s_M)^{3/2}}{3r - s - 2v} k(\rho, \epsilon)\, d\rho, \tag{4.4}$$

where ρ is the value of ϵ at a point in the range of integration and

$$k(\rho, \epsilon) = \left(1 - \frac{ds}{dr}\right) w - \left\{\frac{d}{dr}\{(r - s - v)\, w\} - \frac{r + s}{2}\left(\frac{\partial w}{\partial r} - \frac{\partial w}{\partial s}\frac{ds}{dr}\right)\right\} \frac{dr}{d\rho} .$$

Inspection of $k(\rho, \epsilon)$ shows that it is a bounded function in the range $0 \leqslant \epsilon \leqslant \rho \leqslant \epsilon_M$, and hence that the singularity arises only from the term $3r - s - 2v$ in the denominator of the integral in (4.4). Consequently we will get a uniformly valid approximate solution by expanding $k(\rho, \epsilon)$ as a double power series in ρ and ϵ. We obtain

$$k(\rho, \epsilon) = \frac{3}{2\sqrt{2}}(1 + s_M)^{-3/2}\left\{1 + \frac{5\rho - 9\epsilon}{12} - \frac{2\rho^2 + 10\rho\epsilon - 15\epsilon^2}{32} - \frac{22\rho^3 - 30\rho^2\epsilon - 50\rho\epsilon^2 + 67\epsilon^3}{256} + 0(\epsilon_M{}^4)\right\}. \tag{4.5}$$

In order to solve (4.4) it is desirable to write the kernel as far as possible as products of separate functions and ϵ. The form of (4.5) which we find most convenient leads to

$$\frac{2(r+s)^{-1/2}(r_M + s_M)^{3/2}}{3r - s - 2v} k(\rho, \epsilon) = \frac{1}{\rho}\left(1 - \frac{3}{4}\epsilon + \frac{15}{32}\epsilon^2 - \frac{67}{256}\epsilon^3\right) \times \left(1 + \frac{9}{8}\rho + \frac{21}{64}\rho^2 - \frac{13}{128}\rho^3\right) + \frac{9\rho^2}{128}\epsilon + 0(\epsilon_M{}^4).$$

If we now define

$$M(\epsilon) = \frac{(1 + s_M)^{3/2}\, T(\epsilon)}{1 - \frac{3}{4}\epsilon + \frac{15}{32}\epsilon^2 - \frac{67}{256}\epsilon^3}, \tag{4.6}$$

then

$$M(\epsilon) = 1 - \int_{\epsilon_M}^{\epsilon} \frac{M(\rho)}{\rho}\left\{\left(1 + \frac{3}{8}\rho - \frac{3}{64}\rho^2 - \frac{21}{256}\rho^3\right) + \frac{9\rho^2}{128}\epsilon\right\} d\rho$$

to the given order of approximation. By a double differentiation we find that this is equivalent to the ordinary differential equation

$$\frac{d^2M}{d\epsilon^2} + \frac{1}{\epsilon}\left(1 + \frac{3}{8}\epsilon - \frac{3}{64}\epsilon^2 - \frac{3}{256}\epsilon^3\right)\frac{dM}{d\epsilon} - \frac{1}{\epsilon^2}\left(1 + \frac{3}{64}\epsilon^2 - \frac{15}{128}\epsilon^3\right) M = 0 \tag{4.7}$$

together with the boundary conditions

$$M(\epsilon_M) = 1,$$

$$\left(\frac{dM}{d\epsilon}\right)_{\epsilon=\epsilon_M} = -\frac{1}{\epsilon_M}\left(1 + \frac{3}{8}\epsilon_M - \frac{3}{64}\epsilon_M{}^2 - \frac{3}{256}\epsilon_M{}^3\right).$$

Equation (4.7) has a regular singularity at $\epsilon = 0$. A solution in series is sought by the method of Frobenius. The indicial equation gives exponents ± 1 at the origin, but although these differ by an integer, the resulting series solutions can in fact be generated without recourse to logarithmic singularities. When the solution is carried out to the appropriate degree of approximation and the boundary conditions fitted in we obtain

$$M(\epsilon) = \frac{\epsilon_M}{\epsilon}\left[1 + \frac{3}{8}(\epsilon_M - \epsilon) + \frac{3}{64}(\epsilon_M{}^2 - 3\epsilon_M + 2\epsilon^2) - \frac{1}{1024}(37\epsilon_M{}^3 + 18\epsilon_M{}^2\epsilon - 135\epsilon_M\epsilon^2 + 80\epsilon^3) + \cdots\right],$$

and this gives, from (4.6) and (4.3)

$$t(\epsilon) = \frac{\epsilon^2{}_M t_M}{\epsilon^2}\left[1 + \frac{5}{12}(\epsilon_M - \epsilon) + \frac{1}{288}(9\epsilon_M{}^2 - 50\epsilon_M\epsilon + 41\epsilon^2) - \frac{1}{27648}(999\epsilon_M{}^3 + 360\epsilon_M{}^2\epsilon - 4313\epsilon_M\epsilon^2 + 2954\epsilon^3) + \cdots\right]. \tag{4.8}$$

As in the previous section, we could find $x(\epsilon)$ if required by using (2.7). However we do not give it explicitly here.

