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On maximal injective subalgebras of tensor products of von Neumann algebras

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Abstract

Let $\mathcal{M}_i$ be a von Neumann algebra, and $\mathfrak{B}_i$ be a maximal injective von Neumann subalgebra of $\mathcal{M}_i$, $i = 1, 2$. If $\mathcal{M}_1$ has separable predual and the center of $\mathfrak{B}_1$ is atomic, e.g., $\mathfrak{B}_1$ is a factor, then $\mathfrak{B}_1 \bar{\otimes} \mathfrak{B}_2$ is a maximal injective von Neumann subalgebra of $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$. This partly answers a question of Popa.

Keywords: von Neumann algebras; Maximal injective von Neumann subalgebras; Maximal injective subfactors; Tensor products

0. Introduction

F.J. Murray and J.von Neumann [9–11,19,20] introduced and studied certain algebras of Hilbert space operators. Those algebras are now called von Neumann algebras. They are strong-operator closed self-adjoint subalgebras of all bounded linear transformations on a Hilbert space. *Factors* are von Neumann algebras whose centers consist of scalar multiples of the identity. Every von Neumann algebra is a direct sum (or "direct integral") of factors. Thus factors are the building blocks for all von Neumann algebras.

Murray and von Neumann [9] classified factors by means of a relative dimension function. *Finite factors* are those for which this dimension function has range the closed interval $[0, c]$ for some positive c. For finite factors, this dimension function gives rise to a (unique, when normalized) tracial state. In general, a von Neumann algebra admitting a faithful normal trace is

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said to be *finite*. Finite-dimensional "finite factors" are full matrix algebras $M_n(\mathbb{C})$, $n = 1, 2, \ldots$. Infinite-dimensional "finite factors" are called factors of type II_1. *Infinite factors* are those for which the range of the dimension function includes ∞.

In [11], Murray and von Neumann introduced and studied a family of factors of type II_1 very closely related to matrix algebras. Murray and von Neumann called these factors *approximately finite* since they are the ultraweak closure of the ascending union of a family of finite-dimensional self-adjoint subalgebras. They proved that all "approximately finite" factors of type II_1 are $*$-isomorphic. Since these factors are finite, J. Dixmier [2] considered the term "approximately finite" inappropriate and called them *hyperfinite*. However, for infinite factors possessing the same property, the term "hyperfinite" is also inappropriate. So later on the name *approximately finite-dimensional* (AFD) were introduced for these factors.

A von Neumann algebra $\mathfrak{B}$ acting on a Hilbert space $\mathcal{H}$ is called *injective* if there is a norm one projection from $\mathcal{B}(\mathcal{H})$, the algebra of all bounded linear operators on $\mathcal{H}$, onto $\mathfrak{B}$. Since the intersection of a decreasing sequence of injective algebras is injective, and the commutant of an injective algebra is injective, every AFD factor is injective. In [1], A. Connes proved that a separable injective von Neumann algebra (von Neumann algebra with separable predual) is approximately finite-dimensional. As a corollary, this shows that the hyperfinite type II_1 factor $\mathcal{R}$ is the unique separable injective factor of type II_1. The proof of Connes' result is so deep and rich in ideas and techniques that it remains a basic resource in the subject.

Compare with injective factors, non-injective factors (even non-injective type II_1 factors) are far from being understood. A standard method of investigation in the study of general factors is to study the injective von Neumann subalgebras of these factors. Along this line, we have R. Kadison's question [7, Problem 7]: Does each self-adjoint operator in a II_1 factor lie in some hyperfinite subfactor? Since every separable abelian von Neumann algebra is generated by a single self-adjoint operator, Kadison's question has an equivalent form: Is each separable abelian von Neumann algebra of a II_1 factor contained in some hyperfinite subfactor?

Let $\mathcal{M}$ be a type II_1 factor with a faithful normal trace τ. If $T = T^* = \sum_{k=1}^{n} \lambda_k E_k$ is a self-adjoint operator in $\mathcal{M}$ such that $\sum_{k=1}^{n} E_k = I$ and $\tau(E_k) = 1/n$ for $1 \leqslant k \leqslant n$, then T is in a type I_n subfactor $\mathcal{M}_n$ of $\mathcal{M}$ which has $E_1, E_2, \ldots, E_n$ as diagonals. Since the set $\mathcal{S} = \{T\colon T = T^* = \sum_{k=1}^{n} \lambda_k E_k$ such that $\sum_{k=1}^{n} E_k = I$ and $\tau(E_1) = \cdots = \tau(E_n) = \frac{1}{n}$, $n = 1, 2, \ldots\}$ is dense in the set of self-adjoint operators in $\mathcal{M}$ relative to the strong-operator topology, for each self-adjoint operator T in $\mathcal{M}$ we can choose a sequence $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{S}$ such that T_n converges to T in the strong operator topology. So one may expect that the answer to Kadison's question could be affirmative if one very carefully constructs T_n and $\mathcal{M}_n$ for each n.

Out of expectation, this problem was answered in the negative in a remarkable paper [13] by S. Popa. In [13], Popa showed that if $\mathcal{L}(\mathbb{F}_n)$ is the type II_1 factor associated with the left regular representation λ of the free group $\mathbb{F}_n$ on n generators, $2 \leqslant n \leqslant \infty$, and a is one of the generators of $\mathbb{F}_n$, then the abelian von Neumann subalgebra generated by the unitary $\lambda(a)$ is a maximal injective von Neumann subalgebra of $\mathcal{L}(\mathbb{F}_n)$. So quite surprisingly, a diffuse abelian von Neumann algebra can be embedded in a type II_1 factor as a maximal injective von Neumann subalgebra!

By considering actions of free groups on non-atomic probability spaces, Popa constructed more examples of maximal injective von Neumann subalgebras in factors of type II_1. In [4], L. Ge showed that every non-atomic injective von Neumann algebra with separable predual is maximal injective in its free product with any von Neumann algebra associated with a countable discrete group. Popa raised the following question in [13]: If $\mathcal{M}_1, \mathcal{M}_2$ are type II_1 factors and $\mathfrak{B}_1 \subseteq \mathcal{M}_1, \mathfrak{B}_2 \subseteq \mathcal{M}_2$ are maximal injective von Neumann subalgebras, is $\mathfrak{B}_1 \bar{\otimes} \mathfrak{B}_2$ maximal injective in $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$? He also asked if this is true when we assume that $\mathfrak{B}_1 = \mathcal{M}_1$ is the hyperfinite II_1 factor. This question, if answered in the affirmative, would considerably enlarge our class of examples.

The first break of Popa's question was obtained by Ge and Kadison [4,5]. By applying their remarkable "splitting theorem," Ge and Kadison answered the second part of Popa's question affirmatively. Precisely, Ge and Kadison proved that if $\mathcal{M}_1$ is an injective factor, and $\mathfrak{B}_2$ is a maximal injective von Neumann subalgebra of $\mathcal{M}_2$, then $\mathcal{M}_1 \bar{\otimes} \mathfrak{B}_2$ is maximal injective in $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$. In [16], S. Strătilă and L. Zsidó improved Ge–Kadison's result by removing the factor condition of $\mathcal{M}_1$: if $\mathcal{M}_1$ is an injective von Neumann algebra and $\mathcal{M}_2$ is a von Neumann algebra with separable predual, and $\mathfrak{B}_2$ is a maximal injective von Neumann subalgebra of $\mathcal{M}_2$, then $\mathcal{M}_1 \bar{\otimes} \mathfrak{B}_2$ is maximal injective in $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$.

In this paper we answer Popa's question affirmatively in a more general setting. Our main result is the following. Let $\mathcal{M}_i$ be a von Neumann algebra, and $\mathfrak{B}_i$ be a maximal injective von Neumann subalgebra of $\mathcal{M}_i$, $i = 1, 2$. If $\mathcal{M}_1$ has separable predual and the center of $\mathfrak{B}_1$ is atomic, e.g., $\mathfrak{B}_1$ is a factor, then $\mathfrak{B}_1 \bar{\otimes} \mathfrak{B}_2$ is maximal injective in $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$.

