On the annihilators of local cohomology modules

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Abstract
Let \((R, \mathfrak{m})\) be a commutative Noetherian complete local ring, \(M\) a non-zero finitely generated \(R\)-module of dimension \(d \geq 1\), and \(T_R(M) := \bigcup N: N \leq M \text{ and } \dim N < \dim M\). In this paper we calculate the annihilator of the top local cohomology module \(H^d_{\mathfrak{m}}(M)\). More precisely, we show that \(0: R H^d_{\mathfrak{m}}(M) = 0: R M/T_R(M)\). Moreover, for every positive integer \(n\), we calculate the radical of the annihilator of \(H^n_{\mathfrak{m}}(M)\). More precisely, we prove that if \(H^n_{\mathfrak{m}}(M)\) is not finitely generated then \(\text{Rad}(0: R H^n_{\mathfrak{m}}(M)) = \bigcap_{p \in S} p\), where

\[ S = \{ p \in \text{Spec } R: (H^{n-1}_{\mathfrak{p}}(M))_p \neq 0 \text{ and } \dim R/p = 1 \}. \]

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1. Introduction

Throughout this paper, let \(R\) denote a commutative Noetherian local ring (with identity) and \(I\) an ideal of \(R\). For an \(R\)-module \(M\), the \(i\)th local cohomology module of \(M\) with respect to \(I\) is defined as

\[ H^i_I(M) = \lim_{n \to \infty} \text{Ext}_R^n(R/I^n, M). \]

For an Artinian \(R\)-module \(A\) we denote by \(\text{Att}_R A\) the set of attached prime ideals of \(A\). For each \(R\)-module \(L\), we denote by \(\text{Assh}_R L\) (resp. \(\text{mAss}_R L\)) the set \(\{ p \in \text{Ass}_R L: \dim R/p = \dim L \}\) (resp. the set of minimal primes of \(\text{Ass}_R L\)). Also, for any ideal \(a\) of \(R\), we denote \(\{ p \in \text{Spec } R: p \supseteq a \}\) by \(V(a)\). Finally, for any ideal \(b\) of \(R\), the radical of \(b\), denoted by \(\text{Rad}(b)\), is defined to be the set

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\[ x \in R: \ x^n \in b \text{ for some } n \in \mathbb{N} \]. We refer the reader to [4] or [2] for more details about local cohomology. The main results of this paper are the following:

**Theorem 1.1.** Let \((R, \mathfrak{m})\) be a complete Noetherian local ring and \(M\) be a non-zero finitely generated \(R\)-module of dimension \(d\). Then

\[ 0 :_R H^d_{\mathfrak{m}}(M) = 0 :_R M / T_R(M), \]

where, \(T_R(M) := \bigcup \{ N: N \subseteq M \text{ and } \dim N < \dim M \} \).

**Theorem 1.2.** Let \((R, \mathfrak{m})\) be a complete Noetherian local ring and \(n \geq 1\) be an integer. Let \(M\) be a non-zero finitely generated \(R\)-module such that \(\dim M \geq 1\). Set \(J := \text{Rad}(0 :_R H^n_{\mathfrak{m}}(M))\) and \(S = \{ p \in \text{Spec } R: (H^{n-1}_{\mathfrak{m}}(M))_p \neq 0 \text{ and } \dim R / p = 1 \}\). Then the following statements hold:

(i) If \(H^n_{\mathfrak{m}}(M)\) is not finitely generated, then \(J = \bigcap_{p \in S} p\).

(ii) If \(H^n_{\mathfrak{m}}(M)\) is non-zero and finitely generated, then \(J = \mathfrak{m}\).

**Theorem 1.3.** Let \((R, \mathfrak{m})\) be a Noetherian local ring of dimension \(d\). Then \(0 :_R H^d_{\mathfrak{m}}(R) = T_R(R)\).

For any unexplained notation and terminology we refer the reader to [2] and [5].

2. The results

To prove the main results of this paper, we need the following proposition.

**Proposition 2.1.** Assume that \((R, \mathfrak{m})\) is a Cohen–Macaulay local (Noetherian) ring of dimension \(d\) and for every \(p \in \text{Ass}_R R\) the zero-dimensional local ring \(R_p\) is Gorenstein. Let \(M\) be a finitely generated \(R\)-module such that \(\text{Ass}_R M \subseteq \text{Ass}_R R\). Then there is an exact sequence \(0 \to M \to \bigoplus_{i=1}^n R\) for some positive integer \(n\).

**Proof.** See [3, Theorem 3.5]. □

The following result follows from Proposition 2.1.

**Corollary 2.2.** Assume that \((R, \mathfrak{m})\) is a Gorenstein local (Noetherian) ring and \(I\) be an ideal of \(R\). Then the following statements are equivalent:

(i) \(\text{Ass}_R R / I \subseteq \text{Ass}_R R\).

