Convective stability analysis of a micropolar fluid layer by variational method

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Abstract This paper studies Rayleigh-Bénard convection of micropolar fluid layer heated from below with realistic boundary conditions. A specific approach for stability analysis of a convective problem based on variational principle is applied to characterize the Rayleigh number for quite general nature of bounding surfaces. The analysis consists of replacing the set of field equations by a variational principle and the expressions for Rayleigh number are then obtained by using trial function satisfying the essential boundary conditions. Further, the values of the Rayleigh number for particular cases of large and small values of the microrotation coefficient have been obtained. The effects of wave number and micropolar parameter on the Rayleigh numbers for onset of stationary instability for each possible combination of the bounding surfaces are discussed and illustrated graphically. The present analysis establishes that the nature of bounding surfaces combination and microrotation have significant effect on the onset of convection. © 2011 The Chinese Society of Theoretical and Applied Mechanics. [doi:10.1063/2.1104204]

Keywords critical Rayleigh number, Rayleigh-Bénard convection, micropolar fluid, eigenvalue problem, variational principle

The onset of convective instability in a thin layer of fluid heated from below has been extensively studied by many researchers. Rayleigh-Bénard convection originated from the experimental works of Bénard and theoretical analysis of Rayleigh has also received a considerable importance due to its relevance in various fields such as chemical and industrial engineering, soil mechanics, geophysics and elsewhere. The main objectives of the various studies related to the convective instabilities, in particular, is to determine the critical Rayleigh number at which the onset of instability sets in either as stationary convection or through oscillations. In most of the cases attention is focused on the determination of these stability criteria for the ideal case of both dynamically free boundaries, since in this case the exact solution of the problem can be obtained in closed form leading to a dispersion relation. However, a limited effort has been put to investigate the instability in the cases of physically realistic boundary conditions (i.e. when both boundaries are rigid or combinations of a rigid boundary and a free boundary), since then an exact solution in closed form is not obtainable for these cases of boundary combinations. Chandrasekhar, Drazin and Reid and other authors therefore used the numerical evaluation methods to obtain the approximate values of Rayleigh numbers for these (physically realistic) cases of boundary combinations.

The Rayleigh-Bénard convection in Eringen’s micropolar fluids heated from below has been thoroughly investigated (c.f. Ahmadi, Datta and Sastry, Bhattacharyya and Jena, Sharma and Gupta, Dragomirescu and references therein) due to their applications in engineering and technology, geophysics and in industrial processes. The common characteristics of all these investigations are that the stationary convection is the preferred mode of instability and the microrotation (spin) has a stabilizing effect on the onset of Rayleigh-Bénard convection. Extensive reviews on the micropolar fluid theory and its applications can be found in the recent books by Lukaszewicz and Eringen.

In the present analysis, we have studied the influence of the microrotations of the fluid particles and the natures of the bounding surfaces on the onset of Rayleigh-Bénard convection. Firstly, the principle of exchange of stabilities (PES) is investigated by using the celebrated Pellew and Southwell technique. A specific approach for stability analysis based on variational principles to characterize the “most unstable” solution of the eigenvalue problem is applied. The consists of replacing the set of field equations by a variational principle and the expressions for Rayleigh number are then obtained by using trial function satisfying the essential boundary conditions. Further, the values of the Rayleigh number for particular cases of large and small values of the microrotation coefficient have been obtained. The effects of wave number and micropolar parameter on the Rayleigh numbers for onset of stationary instability for each possible case of boundary combinations are discussed and illustrated graphically.

Following the usual steps of linear stability theory, it is easily seen that the non-dimensional linearized perturbation equations governing the problem of the thermal stability of micropolar fluid layer heated from below with time dependence of the form exp(\(pt\)) \((p = p_r + ip_i)\) (with minor notational changes from Ref. 7) are given by

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(1 + K)(D^2 - a^2) \left[ D^2 - a^2 - \frac{p}{\sigma(1 + K)} \right] w
= -K (D^2 - a^2) G + Ra^2 \theta, \quad (1)

(D^2 - a^2 - p) \theta = -w; \quad (2)

\left[ c_0(D^2 - a^2) - \frac{n_1p + 2K\sigma}{\sigma} \right] G
= K (D^2 - a^2) w, \quad (3)

together with one of the following cases of the boundary conditions:

Case 1. both dynamically free boundaries

w = 0 = \theta = G = D^2 w \text{ at } z = 0 \text{ and } z = 1; \quad (4a)

Case 2. both rigid boundaries

w = 0 = \theta = G = Dw \text{ at } z = 0 \text{ and } z = 1; \quad (4b)

Case 3. lower free and upper rigid boundaries

w = 0 = \theta = G = D^2 w \text{ at } z = 0, \text{ and } w = 0 = \theta = G = Dw \text{ at } z = 1, \text{ or lower rigid and upper free } w = 0 = \theta = G = D^2 w \text{ at } z = 1, \text{ and } w = 0 = \theta = G = Dw \text{ at } z = 0. \quad (4c)

