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A Representation Theory for Noetherian Rings

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INTRODUCTION

The representation theory of artinian rings has long been studied, and seldom more intensely than in the past few years. However, with the exception of integral representations of finite groups, there is no representation theory for noetherian rings even when they are commutative. Our aim in this paper is to develop such a theory. We attempt to pattern it after the known representation theory of artinian rings. For this, we require a notion of indecomposability different from the usual one; and we say that a module is strongly indecomposable if it is noetherian, and no factor by a nontrivial direct sum of submodules has lower Krull dimension than the module. Thus strongly indecomposable modules are indecomposable and, by way of example, any uniform noetherian module is strongly indecomposable.

Our first result states that some factor of a noetherian module by a direct sum of strongly indecomposable submodules has dimension less than the dimension of the module. This decomposition refines indecomposable noetherian modules, but we can say little about it without restricting the underlying ring. Thus we study, in Section 1, α-indecomposable modules; that is, strongly indecomposable α-dimensional modules that have no nonzero submodule of dimension $< \alpha$. The resemblance of these to indecomposable modules of finite length is analogous to that of α-critical modules to simple ones (see [8]). Of course α-indecomposables have the disadvantage that, in general, not all of the strongly indecomposable submodules in the decomposition just described can be chosen to be α-indecomposable for some α. But each factor in the submodule sequence of a noetherian module (see Section 2) can be decomposed in terms of them; and the α-indecomposables appearing in such a decomposition have certain envelopes, typically proper submodules

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of the injective envelope, which are determined by the module up to order and isomorphism. This is proved in Section 1 along with the fact, and its consequences, that the envelopes have endomorphism rings that are local with nilpotent radical. Further, the endomorphism ring of an α-indecomposable module itself has properties similar to a commutative primary ring (forgetting the commutativity).

The next section begins with a characterization of those noetherian modules every strongly indecomposable submodule of which is α-indecomposable for some α. We then prove that finitely generated modules over fully bounded noetherian rings (FBN rings as defined in, say, [10]) have this property. Thus, for finitely generated modules over such rings, the two concepts, α-indecomposable and strongly indecomposable, coincide. Hence, if R is an FBN ring, and M a finitely generated α-dimensional module, we can check whether M is strongly indecomposable by first requiring that it have no nonzero submodule of dimension $< \alpha$. We call such modules α-Macaulay; cf., [6, 7]. Then we check (see Proposition 1.9) whether $M \otimes_R \Lambda$ is an indecomposable Λ-module, where Λ is the artinian classical quotient ring of $R/\text{ann } M$ (which exists by [5, 6]).

The rest of the paper is mainly concerned with the representation theory of FBN rings. Section 3 deals with the special case of noetherian rings that are finitely generated modules over their center–we use the term "Noether algebra." We prove that a Macaulay module over a Noether algebra is strongly indecomposable if and only if its endomorphism ring is an order in a local Artin algebra. However, the main result of the section is that decompositions of a finitely generated module can be specified so that the strongly indecomposable submodules that occur are unique up to order and subisomorphism. We conjecture (Conjecture 1.8) that the same is true of finitely generated modules over an arbitrary FBN ring; but our only result in this generality is a special case (see Proposition 5.8).

At the end of Section 3, we define a notion of finite representation type for FBN rings. This is done by noting that associated with an FBN ring is a certain finite set of artinian rings, derived from its ideal sequence; and we say that the ring has finite associated representation type when each of those artinian rings has finite representation type. We immediately exploit theorems of Auslander [1] and Roiter [13] (characterizing Artin algebras of finite representation type) to characterize Noether algebras of finite associated representation type. One characterization is in terms of the lengths of critical composition series of certain strongly indecomposable modules being bounded, and the other in terms of certain torsionfree modules (in a prescribed torsion theory) being "torsion" modulo a direct sum of finitely generated submodules.

Section 4 is largely devoted to an analysis of the strongly indecomposable modules that arise in the calculation of the associated representation type of an FBN ring. Our analysis is incomplete, even when the ring is commutative. But in Section 5, where we study FBN orders in artinian rings, we show that our notions of indecomposability and representation type are compatible with those obtained naturally from the artinian theory. More precisely, if R is a fully bounded noetherian order in an artinian ring Λ, we prove that the torsionfree strongly indecomposable R-modules are precisely the torsionfree modules that becomes indecomposable when tensored with Λ and are, morever, exactly the strongly indecomposable modules used in determining the associated representation type of R.

1. α-Indecomposable Modules

In this section, we introduce the notions of α-indecomposable and strongly indecomposable modules. After defining strongly indecomposable modules, we defer their further discussion to Section 2, preferring to study the more restrictive concept first. In the balance of the section, we prove some basic results about α-indecomposable modules, their endomorphism rings, decomposing modules in terms of them, and the uniqueness of the decomposition. Our work will require a knowledge of the theory of Krull dimension for noncommutative rings, and we refer the reader to [8] for the relevant definitions and results.

We say that a nonzero module I is *strongly indecomposable* if it is noetherian and does not contain a direct sum $I' \oplus I''$ of nonzero submodules I' and I'' such that $K \dim I/I' \oplus I'' < K \dim I$. It is obvious that, as the name implies, any strongly indecomposable module is indecomposable. Equally clear is that noetherian uniform modules are strongly indecomposable. In the same vein, if J is a submodule of a strongly indecomposable module I of Krull dimension α such that $K \dim I/J < \alpha$, then J is strongly indecomposable.

Before turning to α-indecomposable modules, we prove the following fundamental result.

PROPOSITION 1.1. *If M is a nonzero noetherian module, then M contains a finite direct sum of strongly indecomposable submodules such that*

$$K \dim M/\coprod I_j < K \dim M.$$

Proof. Let $K \dim M = \alpha$. If M is not already strongly indecomposable, then M contains a nontrivial direct sum $I_1 \oplus I_1'$ such that $K \dim M/I_1 \oplus I_1' < \alpha$. Next if, say, I_1' is not strongly indecomposable, then choose a nontrivial direct sum $I_2 \oplus I_2'$ such that $K \dim I_1'/I_2 \oplus I_2' < K \dim I_1'$. But $K \dim I_1' \leqslant \alpha$, and so each factor in the chain

$$M \supseteq I_1 \oplus I_1' \supseteq I_1 \oplus I_2 \oplus I_2'$$

has Krull dimension $< \alpha$. Therefore,

$$K \dim M/I_1 \oplus I_2 \oplus I_2' < \alpha.$$

This process must terminate with the desired direct sum since M, being noetherian, can contain no infinite direct sum of nonzero submodules. ▮

An *α-indecomposable* module is a strongly indecomposable module of Krull dimension α that has no nonzero submodule of Krull dimension $<\alpha$. Note that every α-critical noetherian module is α-indecomposable, and that 0-indecomposable modules are precisely indecomposable modules of finite length.

We should point out that, although the notions of α-indecomposable and strongly indecomposable are equivalent for finitely generated modules over FBN rings (see Section 2), over right FBN rings, the latter notion is weaker. For there can be finitely generated uniform modules over fully bounded right noetherian rings that are not α-indecomposable for any α.

To study α-indecomposable R-modules, we must use the quotient category $\mathbb{R}/\mathbb{R}_\alpha$, where $\mathbb{R} = \operatorname{Mod} R$ and $\mathbb{R}_\alpha$ is the smallest localizing subcategory containing the Serre subcategory of R-modules of Krull dimension $< \alpha$. If $M \in \mathbb{R}$, we denote the largest submodule of M killed by the canonical functor $T_\alpha: \mathbb{R} \to \mathbb{R}/\mathbb{R}_\alpha$ by $\tau_\alpha M$. We note that, if M is noetherian, then $\tau_\alpha M$ is the largest submodule of M of Krull dimension $< \alpha$; and this makes the following useful result evident.

LEMMA 1.2. *If $0 \neq M \in \mathbb{R}$ is noetherian, then M is α-indecomposable if and only if $T_\alpha M$ is an indecomposable object of $\mathbb{R}/\mathbb{R}_\alpha$ of finite length and $\tau_\alpha M = 0$.*

It is well known that the endomorphism ring of an indecomposable module of finite length is a local ring with nilpotent radical. Our next result exploits this knowledge.

PROPOSITION 1.3. *If S is the endomorphism ring of an α-indecomposable module and P its set of nilpotent elements then P is a nilpotent ideal, S/P is a domain and $S - P$ is the set of regular elements of S.*

Proof. Let S be the endomorphism ring of I, where $I \in \mathbb{R}$ is α-indecomposable. By 1.2, $L = \operatorname{End} T_\alpha I$ is a local ring with nilpotent radical, J, say. Now, since $\tau_\alpha I = 0$, $T_\alpha: S \to L$ is a monomorphism. Similarly, if $s \in S$, then

$T_\alpha s$ is a unit if and only if $s \notin P$. Thus, if we identify S with a subring of L via the embedding T_α, then $P = J \cap S$ and $S - P$ is the intersection of S with the set of units, $L - J$, of L. ▮

We say that a submodule N of a noetherian module M has *codimension* $< \alpha$ if $K \dim M/N < \alpha$. For example, if $K \dim M = \alpha$, and M embeds in N, then N has codimension $< \alpha$. If M embeds in each of its submodules of codimension $< \alpha$, then we say that M is *α-compressible.*

Since we wish to state our next results in terms of $\mathbb{R}$ rather than $\mathbb{R}/\mathbb{R}_\alpha$, we must use the $\mathbb{R}_\alpha$-envelope of a module M with $\tau_\alpha M = 0$. It is the module $E_\alpha(M)$, containing M, defined by the requirement that

$$E_\alpha(M)/M = \tau_\alpha(E(M)/M),$$

where $E(M)$ is the injective envelope of M.

COROLLARY 1.4. *If I is an α-compressible α-indecomposable R-module, then* End I *is a right order in the semiprimary local ring* End $E_\alpha(I)$.