(ii) There is an exact sequence \(0 \to R / I \to \bigoplus_{i=1}^n R\) for some positive integer \(n\).

(iii) There is an ideal \(J\) of \(R\) such that \(I = 0 :_R J\).

(iv) \(I = 0 :_R (0 :_R I)\).

**Proof.** (i) \(\iff\) (ii) The assertion follows from Proposition 2.1.

(ii) \(\implies\) (iii) Let \(f : R / I \to \bigoplus_{i=1}^n R\) be a monomorphism such that \(f(1 + I) = (a_1, \ldots, a_n)\). Set \(J := Ra_1 + \cdots + Ra_n\). It is easy to see that \(I = 0 :_R J\).

(iii) \(\iff\) (iv) Let \(I = 0 :_R J\). Then \(J \subseteq 0 :_R I\) and so \(I = 0 :_R J \supseteq 0 :_R (0 :_R I) \supseteq I\) that implies \(I = 0 :_R (0 :_R I)\).

(iv) \(\iff\) (ii) Let \(0 :_R I = Ra_1 + \cdots + Ra_n\). We define \(f : R / I \to \bigoplus_{i=1}^n R\) with \(f(r + I) = (ra_1, \ldots, ra_n)\). Then \(f\) is a monomorphism. □

The following lemma will be quite useful in the proof of the first main theorem.

**Lemma 2.3.** Let \((R, \mathfrak{m})\) be a local Noetherian ring and \(M\) a finitely generated \(R\)-module. Let \(p\) be a prime ideal of \(R\) such that \(\dim R / p = 1\) and \(t \geq 1\) be an integer. Then the following conditions are equivalent:
Lemma 2.3, \( H_n \) (ring \( T_R \) Set).

(i) First we show that 

\[ \text{Proof.} \]

(ii) \( H_p^{-1}(M) \) is Artinian.

(iii) \( (H_p^{n-1}(M))_{p} = 0. \)

**Proof.** See [1, Lemma 2.1]. □

Now we are ready to state and prove the first main result of this paper.

**Theorem 2.4.** Let \((R, m)\) be a complete Noetherian local ring and \(n \geq 1\) be an integer. Let \(M\) be a non-zero finitely generated \(R\)-module such that \(\dim R \geq 1\). Set \(J := \text{Rad}(0 : R H^0_m (M)) \) and \(S = \{ p \in \text{Spec } R : (H^{n-1}_p (M))_{p} \neq 0 \text{ and } \dim R/p = 1 \} \). Then the following statements hold:

(i) If \(H^0_m (M)\) is not finitely generated, then \(J = \bigcap_{p \in S} p\).

(ii) If \(H^0_m (M)\) is non-zero and finitely generated, then \(J = m\).

**Proof.** (i) First we show that \(S\) is non-empty. By our assumption and by [2, Corollary 7.2.12], \(\text{Att}_R H^0_m (M) \not\subseteq \{m\}\). So, there exists \(q \in \text{Att}_R H^0_m (M)\) such that \(q \neq m\). Consequently, there exists \(p \in V(q)\) such that \(\dim R/p = 1\). Now in view of Melkersson's theorem [6, Theorem 1.6], \(H^0_m (M)\) is not \(p\)-cofinite and hence by Lemma 2.3 it follows that \(p \in S\), that means \(S \neq \emptyset\). Now assume that \(p \in S\). By Lemma 2.3, \(H^0_m (M)\) is not \(p\)-cofinite. Hence by [6, Theorem 1.6], there exists \(q \in \text{Att}_R H^0_m (M)\) such that \(\text{Rad}(p + q) \neq m\). Whence, \(\dim R/(p + q) = \dim R/p = 1\) that implies \(q \subseteq p\). Since \(0 : R H^0_m (M) \subseteq q \subseteq p\), it follows that \(J \subseteq p\) and so \(J \subseteq \bigcap_{p \in S} p\). In order to prove \(\bigcap_{p \in S} p \not\subseteq J\). Since \(J = \bigcap_{q \in \text{Att}_R H^0_m (M)} q\), it follows that there is \(x_1 \in \bigcap_{p \in S} p\) such that \(x_1 \notin q\) for some \(q \in \text{Att}_R H^0_m (M)\). Suppose that \(\dim R/q = t\). Then \(t \geq 1\) and there are elements \(x_2, \ldots, x_t \in m\) such that \(x_1, x_2, \ldots, x_t\) forms a system of parameters for \(R\)-module \(R/q\). Hence \(\dim R/q + (x_2, \ldots, x_t) = 1\) and there exists a prime ideal \(p\) of \(R\) such that \(p \in \text{Ass}_R R/q + (x_2, \ldots, x_t)\) and \(\dim R/p = 1\). Therefore, as \(q \subseteq p\), according to Melkersson's theorem [6, Theorem 1.6], it follows that \(H^0_m (M)\) is not \(p\)-cofinite and so by Lemma 2.3, it follows that \((H^{n-1}_p (M))_{p} \neq 0\). Thus \(p \in S\) and hence \(x_1 \in p\). But \(0 = \dim R/q + (x_1, \ldots, x_t) \geq \dim R/p = 1\), which is a contradiction.