The paper is divided into six sections. Section 1 contains some preliminaries and one key lemma. Another key lemma is proved in Section 2. Some direct applications are also given. In Section 3, we prove our main result in the special case when $\mathfrak{B}_1$ is a factor. The main result is proved in Section 4. In Section 5, we consider the question: If $\mathcal{M}_1, \mathcal{M}_2$ are factors, and $\mathcal{R}_1, \mathcal{R}_2$ are maximal injective subfactors of $\mathcal{M}_1, \mathcal{M}_2$, respectively, is $\mathcal{R}_1 \bar{\otimes} \mathcal{R}_2$ a maximal injective subfactor of $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$? We prove the following result. Let $\mathcal{M}_1, \mathcal{M}_2$ be factors, and $\mathcal{R}_1, \mathcal{R}_2$ be maximal injective subfactors of $\mathcal{M}_1, \mathcal{M}_2$, respectively. If $\mathcal{R}_1' \cap \mathcal{M}_1 \simeq \mathbb{C}^N$ $(1 \leqslant N \leqslant \infty)$ and $\mathcal{R}_2' \cap \mathcal{M}_2 = \mathbb{C}I$, then $\mathcal{R}_1 \bar{\otimes} \mathcal{R}_2$ is a maximal injective subfactor of $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$. In the last section, we mention some questions related to Popa's question.

For the general theory of von Neumann algebras, we refer to [2,8,15].

1. Preliminaries

There are five topics in this section: injective von Neumann algebras, maximal injective von Neumann subalgebras, minimal injective von Neumann algebra extensions, maximal injective subfactors, and two basic theorems on tensor products of von Neumann algebras: Ge–Kadison's splitting theorem and Tomiyama's slice mapping theorem. Lemma 1.2 is one of two key lemmas.

1.1. Injective von Neumann algebras

A *conditional expectation* $\mathbf{E}$ from a von Neumann algebra $\mathcal{M}$ onto a von Neumann subalgebra $\mathcal{N}$ is a positive, linear mapping such that $\mathbf{E}(S_1 T S_2) = S_1 \mathbf{E}(T) S_2$ for all S_1, S_2 in $\mathcal{N}$ and all T in $\mathcal{M}$. J. Tomiyama [17] showed that an idempotent of norm 1 from $\mathcal{M}$ onto $\mathcal{N}$ is a conditional expectation. A von Neumann algebra $\mathfrak{B}$ acting on a Hilbert space $\mathcal{H}$ is called injective if there is a conditional expectation from $\mathcal{B}(\mathcal{H})$ onto $\mathfrak{B}$. If $\mathfrak{B}$ is a von Neumann subalgebra of a von Neumann algebra $\mathcal{M}$ and $\mathfrak{B}$ is injective, there is a conditional expectation from $\mathcal{M}$ onto $\mathfrak{B}$.

Let $\mathfrak{B}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. Then $\mathfrak{B}$ is injective if and only if the commutant $\mathfrak{B}'$ of $\mathfrak{B}$ is injective. Recall that if E is a projection in $\mathfrak{B}$, then the reduced von Neumann algebra of $\mathfrak{B}$ with respect to E is the algebra $\mathfrak{B}_E \triangleq E\mathfrak{B}E$. If $\mathfrak{B}$ is injective and $\mathbf{E}$ is a conditional expectation from $\mathcal{B}(\mathcal{H})$ onto $\mathfrak{B}$, then $\mathbf{E}$ induces a conditional expectation $\mathbf{E}_E$ from $\mathcal{B}(E\mathcal{H})$ onto $\mathfrak{B}_E$ by $\mathbf{E}_E(T) = \mathbf{E}(ETE)$ for any $T \in \mathcal{B}(E\mathcal{H})$. Thus $\mathfrak{B}_E$ is an injective von Neumann algebra.

1.2. Maximal injective von Neumann subalgebras

Let $\mathcal{M}$ be a von Neumann algebra. A von Neumann subalgebra $\mathfrak{B}$ of $\mathcal{M}$ is called *maximal injective* if it is injective and if it is maximal with respect to inclusion in the set of all injective von Neumann subalgebras of $\mathcal{M}$. If $\{\mathfrak{B}_\alpha\}$ is a family of injective von Neumann subalgebras of $\mathcal{M}$ which is inductively ordered by inclusion, then the weak operator closure of $\bigcup_\alpha \mathfrak{B}_\alpha$ is an injective von Neumann subalgebra of $\mathcal{M}$ which contains all $\mathfrak{B}_\alpha$. By Zorn's lemma, $\mathcal{M}$ has maximal injective von Neumann subalgebras.

If $\mathcal{M}$ is a separable type II_1 factor, then $\mathcal{M}$ contains a hyperfinite subfactor $\mathcal{R}$ such that $\mathcal{R}' \cap \mathcal{M} = \mathbb{C}I$ [12, Corollary 4.1]. If $\mathfrak{B}$ is a maximal injective von Neumann subalgebra of $\mathcal{M}$ which contains $\mathcal{R}$, then $\mathfrak{B}' \cap \mathcal{M} \subseteq \mathcal{R}' \cap \mathcal{M} = \mathbb{C}I$. In particular, $\mathfrak{B}$ is an injective factor. By [1], $\mathfrak{B}$ is hyperfinite. So every separable type II_1 factor contains a hyperfinite subfactor as a maximal injective von Neumann subalgebra.

In [13], Popa exhibited concrete examples of maximal injective von Neumann subalgebras of type II_1 factors. Popa showed that if $\mathcal{L}(\mathbb{F}_n)$ is the type II_1 factor associated with the left regular representation λ of the free group $\mathbb{F}_n$ on n $(2 \leqslant n \leqslant \infty)$ generators, and if a is one of the generators of $\mathbb{F}_n$, then the abelian von Neumann algebra generated by the unitary $\lambda(a)$ is a maximal injective von Neumann subalgebra of $\mathcal{L}(\mathbb{F}_n)$. In [4], Ge showed that each non-atomic injective von Neumann algebra with separable predual is maximal injective in its free product with any von Neumann algebra associated with a countable discrete group. Note that any maximal injective von Neumann subalgebra of a type II_1 factor must be non-atomic.

If $\mathfrak{B}$ is a maximal injective von Neumann subalgebra of $\mathcal{M}$, then $\mathfrak{B}$ is singular in $\mathcal{M}$, i.e., its normalizers in $\mathcal{M}$ are unitary elements in $\mathfrak{B}$. Indeed, if U is a unitary element in $\mathcal{M}$ and $U\mathfrak{B}U^* = \mathfrak{B}$, then the von Neumann subalgebra of $\mathcal{M}$ generated by $\mathfrak{B}$ and U is also injective. Since $\mathfrak{B}$ is maximal injective in $\mathcal{M}$, $U \in \mathfrak{B}$. In particular, it follows that $\mathfrak{B}' \cap \mathcal{M} \subseteq \mathfrak{B}$. Let $\mathcal{Z}$ be the center of $\mathfrak{B}$. We have $\mathcal{Z} \subseteq \mathfrak{B}' \cap \mathcal{M} \subseteq \mathfrak{B}' \cap \mathfrak{B} = \mathcal{Z}$, which implies that $\mathcal{Z} = \mathfrak{B}' \cap \mathcal{M}$. We summarize these facts in the following lemma.

Lemma 1.1. *Let $\mathfrak{B}$ be a maximal injective von Neumann subalgebra of $\mathcal{M}$. Then $\mathfrak{B}$ is singular in $\mathcal{M}$. In particular, $\mathcal{Z} = \mathfrak{B}' \cap \mathfrak{B} = \mathfrak{B}' \cap \mathcal{M}$.*

1.3. Minimal injective von Neumann algebra extensions

Let $\mathcal{N}$ be a von Neumann algebra. An injective von Neumann algebra $\mathfrak{A}$ is called a *minimal injective von Neumann algebra extension* of $\mathcal{N}$ if $\mathfrak{A} \supseteq \mathcal{N}$ and if it is minimal with respect to inclusion in the set of all injective von Neumann algebras which contain $\mathcal{N}$.

Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and $\mathfrak{B}$ be a maximal injective von Neumann subalgebra of $\mathcal{M}$. Then $\mathfrak{B}'$, the commutant of $\mathfrak{B}$, is a minimal injective von Neumann algebra extension of $\mathcal{M}'$. Indeed, if $\mathcal{L}'$ is an injective von Neumann algebra such that $\mathcal{M}' \subseteq \mathcal{L}' \subseteq \mathfrak{B}'$, then $\mathcal{L} = (\mathcal{L}')'$ is an injective von Neumann algebra such that $\mathfrak{B} \subseteq \mathcal{L} \subseteq \mathcal{M}$. Since $\mathfrak{B}$ is a maximal injective von Neumann subalgebra of $\mathcal{M}$, $\mathfrak{B} = \mathcal{L}$. By von Neumann's double commutant theorem [19], $\mathfrak{B}' = \mathcal{L}'$.

Let $\mathfrak{A}$ be a minimal injective von Neumann algebra extension of a von Neumann algebra $\mathcal{N}$. Let φ be a faithful normal representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$. Then $(\varphi(\mathfrak{A}))'$ is a maximal injective von Neumann algebra of $(\varphi(\mathcal{N}))'$. Indeed, if $\mathcal{L}$ is an injective von Neumann algebra such that $(\varphi(\mathfrak{A}))' \subseteq \mathcal{L}' \subseteq (\varphi(\mathcal{N}))'$, then $\varphi(\mathcal{N}) \subseteq \mathcal{L} \subseteq \varphi(\mathfrak{A})$ and $\mathcal{L}$ is injective. Thus

$\mathcal{N} \subseteq \varphi^{-1}(\mathcal{L}) \subseteq \mathfrak{A}$ and $\varphi^{-1}(\mathcal{L})$ is injective. Since $\mathfrak{A}$ is a minimal injective von Neumann algebra extension of $\mathcal{N}$, $\varphi^{-1}(\mathcal{L}) = \mathfrak{A}$. Hence $\mathcal{L} = \varphi(\mathfrak{A})$ and $\mathcal{L}' = (\varphi(\mathfrak{A}))'$.

In [6], U. Haagerup proved that any von Neumann algebra is *-isomorphic to a von Neumann algebra $\mathcal{M}$ on a Hilbert space $\mathcal{H}$, such that there is a conjugate linear, isometric involution J of $\mathcal{H}$ and a self-dual cone $\mathcal{P}$ in $\mathcal{H}$ with the properties:

1. $J\mathcal{M}J = \mathcal{M}'$.
2. $JZJ = Z^*$ for Z in the center of $\mathcal{M}$.
3. $J\xi = \xi$, $\xi \in \mathcal{P}$.
4. $XJXJ(\mathcal{P}) \subseteq \mathcal{P}$ for all $X \in \mathcal{M}$.

A quadruple $(\mathcal{M}, \mathcal{H}, J, \mathcal{P})$ satisfying the conditions 1–4 is called a *standard form* of the von Neumann algebra $\mathcal{M}$. Recall that a von Neumann algebra $\mathcal{M}$ acting on a Hilbert space $\mathcal{H}$ is said to be *standard* if there exists a conjugation $J : \mathcal{H} \to \mathcal{H}$, such that the mapping $X \to JX^*J$ is a *-anti-isomorphism from $\mathcal{M}$ onto $\mathcal{M}'$. If $\mathcal{M}$ is standard on $\mathcal{H}$, we can choose J and $\mathcal{P}$ in $\mathcal{H}$, such that $(\mathcal{M}, \mathcal{H}, J, \mathcal{P})$ is a standard form (cf. [6, Theorem 1.1]). Let $\mathcal{M}$ be standard on $\mathcal{H}$, and θ be a *-automorphism of $\mathcal{M}$, then there is a unitary operator U on $\mathcal{H}$ such that $\theta(X) = UXU^*$ for all $X \in \mathcal{M}$ (cf. [6, Theorem 3.2]).

The following lemma, which has an independent interest, is a key lemma.

Lemma 1.2. *Let $\mathcal{N}$ be a von Neumann algebra and $\mathfrak{A}$ be a minimal injective von Neumann algebra extension of $\mathcal{N}$. If $\theta \in Aut(\mathfrak{A})$ (the group of all *-automorphisms of $\mathfrak{A}$) satisfies $\theta(X) = X$ for all $X \in \mathcal{N}$. Then $\theta(Y) = Y$ for all $Y \in \mathfrak{A}$.*

Proof. We can assume that $\mathfrak{A}$ is standard on a Hilbert space $\mathcal{H}$. Then there is a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that $\theta(Y) = UYU^*$ for all $Y \in \mathfrak{A}$. Since for all $X \in \mathcal{N}$ we have $\theta(X) = X$, $UXU^* = X$. Thus, $U \in \mathcal{N}'$. Define $\theta'(Y') = UY'U^*$ for $Y' \in \mathfrak{A}'$. Note that for all $Y \in \mathfrak{A}$, $Y' \in \mathfrak{A}'$, $\theta'(Y')\theta(Y) = \theta(Y)\theta'(Y')$. Since $\theta(\mathfrak{A}) = \mathfrak{A}$, $\theta'(Y') \in \mathfrak{A}'$ for all $Y' \in \mathfrak{A}'$, i.e., $U\mathfrak{A}'U^* \subseteq \mathfrak{A}'$. Note that $\theta^{-1}(Y) = U^*YU$ is also a *-isomorphism of $\mathfrak{A}$. Same arguments as above show that $U^*\mathfrak{A}'U \subseteq \mathfrak{A}'$. So $U\mathfrak{A}'U^* = \mathfrak{A}'$. This implies that $U \in \mathcal{N}'$ is in the normalizer of $\mathfrak{A}'$. Note that $\mathfrak{A}'$ is maximal injective in $\mathcal{N}'$. By Lemma 1.1, $U \in \mathfrak{A}'$. So $\theta(Y) = UYU^* = Y$ for all $Y \in \mathfrak{A}$. □

Lemma 1.3. *If $\mathfrak{A}$ is a minimal injective von Neumann algebra extension of a von Neumann algebra $\mathcal{N}$, and P, Q are non-zero central projections in $\mathfrak{A}$ such that $PQ = 0$, then there does not exist a *-isomorphism ϕ from $\mathfrak{A}_P$ onto $\mathfrak{A}_Q$ such that $\phi(PX) = QX$ for all $X \in \mathcal{N}$.*

Proof. Otherwise, assume ϕ is a *-isomorphism from $\mathfrak{A}_P$ onto $\mathfrak{A}_Q$ such that $\phi(PX) = QX$ for all $X \in \mathcal{N}$. For any $Y \in \mathfrak{A}$, $Y = PY + QY + (I - P - Q)Y$. Define θ from $\mathfrak{A}$ to $\mathfrak{A}$ by $\theta(Y) = \phi(PY) + \phi^{-1}(QY) + (I - P - Q)Y$. Since P, Q are mutually orthogonal central projections in $\mathfrak{A}$ and ϕ is a *-isomorphism from $\mathfrak{A}_P$ onto $\mathfrak{A}_Q$, $\theta \in Aut(\mathfrak{A})$. Note that for any $X \in \mathcal{N}$, $\theta(X) = \phi(PX) + \phi^{-1}(QX) + (I - P - Q)X = QX + PX + (I - P - Q)X = X$. Since $\mathfrak{A}$ is a minimal injective von Neumann algebra extension of $\mathcal{N}$, by Lemma 1.2, $\theta(Y) = Y$ for all $Y \in \mathfrak{A}$. Therefore, $P = \theta(P) = \phi(P) = Q$. Now we have $P = PQ = 0$. It contradicts to the assumption that $P \neq 0$. □

Corollary 1.4. *Let $\mathfrak{A}$ be a minimal injective von Neumann algebra extension of a von Neumann algebra $\mathcal{N}$, and P, Q be central projections in $\mathfrak{A}$. If there is a *-isomorphism ϕ from $\mathfrak{A}_P$ onto $\mathfrak{A}_Q$ such that $\phi(PX) = QX$ for all $X \in \mathcal{N}$, then $P = Q$ and $\phi(PY) = PY$ for all $Y \in \mathfrak{A}$.*