Examination of (4.8) shows that it has the anticipated behavior near $\epsilon = 0$. As we shall see later it is also accurate near $\epsilon = \epsilon_M$ during the early stages of decay. The corresponding result obtained by Friedrichs with the simple wave approximation agrees with (4.8) up to the linear term in the square bracket, but the quadratic term in ϵ and ϵ_M is different. Inspection of Friedrichs' result shows that in deriving the quadratic terms he has already neglected terms of comparable order. Thus we confirm the statement made earlier that the present approach leads to an approximation two degrees better than that of the simple wave theory, although the initial assumption only involved retaining one extra power of ϵ_M. The method of solution, however, now takes into account the cumulative effects of the error introduced by the approximation.

While it is believed that this will give a good assumption for the decay of shocks of moderate strength, it cannot be strictly applied for very large values of t, since, by stopping the piston, we must eventually cause secondary shocks which will interact with the main shock. These will arise because the expansion wave sent back from the shock is reflected back from the piston as a compression wave which ultimately leads to shock formation. The above theory is, however, applicable strictly to the physical situation when the piston is removed altogether at 0 or at any rate suddenly withdrawn at sufficient speed to leave a vacuum bounding the water column on the left.

It is then easy to verify that the motion is nowhere compressive and thus no secondary shocks can form. The boundary value problem of Fig. 4 is then exact for the determination of the flow behind the shock in this case.

5. The Focusing Equations

We shall conclude by discussing the initial stages of decay of shocks which which are not limited in strength in any way. It would be possible to obtain from the integral equation (4.2) a power series representation of the form

$$t = \sum a_n(\epsilon_M - \epsilon)^n.$$

The method would be essentially similar to that used in the shock formation problem (Section 3). It would be useful for describing the early stages of decay but could not be used for large values of t, since the series would become divergent when $\epsilon \to 0$.

Rather than use this method, however, we shall obtain t in the early stages of decay by a method based on the focusing equations introduced by Meyer [4]. This method, too, cannot be used for large values of t, but it has two advantages over a direct expansion in the integral equation. First it does not use the Riemann function at all and hence avoids complicated expansions of hypergeometric series. Secondly it can be used to describe the entire flow behind the shock (for values of t sufficiently close to t_M), whereas the integral equation has been formulated in such a manner as to give the shock path only.

The focusing, equations are derived from (2.5). If we define

$$U = \left(\frac{r+s}{r_M+s_M}\right)^{3/2} \frac{\partial t}{\partial r}, \qquad V = \left(\frac{r+s}{r_M+s_M}\right)^{3/2} \frac{\partial t}{\partial s},$$

then

$$\frac{\partial U}{\partial s} + \frac{3V}{2(r+s)} = 0, \qquad \frac{\partial V}{\partial r} + \frac{3U}{2(r+s)} = 0. \tag{5.1}$$

For the shock decay problem we know from (4.1) the value of U on MY_1, that is, on $s = s_M$. Further we have the shock relation (2.3) which holds on MN. Specifically these relations are

$$U = -\frac{3t_M}{2(r+s_M)} \qquad \text{on} \qquad s = s_M, \tag{5.2}$$

$$V = G(r, s)\, U \qquad \text{on} \qquad P(r, s) = 0. \tag{5.3}$$

We seek to obtain the solution in some neighborhood of MY_1 (Fig. 4) for general values of r. To this end we look for a solution which is an expansion in powers of $s - s_M$. Formally we write

$$U(r, s) = \sum_k U_k(r)\,(s - s_M)^k,$$

$$V(r, s) = \sum_k V_k(r)\,(s - s_M)^k,$$

where U_0 is given by (5.2). When these are substituted in (5.1) and coefficients of $(s - s_M)^k$ equated, then

$$(k + 1)\,(r + s_M)\,U_{k+1} + kU_k = -\tfrac{3}{2}\,V_k\,, \qquad (k \geqslant 0) \qquad (5.4)$$

$$(r + s_M)\,V_k' + V_{k-1}' = -\tfrac{3}{2}\,U_k\,, \qquad (k \geqslant 1) \qquad (5.5)$$

with

$$(r + s_M)\,V_0' = -\tfrac{3}{2}\,U_0\,. \qquad (5.6)$$

We already know U_0. Then (5.6) gives V_0' from which we find V_0. We then obtain U_1 from (5.4) and this gives V_1' from (5.5). Thus the two sequences of U and V functions are built up. At any stage U_k is obtained by a purely algebraic relation, but V_k has to be found as the solution of a first order ordinary differential equation. The boundary condition which fixes V_k is determined by equation (5.3). In practice we expand (5.3) about the point M, and this leads to the system

$$V_k(r_M) = \sum_{i=0}^{k} U_i(r_M)\,G_{k-i}\,, \qquad (5.7)$$

where

$$G(r, s) = \sum_{j=0}^{\infty} G_j(s - s_M)^j \qquad \text{on} \qquad P(r, s) = 0.$$