(ii) □

**Definition 2.5.** Let \((R, m)\) be a Noetherian local ring and \(M\) be a non-zero finitely generated \(R\)-module. Set \(T_R(M) := \bigcup \{N : N \subseteq M \text{ and } \dim N < \dim M\}\). It is clear that \(T_R(M)\) is a submodule of \(M\) and \(\text{Ass}_R M/T_R(M) = \text{Ass}_R M\).

The following theorem is the second main result of this paper.

**Theorem 2.6.** Let \((R, m)\) be a complete Noetherian local ring and \(M\) be a non-zero finitely generated \(R\)-module of dimension \(d\). Then \(0 : R H^d_m (M) = 0 : R M/T_R(M)\).

**Proof.** Since \(R\) is a complete, in view of Cohen's structure theorem there is a complete regular local ring \((S, n)\) such that \(R\) is a homomorphic image of \(S\). Consider the exact sequence:

\[ 0 \rightarrow T_R(M) ightarrow M ightarrow M/T_R(M) ightarrow 0 \]

that induces an exact sequence:

\[ H^d_m (T_R(M)) \rightarrow H^d_m (M) \rightarrow H^d_m (M/T_R(M)) \rightarrow H^{d+1}_m (T_R(M)). \]

As, \(\dim T_R(M) < d\) it follows from Grothendieck's vanishing theorem [2, Theorem 6.1.2], that \(H^d_m (T_R(M)) = 0 = H^{d+1}_m (T_R(M))\) and consequently \(H^d_m (M) \cong H^d_m (M/T_R(M))\).
Let \( J := 0 :_S M/T_R(M) \) and \( \text{height}_S J = t \). Then \( \text{grade}(J, S) = t \). Let \( x_1, \ldots, x_t \) be a maximal \( S \)-sequence in \( J \). Set \( R_1 = S/(x_1, \ldots, x_t) \) and \( n_1 = n/(x_1, \ldots, x_t) \). Now \((R_1, n_1)\) is a Gorenstein local ring and we have the following:

\[
H^d_{n_1}(M/T_R(M)) \cong H^n_{n_1}(M/T_R(M)) \cong H^n_{n_1}(R_1) \otimes_R M/T_R(M).
\]

To prove \( 0 :_R H^d_{n_1}(M) = 0 :_R M/T_R(M) \), it is clear that \( JR \subseteq 0 :_R H^d_{n_1}(M) \), so it is enough to show that \( 0 :_R H^d_{n_1}(M) \subseteq JR \). For any \( p \in \text{Ass}_R M/T_R(M) \) we have \( \dim R/p = d \) and so \( p \in \text{Ass}_R R_1 \). Hence by Proposition 2.1 there is a monomorphism \( f : M/T_R(M) \rightarrow \bigoplus_{i=1}^t R_1 \), for some positive integer \( n \). Let \( r \in n \setminus J \). If \( rf = 0 \), then \( f(r(M/T_R(M))) = 0 \). Since \( f \) is a monomorphism it follows that \( r \in J = 0 :_S M/T_R(M) \), which is a contradiction. Thus, we have \( rf \neq 0 \) and so \( r \text{Hom}_{R_1}(M/T_R(M), \bigoplus_{i=1}^t R_1) \neq 0 \). This shows that \( 0 :_R \text{Hom}_{R_1}(M/T_R(M), \bigoplus_{i=1}^n R_1) = J/(x_1, \ldots, x_t) = 0 :_R M/T_R(M) \). Now we have the following:

\[
0 :_{R_1} H^d_{n_1}(M) \subseteq 0 :_{R_1} \text{Hom}_{R_1}\left( H^d_{n_1}(M), \bigoplus_{i=1}^n E_{R_1}(R_1/n_1) \right).
\]

On the other hand

\[
\text{Hom}_{R_1}\left( H^d_{n_1}(M), \bigoplus_{i=1}^n E_{R_1}(R_1/n_1) \right) \cong \text{Hom}_{R_1}\left( H^d_{n_1}(R_1) \otimes_{R_1} M/T_R(M), \bigoplus_{i=1}^n E_{R_1}(R_1/n_1) \right)
\]

\[
\cong \text{Hom}_{R_1}\left( M/T_R(M), \text{Hom}_{R_1}\left( H^d_{n_1}(R_1), \bigoplus_{i=1}^n E_{R_1}(R_1/n_1) \right) \right).
\]