Proof. We regard $\mathbb{R}/\mathbb{R}_\alpha$ as a full subcategory of $\mathbb{R}$ in the usual way, by saying that $T_\alpha M = E_\alpha(M/\tau_\alpha M)$ for each $M \in \mathbb{R}$. One concludes, as in the proof of 1.3, that $S = \operatorname{End} I$ is a subring of $L = \operatorname{End} E_\alpha(I)$. In the same manner, every regular element of S is a unit of L. Thus, it remains to show that if $f \in L$, then there is a regular element, g say, of S such that $fg \in S$.

Let f_0 be the restriction of f to I, where $f \in L$, and let $I_0 = f_0^{-1}(I \cap \operatorname{im} f_0)$. However $I/I_0 \simeq \operatorname{im} f_0/I \cap \operatorname{im} f_0$, which embeds in $E_\alpha(I)/I$ and so, since $E_\alpha(I)/I \in \mathbb{R}_\alpha$ and I/I_0 is noetherian, $K \dim I/I_0 < \alpha$. Thus, by hypothesis, there is a monomorphism $g: I \to I_0$. Moreover, $fg \in S$ by construction, and by Proposition 1.3, g is a regular element of S. ▮

The reason for our interest in this result will become apparent later in the section. For now, we exploit the classical Krull–Schmidt theorem.

PROPOSITION 1.5. *Let M be a nonzero α-dimensional noetherian R-module such that $\tau_\alpha M = 0$.*

(i) *M contains a finite direct sum of α-indecomposable submodules I_j such that $K \dim M/\coprod I_j < \alpha$.*

(ii) *The $\mathbb{R}_\alpha$-envelopes of the α-indecomposable submodules of* (i) *are determined by M up to order and isomorphism.*

(iii) *M is α-indecomposable if and only if* End $E_\alpha(M)$ *is local.*

Proof. Of course (i) is an immediate consequence of Proposition 1.1, using Lemma 1.2. More directly, we can observe, that $T_\alpha M$ has finite length;

and that T_α induces a surjection from decompositions of M, as described in (i), to decompositions of $T_\alpha M$ into a direct sum of indecomposable objects of $\mathbb{R}/\mathbb{R}_\alpha$. But this also makes (ii) and (iii) clear, from the properties of objects of finite length, by embedding $\mathbb{R}/\mathbb{R}_\alpha$ as a full subcategory of $\mathbb{R}$. ∎

We point out that the number of α-indecomposables in the decomposition (i) is unique. Concerning (ii), it would be desireable to know when modules have isomorphic envelopes. The following result gives a criterion.

PROPOSITION 1.6. *Two noetherian R-modules M_1 and M_2, with $\tau_\alpha M_i = 0$, have isomorphic $\mathbb{R}_\alpha$-envelopes if and only if there exist submodules M_i' of M_i such that $M_1' \simeq M_2'$ and $K \dim M_i/M_i' < \alpha$, for $i = 1, 2$.*

Proof. $\Leftarrow$ This is clear, since $E_\alpha(M_i') = E_\alpha(M_i)$.

$\Rightarrow$ Let $f: E_\alpha(M_1) \to E_\alpha(M_2)$ be the isomorphism and note that $E_\alpha(M_1)/M_1 \simeq E_\alpha(M_2)/fM_1$, and that $M_2/fM_1 \cap M_2$ is isomorphic to a submodule of $E_\alpha(M_2)/fM_1$. Thus, $K \dim M_2/fM_1 \cap M_2 < \alpha$. But $M_2' = fM_1 \cap M_2 \simeq M_1' = f^{-1}M_2'$, and by symmetry, $K \dim M_1/M_1' < \alpha$. ∎

We recall that two modules are subisomorphic if each is isomorphic to a submodule of the other.

COROLLARY 1.7. *The following statements are equivalent.*

(i) *For all α-dimensional noetherian R-modules M, if $I_1, \ldots, I_n$ are α-indecomposable submodules of M such that $K \dim M/\coprod I_j < \alpha$, then the I_j are determined up to order and subisomorphism by M.*

(ii) *Every α-indecomposable R-module is α-compressible.*

Proof. The implication (i) $\Rightarrow$ (ii) is noted above Proposition 1.1, and the other implication follows from Propositions 1.5 and 1.6 by factoring out $\tau_\alpha M$. ∎

Thus, we see that the type of uniqueness of "indecomposable components" described in (i) requires of noetherian α-critical modules that they be α-compressible. This in turn is known to not always be the case for modules over prime right FBN rings (see [8, Example 6.9]). However, a nontrivial result of Jategaonkar [10] states that every finitely generated α-critical module over an FBN ring is α-compressible.

We wish to make the following conjecture.

CONJECTURE 1.8. *If I is an α-indecomposable module over an* FBN *ring, then I is α-compressible.*

The best new evidence we have for the truth of this conjecture is given in Section 3, where we show that it is valid when the ring is finitely generated over its center.

The next result exposes the special nature of α-indecomposables over FBN rings.

PROPOSITION 1.9. *If M is a nonzero finitely generated α-dimensional module over an* FBN *ring R such that $\tau_\alpha M = 0$, then $R/\text{ann}\, M$ is an order in an artinian ring Λ, and M is α-indecomposable if and only if $M \otimes_R \Lambda$ is an indecomposable Λ-module.*

Proof. We can assume that M is faithful and then, by [6, Theorem 2.3] and [5, Theorem 1], that R is an order in the artinian ring Λ. But then, it follows from [5, Corollary 5] that $\text{Mod}\, \Lambda$ is equivalent to the quotient category $\mathbb{R}/\mathbb{R}_\alpha$. Further, the canonical functor $\mathbb{R} \to \text{Mod}\, \Lambda$ is $-\otimes_R \Lambda$; and so 1.2 completes the proof. ▮

We emphasize immediately that this result shows it would suffice to verify Conjecture 1.8 for FBN Macaulay rings (which will be discussed in the next section). The result also makes possible a restatement of Corollary 1.4 that might be of use in finding a counterexample to Conjecture 1.8.

COROLLARY 1.10. *If R is an* FBN *ring and I is a faithful α-indecomposable α-compressible module, then* End I *is a right order in* End $I \otimes_R \Lambda$, *where Λ is the quotient ring of R.*

2. STRONGLY INDECOMPOSABLE MODULES

Our main aim in this section is to show that every finitely generated module over an FBN ring can be decomposed in terms of α-indecomposable submodules for certain prescribed ordinals α. We do this by characterizing noetherian modules such that each strongly indecomposable submodule is α-indecomposable for some α; and then we show that finitely generated modules over FBN rings have this property. The notation set forth in the next paragraph will be used throughout the section.

Let M be a nonzero noetherian module. The *submodule sequence* of M is the finite ascending chain

$$M_1 \subset \cdots \subset M_n = M, \qquad (M_0 = 0),$$

of invariant submodules M_i of M defined by saying that M_i/M_{i-1} is the largest submodule of M/M_{i-1} of Krull dimension the least ordinal amongst

Krull dimensions of its nonzero submodules. (If M is a ring, we use the term "ideal sequence.") The ***Krull dimension sequence*** of M is the ascending sequence of ordinals

$$\alpha_1 < \cdots < \alpha_n ,$$

where $\alpha_i = K\dim M_i$ (cf., [4]). Of course, $K\dim M_i/M_{i-1} = \alpha_i$ and $\tau_{\alpha_i}(M_i/M_{i-1}) = 0$.

We need a preparatory result.

LEMMA 2.1. *If H_i is a submodule of M_i such that $\tau_{\alpha_i} H_i = 0$ and $K\dim M_i/H_i < \alpha_i$ then the sum $H_1 + \cdots + H_n$ is direct, $K\dim M/\coprod H_i < K\dim M$ and $\coprod H_i$ is essential in M.*

Proof. To see that the sum is direct, note that if, say, $H_1 \cap \sum_{i\neq 1} H_i \neq 0$, then some H_i for $i \neq 1$ would contain a nonzero submodule of dimension α_1.

Next, we note that H_n has codimension $< \alpha_n = K\dim M$ and so, since $M/\coprod H_i$ is a homomorphic image of M/H_n, $\coprod H_i$ must have codimension $< K\dim M$.

Finally, to see that $\coprod H_i$ is essential, note first that H_1 is essential in M_1. For otherwise, some nonzero submodule of M_1 would embed in M_1/H_1. Hence, assume that $H_1 + \cdots + H_{i-1}$ is essential in M_{i-1}. This makes $H_1 + \cdots + H_i$ essential in $M_{i-1} + H_i$ because both sums are direct. Thus, if $M_{i-1} + H_i$ is essential in M_i, then so too is $H_1 + \cdots + H_i$. But $\bar{H}_i$ is essential in $\bar{M}_i = M_i/M_{i-1}$ since $\bar{M}_i/\bar{H}_i$, being a homomorphic image of M_i/H_i, has dimension $< \alpha_i$, and $\tau_{\alpha_i}\bar{M}_i = 0$. Now, if $(M_{i-1} + H_i) \cap N = 0$, N a submodule of M_i, then $\bar{H}_i \cap \bar{N} = 0$. Thus, $N \subseteq M_{i-1}$ and so $N = 0$. We conclude, by induction, that $\coprod H_i$ is essential in M. ∎

THEOREM 2.2. *Every strongly indecomposable submodule of M is α-indecomposable for some α if and only if each M_i has a submodule H_i as specified in the preceding lemma.*

Proof. $\Rightarrow$ Suppose that, for $1 \leqslant i \leqslant n-1$, submodules H_i of M_i have been constructed so that $K\dim M_i/H_i < \alpha_i$ and $\tau_{\alpha_i} H_i = 0$. Using Proposition 1.1, we can choose a direct sum of strongly indecomposable submodules of M such that $K\dim M/\coprod I_j < \alpha_n$. Let

$$H_n = \coprod\{I_j \mid K\dim I_j = \alpha_n\}.$$

Then, our hypothesis implies that $\tau_{\alpha_n} H_n = 0$. Further, since

$$K\dim \coprod I_j/H_n < \alpha_n$$

by construction, $K\dim M/H_n < \alpha_n$.