Proof. Suppose $P \neq Q$. Let $R = PQ$. Without loss of generality, we can assume that $P_1 = P - R > 0$. Let $Q_1 = \theta(P_1) \leqslant Q$. Then P_1, Q_1 are non-zero central projections in $\mathfrak{A}$ and $P_1 Q_1 = Q_1 P_1 = 0$. Since ϕ is a *-isomorphism from $\mathfrak{A}_P$ onto $\mathfrak{A}_Q$, ϕ induces a *-isomorphism ψ from $\mathfrak{A}_{P_1}$ onto $\mathfrak{A}_{Q_1}$ such that $\psi(P_1 Y) = \phi(P_1 Y)$ for all $Y \in \mathfrak{A}$. Since for any $X \in \mathcal{N}$, $\psi(P_1 X) = \phi(P_1 X) = \phi(P_1)\phi(PX) = Q_1 Q X = Q_1 X$. It contradicts to Lemma 1.3. Thus $P = Q$. Define $\theta(Y) = \phi(PY) + (I - P)Y$, then $\theta \in Aut(\mathfrak{A})$ and $\theta(X) = X$ for any $X \in \mathcal{N}$. Since $\mathfrak{A}$ is a minimal injective von Neumann algebra extension of $\mathcal{N}$, by Lemma 1.2, $\theta(Y) = Y$ for all $Y \in \mathfrak{A}$. Hence, $PY = \theta(PY) = \phi(PY)$ for all $Y \in \mathfrak{A}$. □

1.4. Maximal injective subfactors

In [3], Fuglede and Kadison established the existence of *maximal hyperfinite subfactors* of a type II_1 factor. Since a separable type II_1 factor is injective if and only if it is hyperfinite, a subfactor of a separable type II_1 factor is a maximal injective subfactor if and only if it is a maximal hyperfinite subfactor. So every separable type II_1 factor has maximal injective subfactors.

In [3], Fuglede and Kadison also asked if each maximal hyperfinite subfactor of a II_1 factor has a trivial relative commutant (that is, only the scalars in the factor commute with the subfactor). In [13], Popa answered this question negatively. Indeed, Popa constructed examples of maximal hyperfinite II_1 subfactors with relative commutant isomorphism to $\mathbb{C}^n$ for any $n \geqslant 1$ and hyperfinite II_1 subfactors with noncommutative relative commutant. In [4], Ge constructed a maximal hyperfinite II_1 subfactor of a II_1 factor with a non-injective relative commutant!

The following lemmas show the relation between maximal injective von Neumann subalgebras and maximal injective subfactors.

Lemma 1.5. *If $\mathcal{M}$ is a factor and $\mathcal{R}$ is a maximal injective subfactor of $\mathcal{M}$, then $\mathcal{R}$ is a maximal injective von Neumann subalgebra of $\mathcal{M}$ if and only if $\mathcal{R}$ is irreducible in $\mathcal{M}$, i.e, $\mathcal{R}' \cap \mathcal{M} = \mathbb{C}I$.*

Proof. If $\mathcal{R}$ is a maximal injective von Neumann subalgebra of $\mathcal{M}$, then by Lemma 1.1, the center $\mathcal{Z}$ of $\mathcal{R}$ is $\mathcal{R}' \cap \mathcal{M}$. Since $\mathcal{R}$ is a factor, $\mathcal{R}' \cap \mathcal{M} = \mathbb{C}I$. Conversely, suppose $\mathcal{R}' \cap \mathcal{M} = \mathbb{C}I$. For any injective von Neumann algebra $\mathfrak{B}$ such that $\mathcal{R} \subseteq \mathfrak{B} \subseteq \mathcal{M}$, we have $\mathfrak{B}' \cap \mathfrak{B} \subseteq \mathcal{R}' \cap \mathcal{M} = \mathbb{C}I$. Therefore, $\mathfrak{B}$ is an injective subfactor of $\mathcal{M}$. Since $\mathcal{R}$ is a maximal injective subfactor of $\mathcal{M}$, $\mathcal{R} = \mathfrak{B}$. □

Lemma 1.6. *Let $\mathcal{M}$ be a factor and $\mathfrak{B}$ be a maximal injective von Neumann subalgebra of $\mathcal{M}$. Let $\mathcal{Z}$ be the center of $\mathfrak{B}$. If $\mathcal{Z}$ is atomic and $P_1, P_2, \ldots,$ are minimal projections in $\mathcal{Z}$ such that $\sum P_i = I$. Then $\mathfrak{B}_i = \mathfrak{B}_{P_i}$ is a maximal injective subfactor of $\mathcal{M}_{P_i}$ such that $\mathfrak{B}_i' \cap \mathcal{M}_{P_i} = \mathbb{C}P_i$ for all i.*

Proof. Since $\mathcal{Z}$ is atomic and $P_1, P_2, \ldots,$ are minimal projections in $\mathcal{Z}$, $\mathfrak{B}_i$ is a subfactor of $\mathcal{M}_{P_i}$. Since $\mathfrak{B}$ is injective, $\mathfrak{B}_i$ is injective. If $\mathcal{L}_i$ is an injective von Neumann algebra such that $\mathfrak{B}_i \subseteq \mathcal{L}_i \subseteq \mathcal{M}_{P_i}$, then $\mathfrak{B}_1 \oplus \cdots \oplus \mathcal{L}_i \oplus \cdots$ is an injective von Neumann subalgebra of $\mathcal{M}$ such that $\mathfrak{B} \subseteq \mathfrak{B}_1 \oplus \cdots \oplus \mathcal{L}_i \oplus \cdots \subseteq \mathcal{M}$. Since $\mathfrak{B}$ is a maximal injective von Neumann subalgebra of $\mathcal{M}$, $\mathfrak{B} = \mathfrak{B}_1 \oplus \cdots \oplus \mathcal{L}_i \oplus \cdots$. This implies that $\mathfrak{B}_i = \mathcal{L}_i$. So $\mathfrak{B}_i$ is a maximal injective von Neumann subalgebra of $\mathcal{M}_{P_i}$. Since $\mathfrak{B}_i$ is a factor, $\mathfrak{B}_i$ is irreducible in $\mathcal{M}_{P_i}$ by Lemma 1.5. □

1.5. On tensor products of von Neumann algebras

In [5], Ge and Kadison proved the following basic theorem for tensor products of von Neumann algebras.

Ge–Kadison's Splitting Theorem. *If $\mathcal{M}_1$ is a factor and $\mathcal{M}_2$ is a von Neumann algebra, and $\mathcal{M}$ is a von Neumann subalgebra of $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$ which contains $\mathcal{M}_1 \bar{\otimes} \mathbb{C}I$, then $\mathcal{M} = \mathcal{M}_1 \bar{\otimes} \mathcal{N}_2$ for some $\mathcal{N}_2$, a von Neumann subalgebra of $\mathcal{M}_2$.*

Slice-map technique of Tomiyama [18] plays a key role in the proof of Ge–Kadison's splitting theorem. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be von Neumann algebras. With ρ in $(\mathcal{M}_1)_\#$ (the predual of $\mathcal{M}_1$), σ in $(\mathcal{M}_2)_\#$ and $T \in \mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$, the mapping $\sigma \to (\rho \bar{\otimes} \sigma)(T)$ is a bounded linear functional on $(\mathcal{M}_2)_\#$, hence, an element $\Psi_\rho(T)$ in $\mathcal{M}_2$. Symmetrically, we construct an operator $\Phi_\sigma(T)$ in $\mathcal{M}_1$. The mappings Ψ_ρ and Φ_σ are referred to as *slice mappings* (of $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$ onto $\mathcal{M}_2$ and $\mathcal{M}_1$ corresponding to ρ and σ, respectively). Tomiyama's slice mapping theorem [18] says that if $\mathcal{N}_1$ and $\mathcal{N}_2$ are von Neumann subalgebras of $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, and $T \in \mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$, then $T \in \mathcal{N}_1 \bar{\otimes} \mathcal{N}_2$ if and only if $\Phi_\sigma(T) \in \mathcal{N}_1$ and $\Psi_\rho(T) \in \mathcal{N}_2$ for each $\sigma \in (\mathcal{M}_2)_\#$ and $\rho \in (\mathcal{M}_1)_\#$. For a generalization of Ge–Kadison's splitting theorem, we refer to [16].