In the foregoing equations, \( D \equiv d/dz \) is the differentiation with respect to \( z \), \( z \) is the real independent variable; \( a^2 \) is the square of the wave number, \( \sigma \) is the thermal Prandtl number; \( R = ga\beta a^4/\kappa' \) is the Rayleigh number, where \( g \) is the gravitational acceleration, \( \alpha \) is the coefficient of thermal expansion, \( \beta \) is the adverse temperature gradient, \( d \) is the depth of the layer, \( \kappa' \) is the thermometric conductivity and \( \nu \) is the kinematic viscosity; \( K = \kappa/\mu \), \( n_1 = J/d^2 \) and \( c_0 = \gamma/\mu d^2 \) are positive constants; \( \mu \) is the dynamic Newtonian viscosity; \( \kappa \) is the dynamic microrotation viscosity; \( J \) is the microinertia; \( \gamma \) is a constant and stands for coefficient of angular viscosity; \( p = (p_r + i p_i) \) is the complex growth rate, \( w, \theta \) and \( G \) are the perturbations in the vertical velocity, temperature and microrotation respectively.

The system of Eqs. (1)–(3) together with either case of boundary conditions (4a)–(4c) constitutes an eigenvalue problem for \( R \) for the given values of other parameters \( \sigma, c_0, n_1, a^2 \) and \( K \), and a given state of the system is stable, neutral or unstable as \( p_r \) is negative, zero or positive, respectively. Further, if \( p_r \geq 0 \Rightarrow p_i = 0 \forall a^2 \), then for neutral stability \( (p_r = 0) \), we have \( p = 0 \). This is called as PES. The validity of this principle in convective instability problems leads to notable mathematical simplifications since the transition from stability to instability occurs via a marginal stationary state characterized by \( p = 0 \). Mathematically, this means that the marginally stable modes with \( p_r = 0 \) also have \( p_i = 0 \). Moreover, by setting \( p = 0 \), the original evolution problem for the perturbations reduces to an eigenvalue problem for the Rayleigh number or any other relevant parameter of interest.

Multiplying Eq. (1) by \( w^* \) (the complex conjugate of \( w \)) and integrating the resulting equation by parts a suitable number of times over the vertical range of \( z \), namely \( 0 \leq z \leq 1 \), using the relevant boundary condition (4) on \( w \), we obtain

\[ \sigma(1 + K) \int_0^1 \left( |D^2w|^2 + a^4 |w|^2 + 2a^2 |Dw|^2 \right) dz + \]

\[ 2a^2 |Dw|^2 dz + p \int_0^1 \left( |Dw|^2 + a^2 |w|^2 \right) dz = Ra^2 \sigma \int_0^1 w^* (D^2 - a^2) G dz \]

(5)

We shall now evaluate each of the integral on the right hand side of Eq. (5) separately. For this, taking the complex conjugate of Eq. (3), multiplying the resulting equation by \( G \) on both sides and then integrating it by parts a suitable number of times over the range of \( z \) and using the relevant boundary conditions from (4), we get

\[ \int_0^1 \left( |DG|^2 + a^2 |G|^2 + \frac{n_1p^* + 2K\sigma}{\sigma c_0} |G|^2 \right) dz = -K \int_0^1 G (D^2 - a^2) w^* dz. \]

(6)

Also, taking the complex conjugate of Eq. (2), multiplying the resulting equation by \( \theta \) on both sides and then integrating it by parts a suitable number of times over the range of \( z \) and using the relevant boundary conditions from (4a)–(4c), we have

\[ \int_0^1 \left( |D\theta|^2 + a^2 |\theta|^2 + p^* |\theta|^2 \right) dz = \int_0^1 w^* \theta dz. \]

(7)

Making use of Eqs. (6) and (7) in the Eq. (5), we obtain

\[ \sigma(1 + K) \int_0^1 \left( |D^2w|^2 + a^4 |w|^2 + 2a^2 |Dw|^2 \right) dz + \]

\[ p \int_0^1 \left( |Dw|^2 + a^2 |w|^2 \right) dz = Kc_0 \sigma \int_0^1 \left( |DG|^2 + a^2 |G|^2 + \frac{n_1p^* + 2K\sigma}{\sigma c_0} |G|^2 \right) dz + \]

\[ Ra^2 \sigma \int_0^1 \left( |D\theta|^2 + a^2 |\theta|^2 + p^* |\theta|^2 \right) dz \]

Now, equating the imaginary parts of the above equation, we get

\[ p_i \int_0^1 \left( |Dw|^2 + a^2 |w|^2 + K n_1 |G|^2 + \right. \]

\[ Ra^2 \sigma |\theta|^2 \right) dz = 0. \]

(8)

Keeping in view that \( K, \sigma, n_1 \) and \( R \) are positive, we find that the above equation clearly implies that \( p_i \) must
vanish, i.e. \( p_i = 0 \), irrespective of the nature of \( p_r \) (i.e. whether \( p_r \) is negative, zero or positive).