$\Leftarrow$ We first reduce to the case when M itself is strongly indecomposable. For this, let $0 \neq M'$ be a submodule of M. It follows from [3, p. 250] that there exist integers $1 \leqslant i_1 < \cdots < i_{n'} \leqslant n$ such that $\alpha_{i_1} < \cdots < \alpha_{i_{n'}}$ is the Krull dimension sequence of M'. It is then readily checked that

$$M_1' \subset \cdots \subset M_{n'}' = M'$$

is the submodule sequence of M', where, of course, $M_j' = M_{i_j} \cap M'$. We write $H_j' = H_{i_j} \cap M'$ and $\alpha_j' = \alpha_{i_j}$. Then clearly, $H_j' \subseteq M_j'$, $\tau_{\alpha_j'} H_j' = 0$, and since M_j'/H_j' is isomorphic to a submodule of M_{i_j}/H_{i_j}, $K \dim M_j'/H_j' < \alpha_j'$. This completes the reduction.

Now, suppose that M is strongly indecomposable. But then, it follows from 2.1 that $n = 1$. Therefore, M is α_1-indecomposable by definition. ∎

COROLLARY 2.3. *If every strongly indecomposable submodule of M is α-indecomposable for some α, then*

(i) *M contains an essential finite direct sum of strongly indecomposable submodules I_j such that*

$$K \dim M_i/\coprod\{I_j \mid K \dim I_j = \alpha_i\} < \alpha_i\ ;$$

(ii) *The $\mathbb{R}_{\alpha_i}$-envelopes of the submodules I_j in* (i) *of Krull dimension α_i are determined by M up to order and isomorphism.*

Proof. Let H_i be a submodule of M_i such that $\tau_{\alpha_i} H_i = 0$, and $K \dim M_i/H_i < \alpha_i$. By Proposition 1.5(i), some submodule H_i' of H_i of codimension $< \alpha_i$ is a direct sum of α_i-indecomposable modules. Thus, by Lemma 2.1, $H_1' + \cdots + H_n'$ is a direct sum of strongly indecomposable submodules of M as required in (i). Further, since $H_i' \cap M_{i-1} = 0$, H_i' is isomorphic to its canonical image in $\overline{M}_i = M_i/M_{i-1}$. Thus, (ii) follows from 1.5(ii), because $K \dim \overline{M}_i/\overline{H}_i' < \alpha_i$, and the $\overline{M}_i$ depend only on M. ∎

We remark that [6, Example 4.1] gives one instance of the nonexistence of the decomposition in (i) for a finitely generated module over a primary right FBN ring.

For the rest of the section, and those that follow, we must assume a familiarity with the theory of primary decomposition for FBN rings as developed in [6]. Briefly, if M is finitely generated over an FBN ring R, we will call it *α-Macaulay* if $K \dim R/P = \alpha$ for every $P \in \operatorname{Ass} M$, $\operatorname{Ass} M$ being the set of associated prime ideals of M. (Then, M is α-Macaulay if and only if $K \dim M = \alpha$, and $\tau_\alpha M = 0$.) We say that M is *Macaulay* if it is α-Macaulay for some α. Any (finitely generated nonzero) primary module is Macaulay since, by definition, it has a unique associated prime.

It will be useful to be able to identify the submodule sequence in terms of a primary decomposition. For this, we recall first that the terms α_i of the Krull dimension sequence are precisely the codimensions of the associated primes.

LEMMA 2.4. *If M is finitely generated over an* FBN *ring, and $\bigcap N_j = 0$ is a primary decomposition of M, then*

$$M_i = \bigcap\{N_j \mid K \dim M/N_j > \alpha_i\}.$$

Proof. Let

$$L_i = \bigcap\{N_j \mid K \dim M/N_j > \alpha_i\}.$$

An easy induction shows that it is enough to prove that each L_i/L_{i-1} is α_i-Macaulay. Now, it is clear, from the usual properties of a primary decomposition (see [6, Lemma 1.4]), that $L_i/L_{i-1} \neq 0$. But if we write $L_{i-1} = L_i \cap L_i'$, where

$$L_i' = \bigcap\{N_j \mid K \dim M/N_j = \alpha_i\},$$

then we see that L_i/L_{i-1} embeds in $\coprod\{M/N_j \mid K \dim M/N_j = \alpha_i\}$; and this is an α_i-Macaulay module. Thus, L_i/L_{i-1} is α_i-Macaulay. ∎

THEOREM 2.5. *A strongly indecomposable module of Krull dimension α over an* FBN *ring is α-indecomposable.*

Proof. Let M be the module. By Theorem 2.2, it suffices to find α_i-Macaulay submodules H_i of M_i of codimension $< \alpha_i$, and for this, it suffices to produce H_n. But consider the module

$$H_n = \bigcap\{N_j \mid K \dim M/N_j \leqslant \alpha_{n-1}\},$$

where $\bigcap N_j = 0$ is a primary decomposition of M. By Lemma 2.4, $M_{n-1} \cap H_n = 0$. Further, H_n embeds in M/M_{n-1} and M/H_n embeds in $\coprod\{M/N_j \mid K \dim M/N_j \leqslant \alpha_{n-1}\}$. The former embedding shows that H_n is an α_n-Macaulay submodule of $M = M_n$ and the latter shows that

$$K \dim M/H_n \leqslant \alpha_{n-1} < \alpha_n . \quad ∎$$

This result has a consequence that we shall need in Section 5.

COROLLARY 2.6. *If M is finitely generated over an* FBN *ring R, and* if $\mathscr{T}$ *is a torsion theory on* Mod R *with respect to which M is torsionfree, then the same is true of each factor module M_i/M_{i-1}.*

Proof. Choose an α_i-Macaulay submodule H_i of M_i with

$$K \dim M_i/H_i < \alpha_i .$$

Of course, H_i is $\mathscr{T}$-torsionfree since it is a submodule of M. Now, we have already seen, in the proof of 2.1, that $H_i + M_{i-1}/M_{i-1}$ is an essential submodule of M_i/M_{i-1}; and we have also seen that $H_i \simeq H_i + M_{i-1}/M_{i-1}$. Therefore, M_i/M_{i-1} is $\mathscr{T}$-torsionfree, insomuch as it is an essential extension of a $\mathscr{T}$-torsionfree module. ∎

We wish to partially restate Corollary 2.3 in the context of FBN rings for emphasis. Using Corollary 1.7 we have

COROLLARY 2.7. *Let M be finitely generated over an* FBN *ring.*

(i) *M contains an essential finite direct sum of strongly indecomposable submodules I_j such that*

(a) $K \dim M/\coprod I_j < K \dim M$,

(b) $K \dim M_i/\coprod\{I_j \mid K \dim I_j = \alpha_i\} < \alpha_i$, *for* $1 \leqslant i \leqslant n$.

(ii) *Moreover, granted Conjecture* 1.8, *the submodules I_j in* (i) *are determined by M up to order and subisomorphism.*

One cannot expect (ii) to follow in the absence of the stipulation (b) of (i). For an example, let k be a field, and choose R to be the polynomial ring $k[x]$ localized at (x), and M to be $R/(x)^2 \oplus R$. Clearly, $R/(x)^2$ and $(x)/(x)^2$ are 0-indecomposables, and R is a 1-indecomposable. Also, $N = (x)/(x)^2 \oplus R$ is an essential submodule of M, and $K \dim M/N < K \dim M$. But $R/(x)^2$ and $(x)/(x)^2$ are not subisomorphic. Moreover, since R is a commutative noetherian ring, Conjecture 1.8 is valid. We shall see that this is so in the next section. In Corollary 5.7 we shall see, incidentally, that the module M in the example above cannot be chosen to be torsionfree.

3. NOETHER ALGEBRAS

We recall that an Artin algebra is an artinian ring that is a finitely generated module over its center. If a noetherian ring is similarly finitely generated over its center, then we say that it is a *Noether algebra*. We remark that, since the center of a Noether algebra is noetherian [2], a ring R is a Noether algebra if and only if it is a finitely generated algebra over some commutative noetherian ring S; and then we say that R is a *Noether S-algebra*. It is well known that any Noether S-algebra is an FBN ring being, in fact, a P.I. ring. (For R is a subring of $\text{End}_S R$, and since R is a finitely generated S-module, $\text{End}_S R$ is a homomorphic image of a subring of a matrix ring over S.)

Throughout this section, R will be a Noether S-algebra. We study α-Macaulay modules over R, showing that they are α-compressible (cf., Conjecture 1.8). We show also that the endomorphism ring is an order in an Artin algebra, and that this Artin algebra is local if and only if the module is strongly indecomposable.

The group ring $R = SG$ of a finite group G is a Noether S-algebra and we show that, in this case, if $W_1 \subset \cdots \subset W_n = S$ is the ideal sequence of S, then $W_1 \otimes_S R \subset \cdots \subset W_n \otimes_S R = R$ is the ideal sequence of R.

At the end of the section, we introduce a notion of "finite representation type" for FBN rings. For now, we begin the section by comparing the Krull dimension of modules over R with their dimension as S-modules.

LEMMA 3.1. *If M is a finitely generated R-module then*

(i) $K \dim M = K \dim M_S$,

(ii) *for each ordinal* α, $\tau_\alpha(M_R) = 0$ *if and only if* $\tau_\alpha(M_S) = 0$.

Proof. (i) Let $A = \operatorname{ann}_R M$, and let $B = \operatorname{ann}_S M$. This makes M an $(S/B, R/A)$-bimodule that is finitely generated and faithful over each ring. Thus, by a result of Jategaonkar [10, Theorem 2.3], the rings S/B and R/A have the same Krull dimension. But $K \dim M_R = K \dim M_{R/A} = K \dim R/A$ by [10, Lemma 2.1], and similarly, $K \dim M_S = K \dim S/B$.

(ii) The implication from right to left follows immediately from (i). For the other implication, suppose that N is an S-submodule of M of Krull dimension $< \alpha$. Choosing generators $r_1, \ldots, r_n$ of R over S, we get that $NR = Nr_1 + \cdots + Nr_n$. But each Nr_i is an S-homomorphic image of N, and as such, has Krull dimension $< \alpha$. It follows, using (i), that the R-submodule NR of M has Krull dimension $< \alpha$. ∎

COROLLARY 3.2. *The submodule sequence of a finitely generated R-module M is equally the submodule sequence of M regarded as an S-module.*

COROLLARY 3.3. *If R is α-Macaulay and faithful for S, then S is α-Macaulay.*

We should mention that a faithful Noether algebra over a Macaulay ring need not itself be Macaulay. However, in the important special case of torsionfree Noether algebras, it is. We will prove a stronger result. First, we remind the reader that a module X over an arbitrary ring D is torsionfree if $xd = 0$, where $x \in X$ and d is a regular element of D, implies that $x = 0$.