Since \( R_1 \) is a Gorenstein ring, it follows that \( H^d_{n_1}(R_1) \cong E_{R_1}(R_1/n_1) \) and since \( R_1 \) is complete it follows that

\[
\text{Hom}_{R_1}\left( H^d_{n_1}(R_1), \bigoplus_{i=1}^n E_{R_1}(R_1/n_1) \right) \cong \bigoplus_{i=1}^n R_1.
\]

Therefore,

\[
\text{Hom}_{R_1}\left( H^d_{n_1}(M), \bigoplus_{i=1}^n E_{R_1}(R_1/n_1) \right) \cong \text{Hom}_{R_1}\left( M/T_R(M), \bigoplus_{i=1}^n R_1 \right),
\]

which implies that

\[
0 :_{R_1} H^d_{n_1}(M) \subseteq 0 :_{R_1} \text{Hom}_{R_1}\left( H^d_{n_1}(M), \bigoplus_{i=1}^n E_{R_1}(R_1/n_1) \right)
\]

\[
= 0 :_{R_1} \text{Hom}_{R_1}\left( M/T_R(M), \bigoplus_{i=1}^n R_1 \right) = J/(x_1, \ldots, x_t).
\]

Therefore, \( 0 :_R H^d_{n_1}(M) \subseteq J/(x_1, \ldots, x_t) \), hence \( 0 :_S H^d_{n_1}(M) \subseteq J \) and consequently \( 0 :_R H^d_{n_1}(M) \subseteq JR \). Therefore, the assertion follows from the fact that \( JR = 0 :_R M/T_R(M) \). \( \Box \)

The following result is an immediately consequence of Theorem 2.6.
Corollary 2.7. Let \((R, m)\) be a Noetherian local ring and \(M\) be a non-zero finitely generated \(R\)-module of dimension \(d\). Then \(0 :_R H^d_m(M) = R \cap 0 :_R (M \otimes_R \hat{R})/T_R(M \otimes_R \hat{R})\).

**Proof.** The assertion follows immediately from Theorem 2.6. \(\Box\)

The following result is an other consequence of Theorem 2.6.

**Theorem 2.8.** Let \((R, m)\) be a Noetherian local ring of dimension \(d \geq 1\). Then \(0 :_R H^d_m(R) = T_R(R)\).

**Proof.** The exact sequence

\[0 \to T_R(R) \to R \to R/T_R(R) \to 0\]

induces an exact sequence

\[H^d_m(T_R(R)) \to H^d_m(R) \to H^d_m(R/T_R(R)) \to H^{d+1}_m(T_R(R)),\]

which using Grothendieck’s vanishing theorem [2, Theorem 6.1.2] implies that \(H^d_m(R) \cong H^d_m(R/T_R(R))\). Therefore \(T_R(R) \subseteq 0 :_R H^d_m(R/T_R(R)) = 0 :_R H^d_m(R)\). Now suppose on the contrary argument that \(0 :_R H^d_m(R) \neq T_R(R)\). Then there exists \(x \in 0 :_R H^d_m(R)\) such that \(x \notin T_R(R)\). It follows easily from the definition of \(T_R(R)\) that \(\dim_R Rx = d\) and hence \(\dim_{\hat{R}} \hat{R}x = d\). Consequently, using Theorem 2.6 it follows that

\[xH^d_m(R) \cong xH^d_m(R)_{\hat{R}} \neq 0.\]

Which is a contradiction. \(\Box\)

The following result is an application of Theorem 2.8.

**Corollary 2.9.** Let \((R, m)\) be a Noetherian local ring of dimension \(d\). Then the following are equivalent:

(i) \(0 :_R H^d_m(R) = 0\).

(ii) \(\text{Ass}_R(R) = \text{Assh}_R(R)\).

**Proof.** (i) \(\Rightarrow\) (ii) Let \(p \in \text{Ass}_R(R)\). There is an ideal \(J\) of \(R\) such that \(J \cong R/p\). Now it follows from Theorem 2.8 that \(\dim_R R/p = \dim J = d\), and hence \(p \in \text{Assh}_R(R)\).

(ii) \(\Rightarrow\) (i) Suppose that on the contrary, \(0 :_R H^d_m(R) \neq 0\). Therefore, by Theorem 2.8 there is a non-zero ideal \(I\) of \(R\) such that \(\dim I < d\). As the set \(\text{Ass}_R(I)\) is not empty, it follows that there exists a prime ideal \(p \in \text{Ass}_R(I) \subseteq \text{Ass}_R(R)\). Now it is clear that \(\dim_R R/p \leq \dim I < d\) and hence \(p \in \text{Ass}_R(R) \setminus \text{Assh}_R(R)\), which is a contradiction. \(\Box\)

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