The following lemma is well known. For the sake of completeness, we include the proof here.

Lemma 1.7. *If $\mathcal{M}_1, \mathcal{N}_1$ are von Neumann algebras acting on a Hilbert space $\mathcal{H}$, and $\mathcal{M}_2, \mathcal{N}_2$ are von Neumann algebras acting on a Hilbert space $\mathcal{K}$, then $(\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2) \cap (\mathcal{N}_1' \bar{\otimes} \mathcal{N}_2') = (\mathcal{M}_1 \cap \mathcal{N}_1') \bar{\otimes} (\mathcal{M}_2 \cap \mathcal{N}_2')$.*

Proof. It is obvious that $(\mathcal{M}_1 \cap \mathcal{N}_1') \bar{\otimes} (\mathcal{M}_2 \cap \mathcal{N}_2') \subseteq (\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2) \cap (\mathcal{N}_1' \bar{\otimes} \mathcal{N}_2')$. Conversely, if $T \in (\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2) \cap (\mathcal{N}_1' \bar{\otimes} \mathcal{N}_2')$, then $\Phi_\sigma(T) \in \mathcal{M}_1$ and $\Phi_\sigma(T) \in \mathcal{N}_1'$ for any $\sigma \in \mathcal{B}(\mathcal{K})_\#$ by Tomiyama's slice-mapping theorem. Thus $\Phi_\sigma(T) \in \mathcal{M}_1 \cap \mathcal{N}_1'$. Similarly, for any $\rho \in \mathcal{B}(\mathcal{H})_\#$, $\Psi_\rho(T) \in \mathcal{M}_2 \cap \mathcal{N}_2'$. By Tomiyama's slice-mapping theorem, $T \in (\mathcal{M}_1 \cap \mathcal{N}_1') \bar{\otimes} (\mathcal{M}_2 \cap \mathcal{N}_2')$. Therefore, $(\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2) \cap (\mathcal{N}_1' \bar{\otimes} \mathcal{N}_2') = (\mathcal{M}_1 \cap \mathcal{N}_1') \bar{\otimes} (\mathcal{M}_2 \cap \mathcal{N}_2')$. □

2. Induced conditional expectations

Lemma 2.1. *Let $\mathcal{M}$ be a von Neumann algebra and $\mathcal{L}$, $\mathcal{N}$ be von Neumann subalgebras of $\mathcal{M}$ such that $\mathcal{L} \subseteq \mathcal{N}$. If* **E** *is a conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$, then* **E** *induces a conditional expectation from $\mathcal{L}' \cap \mathcal{M}$ onto $\mathcal{L}' \cap \mathcal{N}$.*

Proof. $\forall S \in \mathcal{L}' \cap \mathcal{M}$ and $T \in \mathcal{L}$, $ST = TS$. Apply the conditional expectation **E** to both sides of $ST = TS$ and note that $\mathcal{L} \subseteq \mathcal{N}$. We have $\mathbf{E}(S)T = T\mathbf{E}(S)$. Thus $\mathbf{E}(S) \in \mathcal{L}' \cap \mathcal{N}$. Since $\mathcal{L}' \cap \mathcal{N} \subseteq \mathcal{L}' \cap \mathcal{M}$, **E** is a conditional expectation from $\mathcal{L}' \cap \mathcal{M}$ onto $\mathcal{L}' \cap \mathcal{N}$ when **E** is restricted on $\mathcal{L}' \cap \mathcal{M}$. □

The following is another key lemma.

Lemma 2.2. *Let $\mathfrak{A}_i$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}_i$, and $\mathcal{N}_i$ be a von Neumann subalgebra of $\mathfrak{A}_i$, $i = 1, 2$. Let $\mathfrak{A}$ be a von Neumann algebra such that $\mathcal{N}_1 \bar{\otimes} \mathcal{N}_2 \subseteq$*

$\mathfrak{A} \subseteq \mathfrak{A}_1 \bar{\otimes} \mathfrak{A}_2$. *If there is a conditional expectation* $\mathbf{E}$ *from* $\mathfrak{A}_1 \bar{\otimes} \mathfrak{A}_2$ *onto* $\mathfrak{A}$, *then* $\mathbf{E}$ *induces a conditional expectation from* $(\mathcal{N}_1' \cap \mathfrak{A}_1) \bar{\otimes} \mathfrak{A}_2$ *onto* $((\mathcal{N}_1' \cap \mathfrak{A}_1) \bar{\otimes} \mathfrak{A}_2) \cap \mathfrak{A}$.

Proof. By Lemma 1.7, $(\mathfrak{A}_1 \bar{\otimes} \mathfrak{A}_2) \cap (\mathcal{N}_1 \bar{\otimes} \mathbb{C}I)' = (\mathfrak{A}_1 \bar{\otimes} \mathfrak{A}_2) \cap (\mathcal{N}_1' \bar{\otimes} \mathcal{B}(\mathcal{K})) = (\mathcal{N}_1' \cap \mathfrak{A}_1) \bar{\otimes} \mathfrak{A}_2$. By Lemma 2.1, $\mathbf{E}$ induces a conditional expectation from $(\mathcal{N}_1' \cap \mathfrak{A}_1) \bar{\otimes} \mathfrak{A}_2$ onto $((\mathcal{N}_1' \cap \mathfrak{A}_1) \bar{\otimes} \mathfrak{A}_2) \cap \mathfrak{A}$. □

Corollary 2.3. *Assume the conditions of Lemma* 2.2 *and* $\mathcal{N}_1' \cap \mathfrak{A}_1 = \mathbb{C}I$. *Let* $\mathcal{L}_2 = \{T \in \mathfrak{A}_2\colon I \otimes T \in \mathfrak{A}\}$. *Then* $\mathbf{E}$ *induces a conditional expectation from* $\mathfrak{A}_2$ *onto* $\mathcal{L}_2$.

As an application of Lemma 2.2 and Corollary 2.3, we give a new proof of Ge–Kadison's splitting theorem in the case when $\mathcal{M}_1$ and $\mathcal{M}_2$ are finite. Let $\mathcal{N}$ be a von Neumann algebra such that $\mathcal{M}_1 \bar{\otimes} \mathbb{C}I \subseteq \mathcal{N} \subseteq \mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$. Then there is a *normal* conditional expectation $\mathbf{E}$ from $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$ onto $\mathcal{N}$. By Corollary 2.3, $\mathbf{E}$ induces a conditional expectation, denoted by $\mathbf{E}_2$, from $\mathcal{M}_2$ onto $\mathcal{N}_2 \triangleq \{T \in \mathcal{M}_2\colon I \otimes T \in \mathcal{N}\}$. Now for any $S \in \mathcal{M}_1$, $T \in \mathcal{M}_2$, we have $\mathbf{E}(S \otimes T) = S \otimes \mathbf{E}_2(T) \in \mathcal{M}_1 \bar{\otimes} \mathcal{N}_2$. Since $\mathbf{E}$ is normal, $\mathcal{N} = \mathbf{E}(\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2) \subseteq \mathcal{M}_1 \bar{\otimes} \mathcal{N}_2$. Since $\mathcal{N} \supseteq \mathcal{M}_1 \bar{\otimes} \mathcal{N}_2$, $\mathcal{N} = \mathcal{M}_1 \bar{\otimes} \mathcal{N}_2$.

As another application of Lemma 2.2 and Corollary 2.3, we give a new proof of [16, Theorem 6.7].