PROPOSITION 3.4. *If M is a torsionfree S-module and α is an ordinal such that $\tau_\alpha S = 0$, then $\tau_\alpha M = 0$.*

Proof. Let L be a finitely generated submodule of M such that $K\dim L < \alpha$. Now, if $A = \operatorname{ann} L$ contains no regular element, then $A \subseteq \bigcup\{P \mid P \in \operatorname{Ass} S\}$; and so $A \subseteq P$ for some $P \in \operatorname{Ass} S$. But S/P embeds in S and $K \dim S/A = K \dim L < \alpha$. Consequently, the image of S/A in the composite map $S/A \to S/P \hookrightarrow S$ is a nonzero ideal of S of Krull dimension $< \alpha$. This contradicts the assumption that $\tau_\alpha S = 0$. Thus, L is killed by a regular element of S, and so $L = 0$. ∎

We point out that if R is the group ring SG of a finite group G, then R is a torsionfree S-module. Thus, we can expect more complete results for group rings, and the following theorem confirms this. It might be of interest to the reader that the theorem implies, for example, that $\mathbb{Z}G$ is a 1-Macaulay ring.

THEOREM 3.5. *If M is a nonzero finitely generated S-module, and G is a finite group, then M is α-Macaulay if and only if $MG = M \otimes_S SG$ is an α-Macaulay SG-module.*

Proof. $\Leftarrow$ This follows from Lemma 3.1, since M is a nonzero S-submodule of MG.

$\Rightarrow$ Using 3.1, it suffices to prove that MG is an α-Macaulay S-module· It is clearly enough to do so when M is faithful for S; and this makes S α-Macaulay by [6]. In particular, M is torsionfree over S because, if c is a regular element of S, then $K \dim S/(c) < \alpha$. Thus, plainly, MG is a torsionfree S-module. Hence, MG is α-Macaulay by 3.4 since MG, being faithful for S, has dimension α. ∎

With the same hypothesis, we have

COROLLARY 3.6. *If $M_1 \subset \cdots \subset M_p = M$ is the submodule sequence of M, then $M_1G \subset \cdots \subset M_pG = MG$ is the submodule sequence of MG.*

Proof. This is immediate from the fact that

$$M_iG/M_{i-1}G \simeq (M_i/M_{i-1})G. \quad ∎$$

We remark that the Krull dimension sequences of M and MG are identical.

We next verify the claims made of Macaulay modules over Noether algebras at the start of the section.

THEOREM 3.7. *If M is an α-Macaulay R-module, then M is α-compressible.*

Proof. By Lemma 3.1, M is an α-Macaulay S-module. We assume, for convenience, that M is faithful for S. Then, S is α-Macaulay by [6, Theorem 2.3], or directly. (For $K \dim S = \alpha$, and if $I \neq 0$ is an ideal of S of dimension $< \alpha$, then $0 \neq MI$ has dimension $< \alpha$.)

Let M_0 be an R-submodule of M such that $K \dim M/M_0 < \alpha$. Then, M/M_0 has Krull dimension $< \alpha$ as an S-module too, by 3.1. Now, it follows from 3.4 that M/M_0 is a torsion S-module. Thus, since M/M_0 is finitely generated, there is a single regular element, c say, of S, that annihilates it (cf., [6, Corollary 5]). But M is torsionfree over S, as in the proof of 3.5, and so multiplication by c is an R-monomorphism $M \to M_0$. ▮

COROLLARY 3.8. *If M is a Macaulay R-module then*

(i) End *M is an order in an Artin algebra,*

(ii) *M is strongly indecomposable if and only if the Artin algebra of* (i) *is local.*

Proof. We may as well assume that M is a faithful R-module and R a faithful S-module. In particular, M is faithful for S, and so the total quotient ring of S is artinian, by 1.9. (This is evident here, for the associated primes of S must be minimal primes.) Thus, the localization, Λ say, of R with respect to the regular elements of S, is an Artin algebra. Further, since $M \otimes_R \Lambda$ is finitely generated over Λ, it is standard that $\text{End}_\Lambda M \otimes_R \Lambda$ is an Artin algebra. Similarly, $\text{End}_R M$ is a Noether algebra. But the fact that R and S are Macaulay of equal dimension makes it plain that R is torsionfree over S; and so R is an order in Λ. Thus, by the proof of Corollary 1.10, End M is an order in End $M \otimes \Lambda$. This proves (i), and (ii) follows from 1.5(iii). ▮

We hasten to point out that the validity of Conjecture 1.8 for Noether algebras is a special case of Theorem 3.7. Thus, the theorem establishes the uniqueness, up to order and subisomorphism, of the strongly indecomposable submodules occurring in the decomposition 2.7(i) of a finitely generated module over a Noether algebra. What one would like to prove concerning such modules is that they are determined up to subisomorphism by the algebra's strongly indecomposable modules. This is stated more precisely in (i) of the next result.

PROPOSITION 3.9. *The following statements are equivalent.*

(i) *If A and B are finitely generated R-modules with respective submodule sequences*

$$A_1 \subset \cdots \subset A_m = A \qquad and \qquad B_1 \subset \cdots \subset B_m = B,$$

such that A_i/A_{i-1} is subisomorphic to B_i/B_{i-1} for all i, then A is subisomorphic to B.

(ii) *Every finitely generated R-module embeds in the direct sum of the factors of its submodule sequence.*

Proof. Let M be a finitely generated R-module with submodule sequence $\{M_i\}$ and Krull dimension sequence $\{\alpha_i\}$. We note that the factors of the submodule sequences of M and of $\coprod M_i/M_{i-1}$ (where $M_0 = 0$) are identical. We note also, using the proof of Theorem 2.5 (or 2.5 combined with 2.2), that there are α_i-Macaulay submodules H_i of M_i such that $K \dim M_i/H_i < \alpha_i$. But it follows from Theorem 3.7 that M_i/M_{i-1} embeds in H_i. Thus, by 2.1, $\coprod M_i/M_{i-1}$ embeds in M. ▮

This result makes it plain that the R-module R described below Theorem 4.3 is not determined up to subisomorphism by the strongly indecomposable modules of the algebra R. However, part (i) of the result does give a necessary condition for subisomorphism.

PROPOSITION 3.10. *If A and B are subisomorphic noetherian modules with respective submodule sequences*

$$A_1 \subset \cdots \subset A_p = A \qquad \text{and} \qquad B_1 \subset \cdots \subset B_q = B,$$

then $p = q$ and A_i/A_{i-1} and B_i/B_{i-1} are subisomorphic for all i.

Proof. Evidently, A and B have the same Krull dimension sequence (see the proof of 2.2) and so, in particular, $p = q$. Let $f: A \to B$ be a monomorphism. Now, the restriction of f to A_1 is a monomorphism $A_1 \to B_1$. Thus, it suffices, by induction on p, to show that the induced map $A/A_1 \to B/B_1$ is a monomorphism. But if $a \in A$, then, since f is a monomorphism, $aR \simeq f(a)R$, where R is the underlying ring. Hence, if $f(a) \in B_1$, then $a \in A_1$. ▮

Before going on, we should recall that an artinian ring is said to have finite representation type if it has only finitely many nonisomorphic indecomposable modules (which, by convention, are nonzero and finitely generated). We should also mention that, if M is a finitely generated nonzero module over an α-Macaulay ring, then the proof of Proposition 1.9 makes it obvious that M is α-Macaulay if and only if it is torsionfree.

PROPOSITION 3.11. *A Macaulay Noether algebra has only finitely many nonsubisomorphic torsionfree strongly indecomposable modules if and only if its classical quotient ring has finite representation type.*

Proof. We know that if R is a Macaulay Noether algebra then its quotient ring, Λ say, is artinian. Let I_j, $j = 1, 2$, be torsionfree R-modules. Then, I_j is strongly indecomposable if and only if $I_j \otimes_R \Lambda$ is indecomposable, by 1.9. Now, if I_1 and I_2 are subisomorphic, then $I_1 \otimes \Lambda$ and $I_2 \otimes \Lambda$ are subisomorphic, and since they have the same length, they are isomorphic.

Conversely, if $I_1 \otimes \Lambda \simeq I_2 \otimes \Lambda$, then, using Proposition 1.6, I_1 and I_2 are subisomorphic by 3.7. This finishes the proof because if M is an indecomposable Λ-module, then $M = N \otimes \Lambda$ for some finitely generated R-submodule N of M (namely, take a finite set of Λ-generators of M for generators of N); and we just saw that N is strongly indecomposable. ∎

One consequence of the truth of Conjecture 1.8 would be that an arbitrary Macaulay ring whose quotient ring has finite representation type has only finitely many nonsubisomorphic torsionfree strongly indecomposable modules.

We wish to define a notion of finite representation type for any FBN ring R. For this, let

$$W_1 \subset \cdots \subset W_n = R, \qquad (W_0 = 0),$$

be the ideal sequence of R, and let

$$Z_i = \operatorname{ann} W_i/W_{i-1}.$$

We call the Z_i the *associated annihilator ideals* of R. Now, each R/Z_i is a Macaulay ring and thus, an order in an artinian ring (see 1.9), say Λ_i. We say that the Λ_i are the *associated artinian rings* of R. (Note that, as we have seen, if R is a Noether algebra, then the Λ_i are Artin algebras. We note also that if R is commutative and G is a finite group, then by 3.6, the associated artinian rings of RG are the rings $\Lambda_i G$.) We say that R has *finite associated representation type* if each of its associated artinian rings has finite representation type.

In the next two sections, we will try to show that our notion of finite representation type is a reasonable one. We will also try to determine which R-modules are torsionfree R/Z_i-modules. But we first prove a result that, especially when united with the preceding one, shows that to do so would be worthwhile. We use the notation just established and refer the reader to [6, Sect. 1] for the definition and properties of critical composition series, and for further references.