Since, for a given value of k, we obtain U_k before V_k, this means that (5.7) gives us $V_k(r_M)$, which is the required boundary conditions enabling us to solve equation (5.5) for V_k. The functions V_0, U_1 are given by

$$V_0(r) = -\frac{3t_M}{2(r_M + s_M)}\left\{G_0 + \frac{3(r_M - r)}{2(s_M + r)}\right\},$$

$$U_1(r) = \frac{9t_M}{4(r + s_M)\,(r_M + s_M)}\left\{G_0 + \frac{3(r_M - r)}{2(s_M + r)}\right\},$$

where $G_0 = G(r_M, s_M)$. Other U and V functions are easily generated. The expansions then give U and V at any point suitably near to $s = s_M$, and hence x and t can be obtained.

In order to find the equation of the shock wave in the x, t-plane we use the equation

$$dt = \frac{\partial t}{\partial r}\, dr + \frac{\partial t}{\partial s}\, ds = \left(\frac{r_M + s_M}{r + s}\right)^{3/2} [U(r, s)\, dr + V(r, s)\, ds].$$

On the shock, from (5.3) and (2.4) this gives

$$\frac{dt}{dr} = \frac{(r_M + s_M)^{3/2}\, U(r, s)}{(r + s)^{1/2}\, (\frac{3}{2}\, r - \frac{1}{2}\, s - v)}, \tag{5.8}$$

which enables t to be found in terms of the parameter r on the shock once we have found $U(r, s)$ on the shock. The intial rate of decay of the shock is determined from the value of dt/dr at the point M. We have

$$\left(\frac{dt}{dr}\right)_M = -\frac{3}{2}\, t_M \left(\frac{3}{2}\, r_M - \frac{1}{2}\, s_M - v\right)^{-1}. \tag{5.9}$$

For moderately small values of the original parameter ϵ_M this can be expanded to give

$$\left(\frac{dt}{d\epsilon}\right)_{\epsilon=\epsilon_M} = -\frac{2t_M}{\epsilon_M}\left(1 + \frac{5}{24}\,\epsilon_M - \frac{1}{8}\,\epsilon_M{}^2 + \frac{149}{13824}\,\epsilon_M{}^3 + \frac{2839}{331776}\,\epsilon_M{}^4 + \cdots\right).$$

The expansion up to the term in $\epsilon_M{}^3$ agrees with that obtained by differentiating (4.8), the value of $t(\epsilon)$ obtained over the whole MN range by the integral equations method of the previous section. The error term is thus of the order $10^{-2}\, \epsilon_m{}^4$, and it is not unreasonable to suppose that Eq. (4.8) will give an adequate description of the whole history of the shock whenever this expression can be neglected.

The initial stages of decay can be found from (5.8) to any required degree of approximation. Table I shows the values of $(dt/dr)_M$ as calculated by three different formulas for a range of values of $c_M{}^2$, the square of the sound velocity behind the shock immediately before the interaction. The three cases are the exact value as given by (5.9), the values as given by (4.8) from the integral equation method and the value as given by Friedrichs' simple wave theory.[2] As expected the integral equation method is more accurate than that of the simple wave theory, but both approximations are remarkably good except when the shock wave is initially very strong.

[2] The authors accept the responsibility for the accuracy of the results derived from the simple wave theory; as they are not quoted by Friedrichs (1948).

TABLE I

c_M^2	(dt/dr) Exact	(dt/dr) Integral equation	(dt/dr) Simple wave
1.2	10.841	10.905	10.947
1.5	4.804	4.808	4.867
1.8	3.263	3.266	3.342
2.5	2.015	2.027	2.138
3.0	1.638	1.661	1.783

References

1. K. O. Friedrichs. Formation and decay of shock waves. *Comm. Pure Appl. Math.* **1** (1948), 211.
2. M. J. Lighthill. The energy distribution behind decaying shocks. *Phil. Mag.* **41** (1950), 1101.
3. A. F. Pillow. The formation and growth of shock waves in the one-dimensional motion of a gas. *Proc. Cambridge Phil. Soc.* **45** (1949), 558.
4. J. J. Mahony and R. E. Meyer. Analytical treatment of two-dimensional supersonic flow, *Phil. Trans. Roy. Soc. London* **248** (1956), 467.