Lemma 2.4. *Let* $\mathcal{A}$ *be an abelian von Neumann algebra, and* $\mathfrak{A}_2$ *be a minimal injective von Neumann algebra extension of a von Neumann algebra* $\mathcal{N}$. *Suppose* $\mathfrak{A}_2$ *has separable predual. If* $\mathfrak{A}$ *is an injective von Neumann algebra such that* $\mathcal{A} \bar{\otimes} \mathcal{N} \subseteq \mathfrak{A} \subseteq \mathcal{A} \bar{\otimes} \mathfrak{A}_2$, *then* $\mathfrak{A} = \mathcal{A} \bar{\otimes} \mathfrak{A}_2$.

Proof. We can assume that $\mathcal{A}$ and $\mathfrak{A}_2$ are von Neumann algebras acting on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively, in standard form. Then $\mathcal{A}' = \mathcal{A}$ and $\mathcal{K}$ is a separable Hilbert space. By $\mathcal{A} \bar{\otimes} \mathcal{N} \subseteq \mathfrak{A} \subseteq \mathcal{A} \bar{\otimes} \mathfrak{A}_2$, we have $\mathcal{A} \bar{\otimes} \mathfrak{A}_2' \subseteq \mathfrak{A}' \subseteq \mathcal{A} \bar{\otimes} \mathcal{N}'$. Note that $\mathfrak{A}_2'$ is a maximal injective von Neumann subalgebra of $\mathcal{N}'$. By [16, Lemma 6.6], $\mathfrak{A}' = \mathcal{A} \bar{\otimes} \mathcal{N}'$. Therefore, $\mathfrak{A} = \mathcal{A} \bar{\otimes} \mathfrak{A}_2$. □

Lemma 2.4 is almost obvious in the case when $\mathcal{A}$ is atomic. If $\mathcal{A}$ is diffuse, it is natural to consider direct integrals. The proof of [16, Lemma 6.6] is based on direct integrals. It would be interesting if there is a "global proof" of Lemma 2.4. Is Lemma 2.4 true without the assumption that $\mathfrak{A}_2$ has separable predual?

Theorem 2.5. *Let* $\mathcal{M}_1$ *be an injective von Neumann algebra and* $\mathcal{M}_2$ *be a von Neumann algebra with separable predual. If* $\mathfrak{B}_2$ *is a maximal injective von Neumann subalgebra of* $\mathcal{M}_2$, *then* $\mathcal{M}_1 \bar{\otimes} \mathfrak{B}_2$ *is a maximal injective von Neumann subalgebra of* $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$.

Proof. We can assume that $\mathcal{M}_1$ and $\mathcal{M}_2$ are von Neumann algebras acting on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Then $\mathcal{K}$ is a separable Hilbert space. Let $\mathcal{A}$ be the center of $\mathcal{M}_1$. Suppose $\mathfrak{B}$ is an injective von Neumann algebra such that $\mathcal{M}_1 \bar{\otimes} \mathfrak{B}_2 \subseteq \mathfrak{B} \subseteq \mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$. Then we have $\mathcal{M}_1' \bar{\otimes} \mathfrak{B}_2' \supseteq \mathfrak{B}' \supseteq \mathcal{M}_1' \bar{\otimes} \mathcal{M}_2'$. Since $\mathfrak{B}'$ is an injective von Neumann subalgebra of $\mathcal{M}_1' \bar{\otimes} \mathfrak{B}_2'$, there is a conditional expectation $\mathbf{E}$ from $\mathcal{M}_1' \bar{\otimes} \mathfrak{B}_2'$ onto $\mathfrak{B}'$. By Lemma 2.2, $\mathbf{E}$ induces a conditional expectation from $\mathcal{A} \bar{\otimes} \mathfrak{B}_2'$ onto $\mathfrak{A} \triangleq (\mathcal{A} \bar{\otimes} \mathfrak{B}_2') \cap \mathfrak{B}'$. So $\mathfrak{A}$ is an injective von Neumann algebra such that $\mathcal{A} \bar{\otimes} \mathfrak{B}_2' \supseteq \mathfrak{A} \supseteq \mathcal{A} \bar{\otimes} \mathcal{M}_2'$. Since $\mathfrak{B}_2$ is a maximal injective von Neumann subalgebra of $\mathcal{M}_2$, $\mathfrak{B}_2'$ is a minimal injective von Neumann algebra extension of $\mathcal{M}_2'$. By Lemma 2.4, $\mathfrak{A} = \mathcal{A} \bar{\otimes} \mathfrak{B}_2'$. Thus $\mathbb{C}I \bar{\otimes} \mathfrak{B}_2' \subseteq \mathfrak{A} \subseteq \mathfrak{B}'$. So $\mathfrak{B}' = \mathcal{M}_1' \bar{\otimes} \mathfrak{B}_2'$ and $\mathfrak{B} = \mathcal{M}_1 \bar{\otimes} \mathfrak{B}_2$. □

3. Popa's question in the case when $\mathfrak{B}_1$ is a factor

Theorem 3.1. *Let $\mathcal{M}_i$ be a von Neumann algebra, and $\mathfrak{B}_i$ be a maximal injective von Neumann subalgebra of $\mathcal{M}_i$, $i = 1, 2$. If $\mathcal{M}_1$ has separable predual and $\mathfrak{B}_1$ is a factor, then $\mathfrak{B}_1 \bar{\otimes} \mathfrak{B}_2$ is a maximal injective von Neumann subalgebra of $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$.*

Proof. Let $\mathfrak{B}$ be an injective von Neumann algebra such that $\mathfrak{B}_1 \bar{\otimes} \mathfrak{B}_2 \subseteq \mathfrak{B} \subseteq \mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$. To prove the theorem, we need to show that $\mathfrak{B} = \mathfrak{B}_1 \bar{\otimes} \mathfrak{B}_2$. We can assume that $\mathcal{M}_1$ and $\mathcal{M}_2$ are von Neumann algebras acting on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. So we have $\mathfrak{B}_1' \bar{\otimes} \mathfrak{B}_2' \supseteq \mathfrak{B}' \supseteq \mathcal{M}_1' \bar{\otimes} \mathcal{M}_2'$. Since $\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}$ are injective, $\mathfrak{B}_1', \mathfrak{B}_2', \mathfrak{B}'$ are injective. Since $\mathfrak{B}_i$ is a maximal injective von Neumann algebra of $\mathcal{M}_i$, $\mathfrak{B}_i'$ is a minimal injective von Neumann algebra extension of $\mathcal{M}_i'$, $i = 1, 2$.

Since $\mathfrak{B}'$ is an injective von Neumann subalgebra of $\mathfrak{B}_1' \bar{\otimes} \mathfrak{B}_2'$, there is a conditional expectation $\mathbf{E}$ from $\mathfrak{B}_1' \bar{\otimes} \mathfrak{B}_2'$ onto $\mathfrak{B}'$. Let $\mathcal{L}_2 = \{T \in \mathfrak{B}_2' \colon 1 \otimes T \in \mathfrak{B}'\}$. Then $\mathcal{L}_2 \subseteq \mathfrak{B}_2'$. By Lemma 1.1, $(\mathcal{M}_1')' \cap \mathfrak{B}_1' = \mathfrak{B}_1' \cap \mathcal{M}_1 = \mathfrak{B}_1' \cap \mathfrak{B}_1 = \mathbb{C}I$. By Corollary 2.3, $\mathbf{E}$ induces a conditional expectation from $\mathfrak{B}_2'$ onto $\mathcal{L}_2$. Thus $\mathcal{L}_2$ is injective. Since $\mathcal{M}_2' \subseteq \mathcal{L}_2 \subseteq \mathfrak{B}_2'$ and $\mathfrak{B}_2'$ is a minimal injective von Neumann algebra extension of $\mathcal{M}_2'$, $\mathcal{L}_2 = \mathfrak{B}_2'$. So $\mathbb{C}I \bar{\otimes} \mathfrak{B}_2' \subseteq \mathfrak{B}'$. This implies that $\mathfrak{B}_1' \bar{\otimes} \mathfrak{B}_2' \supseteq \mathfrak{B}' \supseteq \mathcal{M}_1' \bar{\otimes} \mathfrak{B}_2'$ and hence $\mathfrak{B}_1 \bar{\otimes} \mathfrak{B}_2 \subseteq \mathfrak{B} \subseteq \mathcal{M}_1 \bar{\otimes} \mathfrak{B}_2$. By Theorem 2.5, $\mathfrak{B} = \mathfrak{B}_1 \bar{\otimes} \mathfrak{B}_2$. □

4. Popa's question in the case when the center of $\mathfrak{B}_1$ is atomic

The following is the main result of this paper.