THEOREM 3.12. *The following are equivalent properties of a Noether algebra R.*

(i) *R has finite associated representation type.*

(ii) *For each $i = 1, \ldots, n$, the lengths of critical composition series of torsionfree strongly indecomposable R/Z_i-modules are bounded.*

(iii) *For all i, if M is a torsionfree R/Z_i-module, then M has a family of finitely generated submodules M_j such that $M/\coprod M_j$ is torsion.*

Proof. We immediately reduce to the case when R is Macaulay. Let Λ be its associated Artin algebra.

(i) $\Leftrightarrow$ (ii) A celebrated result of Roiter [13] states that Λ has finite representation type if and only if there is a bound on the lengths of indecomposable Λ-modules.

Let I be a torsionfree strongly indecomposable R-module. Then, every critical composition factor of I must be $K \dim R$-critical and so, if

$$0 = C_0 \subset \cdots \subset C_k = I$$

is a critical composition series for I, then

$$0 = C_0 \otimes_R \Lambda \subset \cdots \subset C_k \otimes_R \Lambda = I \otimes_R \Lambda$$

is a composition series for $I \otimes \Lambda$. This completes the proof, for, as in the proof of 3.11, any indecomposable Λ-module can be written $J \otimes \Lambda$, where J is a strongly indecomposable R-module.

(i) $\Leftrightarrow$ (iii) A theorem of Auslander [1] states that Λ has finite representation type if and only if every Λ-module is a direct sum of finitely generated Λ-modules. Now, if an R-module M has finitely generated submodules M_i such that $M/\coprod M_i$ is torsion, then

$$M \otimes \Lambda/\coprod(M_i \otimes \Lambda) \simeq (M/\coprod M_i) \otimes \Lambda = 0,$$

and so $M \otimes \Lambda = \coprod M_i \otimes \Lambda$. Thus, since the $M_i \otimes \Lambda$ are finitely generated submodules of $M \otimes \Lambda$, and every Λ-module has the form $F \otimes \Lambda$ for some torsionfree R-module F, Λ has finite representation type.

Conversely, suppose that Λ has finite representation type and let N be a torsionfree R-module. Thus, $N \otimes \Lambda = \coprod L_i$, where the L_i are finitely generated Λ-modules. Now, choose finitely generated R-submodules L_i' of L_i such that $L_i' \otimes \Lambda = L_i$, and let $N_i = L_i' \cap N$. Then, each N_i is finitely generated over R, and we claim that $N/\coprod N_i$ is torsion. To see this, note that $N/\coprod N_i \subseteq \coprod(L_i/N_i)$, and since $(L_i/L_i') \otimes \Lambda \simeq L_i/L_i' \otimes \Lambda = 0$, each L_i/L_i' is torsion. Moreover, $L_i'/N_i \simeq L_i' + N/N \subseteq N \otimes \Lambda/N$, which is torsion. Hence, each L_i/N_i is torsion, and this verifies our claim. ∎

4. Associated Annihilator Ideals

Let R be an FBN ring with Krull dimension sequence

$$\alpha_1 < \cdots < \alpha_n = K \dim R,$$

ideal sequence

$$W_1 \subset \cdots \subset W_n = R,$$

and associated annihilator ideals

$$Z_1, \ldots, Z_n .$$

We will adhere to this notation in the first part of the section where we study torsionfree R/Z_i-modules. We show that they are torsionfree with respect to a certain natural torsion theory on Mod R. We show also that a deeper study depends on the nature of primary decompositions of the ring, and we then establish some relevant facts about primary components. At the end of the section, we give two pathological examples, announced in [6], of primary decomposition in FBN rings.

If S is a semiprime ideal, we say that a module is *S-torsionfree* if it is torsionfree in the torsion theory cogenerated by $E(R/S)$, and we denote the largest S-torsion (right) ideal by $T_S(R)$. It is known [9, 12], that $E(R/S)$ and $\coprod\{E(R/P) \mid P$ a minimal prime over $S\}$ cogenerate the same torsion theory, and that a module is S-torsionfree if and only if none of its nonzero elements are killed by an element that is regular modulo S. We will be concerned with the case when the minimal primes over S are elements of comp R, our notation for the (finite) set of composition series primes of R. These are the (prime) annihilators of critical composition factors of R, and have been studied in [6, 11].

Proposition 4.1. *Let*

$$S_i = \bigcap\{P \in \operatorname{comp} R \mid K \dim R/P = \alpha_i\}.$$

Then, an R-module M is a torsionfree R/Z_i-module if and only if $MZ_i = 0$, and M is S_i-torsionfree.

Proof. By [6, Theorem 3.2(i)], $S_i \supseteq Z_i$, and $\bar{S}_i$ is the prime radical of $\bar{R} = R/Z_i$. Then, we have a commutative square

$$\begin{array}{ccc} R & \longrightarrow & \bar{R} \\ \downarrow & & \downarrow \\ R/S_i & \rightarrow & \bar{R}/\bar{S}_i , \end{array}$$

where the maps are the canonical ones. Thus, by Small's Theorem, $c + S_i$ is a regular element of R/S_i if and only if $\bar{c}$ is a regular element of $\bar{R}$. ▮

Corollary 4.2. *If M is a torsionfree R/Z_i-module, then M is torsionfree with respect to the torsion theory cogenerated by* $\coprod\{E(R/P) \mid P \in \operatorname{comp} R\}$.

We will call an associated prime ideal P of R *isolated* if it is a minimal prime. Otherwise we say that P is *embedded*. There is an example [6, Example 4.2] of a minimal associated prime of a Noether algebra that is embedded.

We wish to consider the case when R is commutative. Then, since comp M = Ass M for a finitely generated module M, a module N is torsionfree in the torsion theory cogenerated by $\coprod\{E(R/P) \mid P \in \text{comp } R\}$ if and only if it is torsionfree. (Note that N is S-torsion, where S is a semiprime ideal, if and only if the localization N_S of N at the complement of the union of the minimal primes over S is 0.) Thus, concerning 4.2, if R has an embedded prime, there will be, for some i, a cyclic torsionfree R-module killed by Z_i that is not torsionfree over R/Z_i (cf., Theorem 5.10). The second part of the theorem below shows, vis-a-vis 4.1, that if, for a fixed i, every S_i-torsionfree module is killed by Z_i, then every associated prime of codimension α_i is isolated. More extensive consequences of the theorem, based on the fact that the factor by a primary ideal belonging to an associated prime of codimension α_i is S_i-torsionfree, will be given in Corollary 4.4.

THEOREM 4.3. *Let R be commutative, let P be an associated prime such that K dim $R/P = \alpha_i$, and let T be a primary ideal belonging to P.*

(i) *If P is isolated, then $(R/T)Z_i = 0$ and Z_i is P-torsion.*

(ii) *If P is embedded and T is a primary component of R, then $(R/T)Z_i \neq 0$.*

Proof. We note that $\bar{R} = R/W_1$ has Krull dimension sequence $\alpha_2 < \cdots < \alpha_n$, ideal sequence $\bar{W}_2 \subset \cdots \subset \bar{W}_n = \bar{R}$, and associated annihilator ideals $\bar{Z}_2, \ldots, \bar{Z}_n$.

(i) Since R/T is a P-torsionfree, it suffices to show that Z_i is P-torsion. This is true if $n = 1$, and so we assume it is true for rings of Krull dimension sequence length $n - 1$, where $n > 1$.

If $i > 1$, then W_1 is P-torsion. For the associated primes of W_1 are precisely the associated primes of R of codimension α_1. Thus, since K dim $R/P > \alpha_1$, no associated prime of W_1 can be contained in P. But also, by Lemma 2.4, $P \supseteq W_1$, and $\bar{P}$ is an isolated prime of $\bar{R}$ of codimension α_i. Thus, it follows from the properties of $\bar{R}$ noted above that $\bar{Z}_i$ is $\bar{P}$-torsion. Consequently,

$$(Z_i)_P/(W_1)_P \simeq (\bar{Z}_i)_{\bar{P}} = 0,$$

and so $(Z_i)_P = (W_1)_P = 0$; that is, Z_i is P-torsion.

If $i = 1$, then R/W_1 is P-torsion. For, every associated prime of R/W_1 has codimension $> \alpha_1 = K$ dim R/P, and P is isolated. But $zW_1 = 0$ for each $z \in Z_1$. Thus, every cyclic submodule of Z_1 is a homomorphic image of R/W_1. It follows that Z_1 is P-torsion.

(ii) We must show that $Z_i \not\subseteq T$. Suppose first that $i > 1$. Then, by 2.4, $\bar{P}$ is an embedded prime of $\bar{R}$ of codimension α_i, and $\bar{T}$ is a $\bar{P}$-primary component of $\bar{R}$. Thus, by induction on n, as in (i), $\bar{Z}_i \not\subseteq \bar{T}$. Hence, $Z_i \not\subseteq T$.

Now suppose that $i = 1$, and that $Z_1 \subseteq T$. Using Lemma 2.4, we choose primary components $T_1, ..., T_q$ such that

$$W_1 \cap T_1 \cap \cdots \cap T_q = 0,$$

$K \dim T/T_j = \alpha_1$, and say, $T_1 = T$. We then have $(T_j)_P = R_P$ for $j \neq 1$ because, if $j \neq 1$, it is clear that R/T_j is P-torsion. Thus, setting $W = W_1$ and $Z = Z_1$, we have $W_P \cap Z_P \subseteq W_P \cap T_P = 0$. But

$$\operatorname{ann}_{R_P} W_P = (\operatorname{ann}_R W)_P = Z_P,$$

and W_P, being isomorphic to a submodule of R_P/T_P, is P_P-primary. Thus, R_P/Z_P is a P_P/Z_P-primary ring, i.e., the maximal ideal of the (artinian) local ring R_P/Z_P is nilpotent. But $W_P + Z_P/Z_P$ is a faithful ideal of this ring, since it is isomorphic to $W_P/W_P \cap Z_P = W_P$. Therefore, $W_P + Z_P = R_P$.

Finally, by hypothesis, P contains properly a minimal prime ideal, Q say, of R. Of course, $Q \in \operatorname{Ass} R$ and so, by 2.4, $Q \supseteq W$ since $K \dim R/Q > \alpha_1$. But then, $R_P = W_P + Z_P \subseteq Q_P + P_P = P_P$. We conclude that $Z_1 \not\subseteq T$. ▌

We hasten to mention that, when R is commutative, every cyclic primary module is strongly indecomposable (by, say, 3.8(ii)). Thus, if P is an embedded prime, then R has infinitely many nonsubisomorphic cyclic P-torsionfree strongly indecomposable modules. For R has infinitely many P-primary components (see 4.6(ii)) and any two subisomorphic cyclic modules have a common annihilator. Thus, one consequence of Theorem 4.3(ii) is that a commutative noetherian ring never has infinite associated representation type merely by virtue of having an embedded prime.

We mention also that the assumption in 4.3(ii), that the primary ideal T is a primary component, is not necessary. To see this, let R be the polynomial ring in indeterminates x and y over a field k subject to $x^2 = xy = 0$. Let $P = (x, y)$, and for $m > 1$, let $T_m = (x, y^m)$. Then, $Z_1 = P$, P is an embedded prime of codimension α_1 $(=0)$, and for all m, R/T_m is P-primary and $(R/T_m)Z_1 \neq 0$. Yet T_m is not a primary component of R for any m. Morever, R has finite associated representation type, its associated artinian rings being k and $k(y)$.

We wish to give another example that illustrates, among other things, why the assumption that T is a primary ideal belonging to P in 4.3(ii) is not sufficient. This time choose an infinite field k, let $R = k[x, y]/(x)(x, y)^2$, and let $P = (x, y)/(x)(x, y)^2$. Again, P is an embedded prime of codimension $\alpha_1 = 0$. But now $Z_1 = P^2$ and so, by the choice of k, the local artinian ring

$R/Z_1 \simeq k[x, y]/(x, y)^2$ has infinitely many nonisomorphic cyclic indecomposable modules. It follows that there are infinitely many primary ideals of R belonging to P and containing Z_1, and of course, R has infinite associated representation type. Further, the cyclic P-primary R-module $k[x, y]/(x, y^3) \simeq k[y]/(y)^3$ has length equal to that of R/Z_1, and yet is not an R/Z_1-module.

Theorem 4.3(ii) allied with the proof of 4.3(i) establishes

COROLLARY 4.4. *If R is commutative and*

$$S_i = \bigcap\{P \in \text{Ass } R \mid K \dim R/P = \alpha_i\},$$

then the following statements are equivalent.

(i) *Every S_i-torsionfree R-module is annihilated by Z_i.*

(ii) $Z_i = T_{S_i}(R)$.

(iii) *Every associated prime of R of codimension α_i is isolated.*

Frequently cited examples show that the hypothesis of commutativity in 4.3(i) and 4.4 is required. The next result is useful in identifying $T_{S_i}(R)$ when R is noncommutative.

LEMMA 4.5. *Let S be a semiprime ideal of R and M an R-module.*

(i) *M is S-torsionfree if and only if every associated prime of M is contained in a minimal prime over S.*

(ii) *M is S-torsion if and only if no composition series prime of any finitely generated submodule of M is contained in a minimal prime over S.*

(iii) $T_S(R) = \bigcap\{A \mid A$ *is an ideal of R with* $\text{Ass } R/A \subseteq \text{Ass } R/S\}$.

Proof. (i) M contains an essential direct sum of critical submodules C_i, and it is clear that M is S-torsionfree if and only if every C_i is S-torsionfree. Now, since R is an FBN ring, the associated primes of M are the primes $P_i = \text{ann } C_i$ and C_i embeds in R/P_i. Further, by Goldie's theorem, R/P_i embeds in a direct sum of copies of C_i. But any critical module, being compressible, is either S-torsionfree or S-torsion; and so C_i and R/P_i are simultaneously either S-torsionfree or S-torsion. Thus, R/P_i is S-torsionfree if and only if $\text{Hom}(R/P_i, E(R/Q)) \neq 0$ for some minimal prime Q over S and, plainly, this is equivalent to saying that $P_i \subseteq Q$.

(ii) M is the direct limit of its finitely generated submodules and a direct limit of S-torsion modules in S-torsion. Thus, we can assume that M is finitely generated. But then M is S-torsion if and only if the same is true of each of its critical composition factors, and so the proof follows from the proof of (i).

(iii) Let D be the specified intersection and let $Q_1, \ldots, Q_s$ be the minimal primes over S. The fact that Ass $R/S = \{Q_1, \ldots, Q_s\}$ shows that $T_S(R) \subseteq D$, by (i).

For the other inclusion, set $A = \operatorname{ann} R/\ker f$, where $f: R \to \coprod E(R/Q_i)$. Then, Ass $R/\ker f \subseteq$ Ass $\coprod E(R/Q_i) = \{Q_1, \ldots, Q_s\}$, and by [6, Corollary 2.6(i)], Ass $R/A \subseteq$ Ass $R/\ker f$. Thus, $D \subseteq A$. But also, $A \subseteq \ker f$, and so the well-known formula

$$T_S(R) = \bigcap\{\ker f \mid f: R \to \coprod E(R)/Q_i)\}$$

shows that $D \subseteq T_S(R)$. ▮

Our next result is well known for commutative noetherian rings. Before stating it, we must remind the reader that in [6], we proved that FBN rings have primary decompositions. That is, if Ass $R = \{P_1, \ldots, P_r\}$, then there are (two-sided!) ideals $T_1, \ldots, T_r$ such that $T_1 \cap \cdots \cap T_r = 0$ and Ass $R/T_i = \{P_i\}$. An ideal such as T_i we call a *P_i-primary component* of R.

COROLLARY 4.6. *Let P be an associated prime of R.*

(i) *If P is isolated, then $T_P(R)$ is the smallest P-primary component of R.*

(ii) *If P is embedded, then any P-primary component contains an infinite descending chain of P-primary components.*

Proof. (i) By 4.5(iii), it suffices to show that $R/T_P(R)$ is P-primary. But if $Q \in$ Ass $R/T_P(R)$, then, since $R/T_P(R)$ is P-torsionfree, $Q \subseteq P$ by 4.5(i). Thus, $Q = P$.

(ii) Let $P' \subset P \in \operatorname{spec} R$, and let T_0 be a P-primary component. Then, $R/T_0 \cap P'$ is P-torsionfree isomuch that it embeds in the P-torsionfree module $R/T_0 \oplus R/P'$. But P is a minimal prime over T_0 (see [6]), and this implies that $T_0 \cap P' \subset T_0$. Thus, $T_0 \not\subseteq T_P(R)$ and so, by 4.5(iii), there is a primary ideal T_0' belonging to P such that $T_0 \not\subseteq T_0'$.

Now, let $T_1 = T_0 \cap T_0'$. Then, $T_0 \supset T_1$, and the fact that R/T_1 is P-primary ensures that T_1 is a primary component. But then we can apply the same process to T_1. ▮

We admit we do not know if the assumption that R is commutative can be omitted from 4.3(ii). However, using 4.6(ii), we can prove a weaker result.

COROLLARY 4.7. *If P is an embedded prime such that $K \dim R/P = \alpha_i$, then there are infinitely many P-primary components that do not contain Z_i.*

Proof. Let

$$T_0 \supset T_1 \supset \cdots \supset T_m$$

be a proper descending chain of P-primary components containing Z_i, and let Λ be the artinian quotient ring of $\bar{R} = R/Z_i$. We claim that

$$\bar{T}_0 \otimes_{\bar{R}} \Lambda \supset \cdots \supset \bar{T}_m \otimes_{\bar{R}} \Lambda$$

is a proper descending chain of ideals of Λ. But this clear, using 4.1, because $\bar{R}/\bar{T}_j$ is isomorphic to the P-torsionfree module R/T_j. ∎

Another consequence of 4.6 is that if R is Macaulay with associated primes $P_1, \ldots, P_r$, then $\bigcap T_{P_i}(R) = 0$ is a primary decomposition. This is the smallest primary decomposition in the sense that if $\bigcap T_i' = 0$ is any primary decomposition such that T_i' belongs to P_i, then $T_{P_i}(R) \subseteq T_i'$. It is easily deduced, using 4.5, that $T_{P_i}(R)$ is the ideal T_i of R maximal with respect to

$$\text{comp } T_i \subseteq \text{comp } R - \{P_i\}.$$

Thus, when R is artinian (see [6, Proposition 3.9]) $T_{P_i}(R)$ is just its largest ideal having no composition factor isomorphic to the unique up to isomorphism minimal right ideal with annihilator P_i. In particular, $T_{P_i}(R)$ contains all but the one homogeneous component of the socle killed by P_i.

These remarks will be useful in identifying primary decompositions vis-a-vis the following examples, referred to at the start of the section. The first example answers [6, Sect. 4, (Q1)] negatively, and the second deals similarly with [6, Sect. 4, (Q4)].

EXAMPLE 4.8. There is an Artin algebra R with infinitely many primary decompositions.

Construction. Let k be an infinite field, let D be the semisimple k-algebra $k \times k$, and let k_{ij} be the simple D-bimodules with $k_{ij} = k$ as k-spaces and D-bimodule action

$$(x_1, x_2)\, a\, (y_1, y_2) = x_i\, a\, y_j,$$

where a, $x_t, y_t \in k$, and $i, j, t = 1, 2$. Let X be the D-bimodule

$$X = k_{21}^{(1)} \oplus k_{21}^{(2)} \oplus k_{21}^{(3)} \oplus k_{12},$$

and let $T_D(X)$ be the tensor algebra

$$T_D(X) = D \oplus X \oplus (X \otimes_D X) \oplus \cdots.$$

Finally, let G be the ideal of $T_D(X)$ generated by X^3, $k_{12}k_{21}^{(1)}$ and $k_{12}k_{21}^{(2)}$, and let $R = T_D(X)/G$.