Theorem 4.1. *Let $\mathcal{M}_i$ be a von Neumann algebra, and $\mathfrak{B}_i$ be a maximal injective von Neumann subalgebra of $\mathcal{M}_i$, $i = 1, 2$. If $\mathcal{M}_1$ has separable predual and the center of $\mathfrak{B}_1$ is atomic, then $\mathfrak{B}_1 \bar{\otimes} \mathfrak{B}_2$ is maximal injective in $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$.*

Proof. Let $\mathfrak{B}$ be an injective von Neumann algebra such that $\mathfrak{B}_1 \bar{\otimes} \mathfrak{B}_2 \subseteq \mathfrak{B} \subseteq \mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$. We can assume that $\mathcal{M}_1$ and $\mathcal{M}_2$ are von Neumann algebras acting on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. To prove the theorem, we need to show that $\mathfrak{B} = \mathfrak{B}_1 \bar{\otimes} \mathfrak{B}_2$. Using Theorem 2.5, it is sufficient to prove that $\mathfrak{B} \subseteq \mathcal{M}_1 \bar{\otimes} \mathfrak{B}_2$ and so, it is sufficient to prove that $\mathbb{C}I \bar{\otimes} \mathfrak{B}_2' \subseteq \mathfrak{B}'$.

Denote $\mathcal{A} = \mathfrak{B}_1 \cap \mathfrak{B}_1' = \mathcal{M}_1 \cap \mathfrak{B}_1'$, which is atomic, with the set of minimal projections $\{P_n \colon n = 1, 2, \ldots\}$. Set $\mathfrak{A} = \mathfrak{B}' \cap (\mathcal{A} \bar{\otimes} \mathfrak{B}_2') = \mathfrak{B}' \cap (\mathcal{M}_1' \bar{\otimes} \mathbb{C}I)'$. As in Section 2 of the paper, $\mathfrak{A}$ is injective and

$$\mathbb{C}I \bar{\otimes} \mathcal{M}_2' \subseteq \mathfrak{A} \subseteq \mathcal{A} \bar{\otimes} \mathfrak{B}_2'.$$

Note that $P_n \otimes I \in \mathfrak{A}'$ for every n. Since $\mathfrak{B}_2'$ is a minimal injective von Neumann algebra extension of $\mathcal{M}_2'$, we have $\mathfrak{A}(P_n \otimes I) = P_n \otimes \mathfrak{B}_2'$ for every n. Denote by Z_n the smallest projection in $\mathcal{Z}(\mathfrak{A})$ satisfying $P_n \otimes I \leqslant Z_n$. We get *-isomorphisms $\theta_n : \mathfrak{B}_2' \to \mathfrak{A} Z_n$ uniquely determined by the formula

$$\theta_n(Y)(P_n \otimes I) = P_n \otimes Y \quad \text{for all } Y \in \mathfrak{B}_2'.$$

Since $\mathbb{C}I \bar{\otimes} \mathcal{M}_2' \subseteq \mathfrak{A}$, it follows that $\theta_n(X) = (I \otimes X) Z_n$ for all $X \in \mathcal{M}_2'$.

So, for every n,m and all $X \in \mathcal{M}_2'$, we have $\theta_n(X)Z_m = \theta_m(X)Z_n$. Since $\mathfrak{B}_2'$ is a minimal injective von Neumann algebra extension of $\mathcal{M}_2'$, by Corollary 1.4, the same formula holds for all $Y \in \mathfrak{B}_2'$ and all n,m. This compatibility formula yields for every $Y \in \mathfrak{B}_2'$ an element $A \in \mathfrak{A}$ such that $AZ_n = \theta_n(Y)$ for all n. In particular, $A(P_n \otimes I) = P_n \otimes Y$, i.e., $A = I \otimes Y$. So, $\mathbb{C}I \mathbin{\bar{\otimes}} \mathfrak{B}_2' \subseteq \mathfrak{A}$, ending the proof. □

Replacing the use of the minimal projections P_n by a careful analysis of "infinitesimal projections" (i.e. using direct integral techniques), the same kind of idea maybe allows to prove the general case, not assuming $\mathcal{Z}(\mathfrak{B}_1)$ to be atomic.

5. A result on maximal injective subfactor of tensor products of von Neumann algebras

Theorem 5.1. *Let $\mathcal{M}_i$ be a factor, and $\mathcal{R}_i$ be a maximal injective subfactor of $\mathcal{M}_i$, $i = 1,2$. If $\mathcal{R}_1' \cap \mathcal{M}_1 \simeq \mathbb{C}^N$ $(1 \leqslant N \leqslant \infty)$ and $\mathcal{R}_2' \cap \mathcal{M}_2 = \mathbb{C}I$, then $\mathcal{R}_1 \mathbin{\bar{\otimes}} \mathcal{R}_2$ is a maximal injective subfactor of $\mathcal{M}_1 \mathbin{\bar{\otimes}} \mathcal{M}_2$.*

Proof. Consider an injective factor $\mathcal{R}$ such that $\mathcal{R}_1 \mathbin{\bar{\otimes}} \mathcal{R}_2 \subseteq \mathcal{R} \subseteq \mathcal{M}_1 \mathbin{\bar{\otimes}} \mathcal{M}_2$. We need to show that $\mathcal{R} = \mathcal{R}_1 \mathbin{\bar{\otimes}} \mathcal{R}_2$. We can assume that $\mathcal{M}_1$ and $\mathcal{M}_2$ are von Neumann algebras acting on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. So we have $\mathcal{R}_1' \mathbin{\bar{\otimes}} \mathcal{R}_2' \supseteq \mathcal{R}' \supseteq \mathcal{M}_1' \mathbin{\bar{\otimes}} \mathcal{M}_2'$. By Lemma 1.5, $\mathcal{R}_2$ is a maximal injective von Neumann subalgebra of $\mathcal{M}_2$ and thus $\mathcal{R}_2'$ is a minimal injective von Neumann algebra extension of $\mathcal{M}_2'$. By assumption, $\mathcal{R}_2' \cap \mathcal{M}_2 = \mathcal{R}_2' \cap \mathcal{R}_2 = \mathbb{C}I$.

Let $\mathbf{E}$ be a conditional expectation from $\mathcal{R}_1' \mathbin{\bar{\otimes}} \mathcal{R}_2'$ onto $\mathcal{R}'$, and $\mathcal{A} = \mathcal{R}_1' \cap \mathcal{M}_1 \simeq \mathbb{C}^N$. By Lemma 2.2, $\mathbf{E}$ induces a conditional expectation from $\mathcal{A} \mathbin{\bar{\otimes}} \mathcal{R}_2'$ onto $\mathfrak{B} \triangleq (\mathcal{A} \mathbin{\bar{\otimes}} \mathcal{R}_2') \cap \mathcal{R}'$. Therefore, $\mathfrak{B}$ is an injective von Neumann algebra such that $\mathcal{R}' \supseteq \mathfrak{B} \supseteq (\mathcal{A} \mathbin{\bar{\otimes}} \mathcal{R}_2') \cap (\mathcal{M}_1' \mathbin{\bar{\otimes}} \mathcal{M}_2') = \mathbb{C}I \mathbin{\bar{\otimes}} \mathcal{M}_2'$.