Proof. Plainly, R is an Artin k-algebra and we write it

$$R = (k_{11} \oplus k_{22}) \oplus (k_{21}^{(1)} \oplus k_{21}^{(2)} \oplus k_{21}^{(3)} \oplus k_{12}) \\ \oplus (k_{21}^{(1)}k_{12} \oplus k_{21}^{(2)}k_{12} \oplus k_{21}^{(3)}k_{12} \oplus k_{12}k_{21}^{(3)}). \tag{1}$$

We observe that the idempotents (1, 0) and (0, 1) of D generate, respectively, modulo rad $R = X + X^2$, simple R-modules C_1 and C_2, say. The C_i are, evidently, apart from isomorphism, the only simple R-modules; and we see that $P_i = \operatorname{ann} C_i$, $i = 1, 2$, are the associated primes of R.

Now, using (1), it can be checked that the smallest primary decomposition of R is given by the homogeneous components of soc $R = X^2$; that is, the smallest P_i-primary components T_i are

$$T_1 = k_{21}^{(1)}k_{12} \oplus k_{21}^{(2)}k_{12} \oplus k_{21}^{(3)}k_{12}, \qquad T_2 = k_{12}k_{21}^{(3)}.$$

But, since k is an infinite field, $k_{21}^{(1)} \oplus k_{21}^{(2)}$ has infinitely many (simple) D-subbimodules. These generate, together with T_1, an infinite family of ideals properly containing T_1 and having trivial intersection with T_2. Using formula (1) again, we compute that the factor ring by each ideal of this family has only copies of C_1 in its socle. ▌

EXAMPLE 4.9. There exists an Artin algebra R with primary components T and T' belonging to the same associated prime such that comp $R/T \neq$ comp R/T'.

Construction. In the notation of the preceding example, let $R = T_D(X)/G$, where $D = k \times k \times k \times k$, $X = k_{12} \oplus k_{23} \oplus k_{34} \oplus k_{42}$, and G is generated by X^4, $k_{42}k_{23}$, and $k_{23}k_{34}k_{42}$. Let T be generated by k_{23} modulo G, and T' by k_{23} and k_{11} mod G.

Proof. The proof proceeds in the vein of the preceding one. We write out

$$R = (k_{11} \oplus k_{22} \oplus k_{33} \oplus k_{44}) \\ \oplus (k_{12} \oplus k_{23} \oplus k_{34} \oplus k_{42}) \oplus (k_{12}k_{23} \oplus k_{23}k_{34} \oplus k_{34}k_{42}) \oplus k_{12}k_{23}k_{34}. \tag{2}$$

Let the idempotents (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), and (0, 0, 0, 1) of D correspond to the simple modules C_1, C_2, C_3, and C_4 of R, so that $P_i = \operatorname{ann} C_i$, $i = 2, 4$, are the associated primes of R. Then, one uses (2) to verify that T and T' are P_2-primary components of R, noting that the ideal generated by k_{42} is a primary ideal belonging to P_4 and having zero intersection with both T and T'. Similarly, one deduces that C_1 is isomorphic to a composition factor of R/T, but not to one of R/T'. ▌

We remark that, although T is the smallest P_2-primary component of R, T is not one of the two homogeneous components of the socle of R.

5. FBN Orders in Artinian Rings

Throughout this section, R will be an FBN ring that is an order in an artinian ring Λ. There is a way of defining a representation theory for R more natural than the one we have considered. What we have in mind is, of course, tensoring nonzero finitely generated torsionfree modules with Λ and requiring that they be indecomposable. However, we will show that the modules so obtained are precisely the torsionfree strongly indecomposable modules. Morever, we show that these coincide with the class of modules that are torsionfree and strongly indecomposable over some R/Z_i, where the Z_i are the associated annihilator ideals of R. Thus, if R is finitely generated over its center, then it has finite associated representation type if and only if it has finite representation type in the "natural" sense–meaning that Λ has finite representation type.

If M is a torsionfree R-module, we will frequently identify $M \otimes_R \Lambda$ with

$$M\Lambda = \{mc^{-1} \mid m \in M \text{ and } c \in R \text{ is regular}\}.$$

Before proving the facts mentioned above, we require some preliminary results.

Lemma 5.1. *If B is the annihilator of a torsionfree Macaulay R-module then*

(i) *every regular element of R is regular modulo B,*

(ii) *R/B is an order in $\Lambda/B\Lambda$.*

Proof. Given (i), (ii) is a standard consequence. Thus, let c be a regular element of R, and suppose that $xc \in B$, where $x \in R$. Then, by hypothesis, $x \in B$. But we know, by 1.9, that R/B is an order in an artinian ring. Thus, $c + B$ is a regular element of R/B. ∎

Corollary 5.2. *The following are equivalent properties of a finitely generated R-module M.*

(i) *M is torsionfree.*

(ii) *Every critical composition factor of M is torsionfree.*

(iii) comp $M \subseteq$ comp R.

Proof. (i) ⇒ (ii) Using Corollary 2.6, the fact that the critical composition factors of M are unique up to subisomorphism enables us to assume that M is Macaulay. But then, since the critical composition factors of M each have Krull dimension $K\dim R/\text{ann}\, M$, each is torsionfree over $R/\text{ann}\, M$; and 5.1(i) implies that each is torsionfree over R.

(ii) ⇒ (iii) If $P \in \operatorname{comp} M$, then the proof of Lemma 4.5(i) shows that R/P is torsionfree. Thus, by Small's Theorem, R/P is S-torsionfree, where S is the prime radical of R. Further, 4.5(i) implies that P is a minimal prime of R. But some product of composition series primes of R is 0. Therefore, $P \in \operatorname{comp} R$.

(iii) ⇒ (i) By hypothesis, every critical composition factor of M is one of R. But R is torsionfree, and we have already proved (i) ⇒ (ii). ∎

COROLLARY 5.3. *Let M be a finitely generated torsionfree R-module.*

(i) *The Krull dimension sequence of M is a subsequence of the Krull dimension sequence of R.*

(ii) *If M is faithful, then* comp M *is the set of minimal primes of R.*

We remark that (ii) of this result strengthens [6, Theorem 3.2(i)].

LEMMA 5.4. *Let M be a finitely generated torsionfree R-module with submodule sequence $\{M_i\}$ and Krull dimension sequence $\{\beta_i\}$, and for each i let H_i be a β_i-Macaulay submodule of M_i. Then $M/\coprod H_i$ is torsion if and only if $K \dim M_i/H_i < \beta_i$ for all i.*

Proof. Let $H = \coprod H_i$, and let p be the length of the sequence $\{\beta_i\}$. We first prove the lemma when $p = 1$. This makes M β_1-Macaulay, and we must prove that $K \dim M/H < \beta_1$ if and only if M/H is torsion. For this, let $B = \operatorname{ann} M$. Then, by Lemma 5.1, R/B is an order in $\Lambda/B\Lambda \simeq R/B \otimes_R \Lambda$. Thus,

$$M/H \otimes_R \Lambda \simeq M/H \otimes_{R/B} \Lambda/B\Lambda.$$

But since $K \dim R/B = \beta_1$, $K \dim M/H < \beta_1$ if and only if

$$M/H \otimes_{R/B} \Lambda/B\Lambda = 0,$$

and this is true precisely when $M/H \otimes_R \Lambda = 0$, as required.

Next, we prove the lemma when $p > 1$. We write

$$H^* = H_1 \oplus \cdots \oplus H_{p-1}.$$

Then, by induction, M_{p-1}/H^* is torsion if and only if $K \dim M_i/H_i < \beta_i$ for $1 \leqslant i \leqslant p-1$. Now, the chain

$$M \supseteq M_{p-1} \oplus H_p \supseteq H^* \oplus H_p = H$$

shows that M/H is torsion if and only if M_{p-1}/H^* and $M/M_{p-1} \oplus H_p$ are torsion. But we know that M/M_{p-1} is torsionfree, by 2.6, and so we know that $M/M_{p-1} \oplus H_p$ is torsion precisely when it has Krull dimension $< \beta_p$.

Finally, since $K \dim M_{p-1} < \beta_p$, it follows that $K \dim M/M_{p-1} \oplus H_p < \beta_p$ if and only if $K \dim M/H_p < \beta_p$. ▮

We continue the section with one of the results promised at the outset.

THEOREM 5.5. *If M is a finitely generated torsionfree R-module, then M is strongly indecomposable if and only if $M\Lambda$ is indecomposable.*

Proof. ⇒ We know that R/B is an order in $\Lambda/B\Lambda$, where $B = \operatorname{ann} M$. But $M\Lambda$ is an $\Lambda/B\Lambda$-module, and as such, it is isomorphic to $M \otimes_{R/B} \Lambda/B\Lambda$ by the proof of 5.4. Thus, $M\Lambda$ is an indecomposable Λ-module, by 1.9.

⇐ Let $\{\beta_i\}$ be the Krull dimension sequence of M, and let $\{M_i\}$ be its submodule sequence. By Corollary 2.7, M contains a direct sum of strongly indecomposable submodules, say

$$M \supseteq I_1 \oplus \cdots \oplus I_m,$$

such that

$$K \dim M_i/\coprod\{I_j \mid K \dim I_j = \beta_i\} < \beta_i.$$

But then, by 5.4, $M/\coprod I_j$ is torsion and so $M\Lambda = I_1\Lambda \oplus \cdots \oplus I_m\Lambda$. Thus, $m = 1$. One concludes that M is Macaulay and that $K \dim M/I_1 < K \dim M$. This makes it plain that M is strongly indecomposable. ▮

COROLLARY 5.6. *A semiprime* FBN *ring has finite associated representation type.*

We note that a more direct proof can be given by observing that the associated artinian rings are semisimple.

COROLLARY 5.7. *Using the notation of Lemma* 5.4, *suppose that M contains a direct sum of strongly indecomposable submodules such that $M/\coprod I_j$ is torsion. Then, the truth of Conjecture* 1.8 *implies that*

$$K \dim M_i/\coprod\{I_j \mid K \dim I_j = \beta_i\} < \beta_i.$$

Proof. Let

$$H = H_1 \oplus \cdots \oplus H_p,$$

where

$$H_i = \coprod\{I_j \mid K \dim I_j = \beta_i\}.$$

By Corollary 2.7, we can choose submodules H_i' of M_i such that H_i' is a direct sum of β_i-indecomposables and $K \dim M_i/H_i' < \beta_i$. We write

$$H' = H_1' \oplus \cdots \oplus H_p'.$$

Now, $H'\Lambda = M\Lambda$ by 5.4, and $H\Lambda = M\Lambda$ since M/H is assumed to be torsion. Further, using 5.4 and 5.5, we see from the proof of 3.9 (employing Conjecture 1.8 in place of Theorem 3.7) that H_i' is subisomorphic to H_i for all i.