Similar arguments as the proof of Theorem 4.1 show that $\mathfrak{B} \supseteq \mathbb{C}I \mathbin{\bar{\otimes}} \mathcal{R}_2'$. So $\mathcal{R}' \supseteq \mathbb{C}I \mathbin{\bar{\otimes}} \mathcal{R}_2'$. By Ge–Kadison's splitting theorem (see 1.4), $\mathcal{R}' = \mathcal{N}' \mathbin{\bar{\otimes}} \mathcal{R}_2'$ for some von Neumann subalgebra $\mathcal{N}'$ of $\mathcal{R}_1'$. Therefore $\mathcal{R} = \mathcal{N} \mathbin{\bar{\otimes}} \mathcal{R}_2$. Since $\mathcal{R}$ is an injective factor, $\mathcal{N}$ is an injective factor such that $\mathcal{R}_1 \subseteq \mathcal{N} \subseteq \mathcal{M}_1$. Since $\mathcal{R}_1$ is a maximal injective subfactor of $\mathcal{M}_1$, $\mathcal{N} = \mathcal{R}_1$. Therefore, $\mathcal{R} = \mathcal{R}_1 \mathbin{\bar{\otimes}} \mathcal{R}_2$. □

Corollary 5.2. *Let $\mathcal{M}_i$ be a factor and $\mathcal{R}_i$ be an irreducible, maximal injective subfactor of $\mathcal{M}_i$, $i = 1,2$. Then $\mathcal{R}_1 \mathbin{\bar{\otimes}} \mathcal{R}_2$ is an irreducible, maximal injective subfactor of $\mathcal{M}_1 \mathbin{\bar{\otimes}} \mathcal{M}_2$.*

6. Concluding remarks

6.1. Let $\mathcal{M}_i$ be a von Neumann algebra and $\mathfrak{B}_i$ be a maximal injective von Neumann subalgebra of $\mathcal{M}_i$, $i = 1,2$. Suppose $\mathcal{M}_1$ is a type II_1 von Neumann algebra with separable predual and $\mathfrak{B}_1$ is an maximal injective von Neumann subalgebra of $\mathcal{M}_1$. By [1], $\mathfrak{B}_1 = (\mathcal{A} \mathbin{\bar{\otimes}} \mathcal{R}) \oplus \bigoplus_{n=1}^{\infty} (\mathcal{A}_n \mathbin{\bar{\otimes}} M_n(\mathbb{C}))$, where $\mathcal{A}, \mathcal{A}_1, \mathcal{A}_2, \ldots,$ are abelian von Neumann algebras and $\mathcal{R}$ is the hyperfinite type II_1 factor. By Theorem 4.1, if $\mathfrak{B}_1 = \mathcal{A} \mathbin{\bar{\otimes}} \mathcal{R}$ is type II_1 and $\mathcal{A}$ is atomic, then $\mathfrak{B}_1 \mathbin{\bar{\otimes}} \mathfrak{B}_2$ is a maximal injective von Neumann subalgebra of $\mathcal{M}_1 \mathbin{\bar{\otimes}} \mathcal{M}_2$. Popa's question remains open for all other cases of $\mathfrak{B}_1$, e.g., $\mathfrak{B}_1$ is abelian. It is still not known that a diffuse abelian von Neumann algebra with separable predual can be embedded into any non hyperfinite separable type II_1 factor as a maximal injective von Neumann subalgebra or not. Recently, J. Shen [14] proved that $\{\mathcal{L}(a)\}'' \mathbin{\bar{\otimes}} \{\mathcal{L}(a)\}''$ is a maximal injective von Neumann algebra of $\mathcal{L}(\mathbb{F}_n) \mathbin{\bar{\otimes}} \mathcal{L}(\mathbb{F}_n)$.

J. Shen also provided the first example of a Mcduff II_1 factor which contains an abelian von Neumann algebra as a maximal injective von Neumann subalgebra.

6.2. Let $\mathcal{M}_i$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}_i$, and $\mathfrak{B}_i$ be a maximal injective von Neumann subalgebra of $\mathcal{M}_i$, $i = 1, 2$. Suppose $\mathfrak{B}$ is an injective von Neumann algebra such that $\mathfrak{B}_1 \bar{\otimes} \mathfrak{B}_2 \subseteq \mathfrak{B} \subseteq \mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$. Then we have $\mathfrak{B}_1' \bar{\otimes} \mathfrak{B}_2' \supseteq \mathfrak{B}' \supseteq \mathcal{M}_1' \bar{\otimes} \mathcal{M}_2'$. Let $\mathbf{E}$ be a conditional expectation from $\mathfrak{B}_1' \bar{\otimes} \mathfrak{B}_2'$ onto $\mathfrak{B}'$. By Lemma 2.2, $\mathbf{E}$ induces a conditional expectation from $\mathcal{Z} \bar{\otimes} \mathfrak{B}_2'$ onto $\mathfrak{A} \triangleq (\mathcal{Z} \bar{\otimes} \mathfrak{B}_2') \cap \mathfrak{B}'$, where $\mathcal{Z} = \mathfrak{B}_1' \cap \mathfrak{B}_1 = \mathfrak{B}_1' \cap \mathcal{M}_1$. Note that $\mathcal{Z} \bar{\otimes} \mathfrak{B}_2' \supseteq \mathfrak{A} \supseteq \mathbb{C}I \bar{\otimes} \mathcal{M}_2'$. This leads to the following question:

Question 1. Suppose $\mathcal{A}$ is an abelian von Neumann algebra and $\mathfrak{A}_2$ is a minimal injective von Neumann algebra extension of a von Neumann algebra $\mathcal{N}_2$. If $\mathfrak{A}$ is an injective von Neumann algebra such that $\mathcal{A} \bar{\otimes} \mathfrak{A}_2 \supseteq \mathfrak{A} \supseteq \mathbb{C}I \bar{\otimes} \mathcal{N}_2$, is $\mathfrak{A} \supseteq \mathbb{C}I \bar{\otimes} \mathfrak{A}_2$?

An affirmative answer to Question 1 would give rise to an affirmative answer to Popa's question (with assumption that $\mathcal{M}_1$ has separable predual). Indeed, if $\mathfrak{A} \supseteq \mathbb{C}I \bar{\otimes} \mathfrak{B}_2'$, then $\mathfrak{B}' \supseteq \mathfrak{A} \supseteq \mathbb{C}I \bar{\otimes} \mathfrak{B}_2'$. Therefore, $\mathfrak{B}_1' \bar{\otimes} \mathfrak{B}_2' \supseteq \mathfrak{B}' \supseteq \mathcal{M}_1' \bar{\otimes} \mathfrak{B}_2'$. Hence, $\mathfrak{B}_1 \bar{\otimes} \mathfrak{B}_2 \subseteq \mathfrak{B} \subseteq \mathcal{M}_1 \bar{\otimes} \mathfrak{B}_2$. Apply Theorem 2.5, $\mathfrak{B} = \mathfrak{B}_1 \bar{\otimes} \mathfrak{B}_2$. So $\mathfrak{B}_1 \bar{\otimes} \mathfrak{B}_2$ is a maximal injective von Neumann subalgebra of $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$.

In Question 1, we may assume that $\mathcal{A}$ and $\mathfrak{A}_2$ are von Neumann algebras acting on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively, in standard form. Then $\mathcal{A}$ is a maximal abelian von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$, i.e., $\mathcal{A}' = \mathcal{A}$. Consider the commutant of $\mathcal{A} \bar{\otimes} \mathfrak{A}_2$, $\mathfrak{A}$, $\mathbb{C}I \bar{\otimes} \mathcal{N}_2$, respectively, Question 1 is equivalent to the following question.

Question 1′. Suppose $\mathcal{A}$ is a maximal abelian von Neumann algebra and $\mathfrak{B}_2$ is a maximal injective von Neumann subalgebra of a von Neumann algebra $\mathcal{M}_2$. If $\mathfrak{B}$ is an injective von Neumann algebra such that $\mathcal{A} \bar{\otimes} \mathfrak{B}_2 \subseteq \mathfrak{B} \subseteq \mathcal{B}(\mathcal{H}) \bar{\otimes} \mathcal{M}_2$, is $\mathfrak{B} \subseteq \mathcal{B}(\mathcal{H}) \bar{\otimes} \mathfrak{B}_2$?

By the proof of Theorem 4.1, if $\mathcal{A}$ is atomic, then the answer to Question 1 is affirmative and thus to Question 1′ is affirmative.

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