We finish the proof, making repeated use of 5.4. We have that $M_i\Lambda = M_{i-1}\Lambda \oplus H_i'\Lambda \simeq M_{i-1}\Lambda \oplus H_i\Lambda$, which is a submodule of $M_i\Lambda$. Thus, by finite length, we must have $M_i\Lambda = M_{i-1}\Lambda \oplus H_i\Lambda$. This implies that $K \dim M_i/M_{i-1} + H_i < \beta_i$, and we conclude that $K \dim M/H_i < \beta_i$. ▌

Our proof of this result shows that the decomposition in the proposition below coincides with the one in Corollary 2.7(i).

PROPOSITION 5.8. *If $I_1, \ldots, I_r$ are strongly indecomposable right ideals of R such that $R/\coprod I_j$ is torsion, then the I_j are determined by R up to order and subisomorphism.*

Proof. The proof elaborates on the known proof when R is semiprime (so that the I_j are just uniform right ideals). It is sufficient to prove that A is subisomorphic to B when A and B are right ideals of R such that $A\Lambda$ and $B\Lambda$ are isomorphic indecomposable direct summands of Λ. Now, it is standard that $A\Lambda \simeq B\Lambda$ if and only if their simple canonical images in $\Lambda/\text{rad}\,\Lambda$ are isomorphic. Equally standard is that $S\Lambda = \text{rad}\,\Lambda$, S the prime radical of R, and that R/S is an order in $\Lambda/\text{rad}\,\Lambda$.

The commutative diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{\text{canon}} & \Lambda/\text{rad}\,\Lambda \\ \uparrow & & \uparrow \\ R & \xrightarrow{\text{canon}} & R/S, \end{array}$$

now makes it clear that $A + S/S$ and $B + S/S$ are subisomorphic (uniform) right ideals of R/S, and as such, do not annihilate each other. In particular, there is an element $a \in A$ such that $aB \not\subseteq S$. But then, taking intersections with R, we see that $aB\Lambda \not\subseteq \text{rad}\,\Lambda$ and $a \notin \text{rad}\,\Lambda$. It follows that a induces an isomorphism $B\Lambda \to A\Lambda$ and thus, a monomorphism $B \to A$. By symmetry, there is a monomorphism $A \to B$. ▌

We proceed in the notation of Section 4: $\alpha_1 < \cdots < \alpha_n$ is the Krull dimension sequence of R, $W_1 \subset \cdots \subset W_n = R$ is its ideal sequence, and $Z_1, \ldots, Z_n$ are its associated annihilator ideals. We have already proved, for commutative noetherian rings, a stronger result than the one that follows (see Corollary 4.4).

LEMMA 5.9. *$Z_i = T_{S_i}(R)$, where*

$$S_i = \bigcap\{P \in \text{comp } R \mid K \dim R/P = \alpha_i\}.$$

Proof. The proof is by induction on n, and follows the outline of the proof of Theorem 4.3(i).

If $n = 1$, then $Z_1 = 0$, and so, by Proposition 4.1, $T_{S_1}(R) = 0$.

If $n > 1$, we will prove that Z_i is S_i-torsion. Since R/Z_i is S_i-torsionfree by 4.1, this suffices to prove the lemma.

Suppose that $i > 1$. Then, by induction, we can assume that $\bar{Z}_i$ is $\bar{S}_i$-torsion over $\bar{R} = R/W_1$. Now, Lemma 4.5(ii) together with Corollary 5.3(ii) show that W_1 is S_i-torsion. Thus, $E_R(R/S_i)$, being S_i-torsionfree, is an $\bar{R}$-module. However, $S_i \supseteq W_1$, by Lemma 2.4, and so $R/S_i \simeq \bar{R}/\bar{S}_i$. It follows that $E_R(R/S_i) \simeq E_{\bar{R}}(\bar{R}/\bar{S}_i)$.

Assume that Z_i is not S_i-torsion. Then, there is a nonzero homomorphism

$$f: Z_i \to E_R(R/S_i).$$

But since W_1 is S_i-torsion, so too is $W_1 \cap Z_i$. Further, $W_1 \cap Z_i \subseteq \ker f$ because $R/\ker f$ is S_i-torsionfree. Thus, f induces a nonzero homomorphism $Z_i/W_1 \cap Z_i \to E_R(R/S_i)$; that is, a nonzero homomorphism $\bar{Z}_i \to E_{\bar{R}}(\bar{R}/\bar{S}_i)$. This contradiction shows that Z_i is S_i-torsion.

Finally, suppose that $i = 1$. Then, since every composition series prime of R/W_1 has codimension $> \alpha_1$, none can be contained in a minimal prime over S_1, by 5.3(ii). This implies that R/W_1 is S_1-torsion by 4.5. Thus, to finish the proof, we need only show that $W_1 \cap Z_1 = 0$; for then, Z_1 is isomorphic to a submodule of R/W_1.

We must show first that R/W_1 is an order in $\Lambda/W_1\Lambda$. To do this, we use [5, Lemma 4(i)] to note that W_1 has left Krull dimension α_1, and is the first term in the ideal sequence of R when R is viewed as a left R-module. Thus, R/W_1 is torsionfree on both sides, by 2.6, and so an order in $\Lambda/W_1\Lambda$, by the proof of 5.1.

Now, if $W_1 \cap Z_1 \neq 0$, then Z_1 contains an α_1-Macaulay left ideal of R, L say, such that $W_1L = 0$. Let B be the left annihilator of L. Then, R/B is an order in $\Lambda/\Lambda B$ by 5.1. Further, the canonical commutative square

$$\begin{array}{ccc} \Lambda/\Lambda W_1 & \rightarrow & \Lambda/\Lambda B \\ \uparrow & & \uparrow \\ R/W_1 & \longrightarrow & R/B, \end{array}$$

induced by the inclusion $W_1 \subseteq B$, shows that every regular element of R/W_1 is a regular element of R/B. But L is a torsionfree R/B-module since it is faithful over R/B and torsionfree over R. Thus, L is a torsionfree left R/W_1-module. By the argument of the preceding paragraph, this contradicts 5.3(i). ▌

THEOREM 5.10. *The following are equivalent properties of a finitely generated R-module M.*

(i) *M is a torsionfree R/Z_i-module.*

(ii) *$MZ_i = 0$ and M is torsionfree.*

(iii) *M is α_i-Macaulay and torsionfree.*

(iv) *M is S_i-torsionfree, where*

$$S_i = \bigcap\{P \in \operatorname{comp} R \mid K \dim R/P = \alpha_i\}.$$

Proof. (ii) $\Rightarrow$ (iii). Let N be a factor in the submodule sequence of M. Then, the argument at the end of the proof of the preceding lemma shows that N is torsionfree over R/Z_i. Thus, since R/Z_i is α_i-Macaulay, so too is N. This implies that M is α_i-Macaulay.

(iii) $\Rightarrow$ (iv) By 5.2, $\operatorname{comp} M \subseteq \operatorname{comp} R$. This ensures that every associated prime of M is a composition series prime of R of codimension α_i. Thus, (iv) follows from Lemma 4.5.

(iv) $\Rightarrow$ (i) This is immediate from 5.9, using Proposition 4.1.

(i) $\Rightarrow$ (ii) This follows from 5.1. ▌

COROLLARY 5.11. *A finitely generated module is torsionfree if and only each factor in its submodule sequence is annihilated by an associated annihilator ideal of codimension equaling the Krull dimension of the factor.*

COROLLARY 5.12. *The torsionfree strongly indecomposable R/Z_i-modules are precisely the torsionfree α_i-indecomposable R-modules.*

Using Theorem 3.7, we get

COROLLARY 5.13. *If R is finitely generated over its center then the following statements are equivalent.*

(i) *R has finite associated representation type.*

(ii) *Λ has finite representation type.*

(iii) *R has only finitely many nonsubisomorphic torsionfree strongly indecomposable modules.*

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References

1. M. Auslander, Representation theory of Artin algebras. IV, *Comm. Algebra*, to appear.
2. D. Eisenbud, Subrings of artinian and noetherian rings, *Math. Ann.* **185** (1970), 247–249.
3. R. Gordon, Gabriel and Krull dimension, *in* "Ring Theory," (B. R. MacDonald, A. R. Magid, K. C. Smith, Eds.), pp. 241–295. Lecture Notes in Pure and Applied Math. 7, Marcel Dekker, New York, 1974.
4. R. Gordon, Nilpotency in endomorphism rings, *Proc. Amer. Math. Soc.* **45** (1974), 38–40.
5. R. Gordon, Artinian quotient rings of FBN rings, *J. Algebra* **35** (1975), 304–307.
6. R. Gordon, Primary decomposition in right noetherian rings, *Comm. Algebra,* **2** (1974), 491–524.
7. R. Gordon, Polynomial modules over Macaulay modules, *Bull. Canad. Math. Soc.*, to appear.
8. R. Gordon and J. C. Robson, Krull dimension, *Mem. Amer. Math. Soc.* **133** (1973).
9. A. V. Jategaonkar, The torsion theory at a semi-prime ideal, *A. Acad. Brasil. Cienc.* **45** (1973), 197–200.
10. A. V. Jategaonkar, Jacobson's conjecture and modules over fully bounded noetherian rings, *J. Algebra* **30** (1974), 103–121.
11. A. V. Jategaonkar, Principal ideal theorem for noetherian P.I. rings, *J. Algebra* **35** (1975), 17–22.
12. J. Lambek and G. Michler, Localization of right noetherian rings at semiprime ideals, *Canad. J. Math.* **26** (1974), 1069–1085.
13. A. V. Roiter, Unboundedness of the dimensions of the indecomposable representations of an algebra which has infinitely many indecomposable representations, *Izv. Akad. SSSR, Ser. Mat.* **32** (1968), 1275–1282.