This establishes the validity of PES for the thermal convection of micropolar fluid layer heated from below. This means that the onset of convection occurs via a marginal stationary state. Further, the above result is an extension of the result in the case of classical fluids (Bénard convection) for micropolar fluids.

Since, PES is shown to be valid for the problem under consideration, therefore setting \( p = 0 \) in Eqs. (1)–(3), we have

\[
(1 + K)(D^2 - a^2)^2 w = -K(D^2 - a^2) G + Ra^2 \theta, \tag{9}
\]

\[
(D^2 - a^2) \theta = -w, \tag{10}
\]

\[
\left[c_0(D^2 - a^2) - 2K\right] G = K(D^2 - a^2) w. \tag{11}
\]

Now, consider

\[
F = (D^2 - a^2)^2 w + \frac{K}{1 + K}(D^2 - a^2) G. \tag{12}
\]

The boundary conditions on \( \theta = 0 \) as given in (4a)-(4c) when utilized in Eqs. (9) and (12) yield the boundary conditions

\[
F = 0 \text{ at } z = 0 \text{ and } z = 1. \tag{13}
\]

In order to formulate the above equations in terms of a variational problem, we first operate Eq. (9) by \((D^2 - a^2)\) and then using Eq. (10) in the resulting equation, we obtain

\[
(D^2 - a^2) F = -\frac{Ra^2}{1 + K} w. \tag{14}
\]

Now, multiplying Eq. (14) by \( F \), using Eq. (12) and then integrating by parts the left hand side of the resulting equation over the range of \( z \) and using boundary conditions (13), we get

\[
(1 + K) \int_0^1 [(DF)^2 + a^2 F^2] \, dz = Ra^2 \left[ \int_0^1 w (D^2 - a^2)^2 w \, dz + K \int_0^1 w(D^2 - a^2) G \, dz \right]. \tag{15}
\]

Integrating the right hand side of Eq. (15) by parts twice and making use of Eq. (3), we have the following expression

\[
R = \frac{(K + 1) \int_0^1 [(DF)^2 + a^2 F^2] \, dz - c_0 K \int_0^1 (DG)^2 + (a^2 + \frac{2K}{c_0}) G^2 \, dz}{a^2 \left[ \int_0^1 (D^2 - a^2)^2 w \, dz - \int_0^1 \frac{c_0}{1 + K} \int_0^1 (DG)^2 + (a^2 + \frac{2K}{c_0}) G^2 \, dz \right]},
\]

\[
= \frac{(K + 1) I_1}{a^2 I_2} \text{ (say).} \tag{16}
\]

The above expression for \( R \) (the Rayleigh number), which is the ratio of two positive definite integrals, is the required functional for the variational treatment of the problem.

Following the variational method of Chandrasekhar\(^3\) for thermal convection problem and proceeding analogously, we can easily prove the stationary property of the functional \( R \) given by expression (16) for the boundary conditions (4a)-(4c) & (13) when the quantities on right hand side are evaluated in terms of true characteristic functions. Also the quantity on the right hand side of Eq. (16) attains its true minimum when \( F \) belongs to \( R_c \), i.e. the lowest characteristic value of \( R \), namely \( R_c \), is indeed a true minimum, i.e.

\[
R_c \leq R = \frac{(1 + K) \int_0^1 [(DF)^2 + a^2 F^2] \, dz}{a^2 \int_0^1 (wF) \, dz}. \tag{17}
\]

Further, it is remarkable to note that the above result is uniformly valid for all cases of boundary conditions (4a)-(4c). This establishes the variational principle for the thermal convection of micropolar fluid layer heated from below.

Now, we shall evaluate the integrals \( I_1 \) and \( I_2 \) by using the trial functions satisfying the given boundary conditions.

Let us consider a trail function \( F = \cos \pi z \), which obviously satisfies the boundary conditions

\[
F = 0 \text{ at } z = -1/2 \text{ and } z = +1/2,
\]

where the origin has been shifted to midway for convenience in computation. Therefore, we have

\[
I_1 = \int_{-1/2}^{1/2} [(DF)^2 + a^2 F^2] \, dz = \frac{\pi^2 + a^2}{2}. \tag{18}
\]

In order to evaluate \( I_2 \), operating on Eq. (12) by \( [D^2 - a^2 - (2K/c_0)] \) and using Eq. (11) in the resulting equation, we get

\[
(D^2 - a^2) \left[ \left( D^2 - a^2 - \frac{2K}{c_0} \right) + \frac{K^2}{c_0(1 + K)} \right] w = \left( D^2 - a^2 - \frac{2K}{c_0} \right) F. \tag{19}
\]

Now, using the above defined value of \( F(= \cos \pi z) \) in Eq. (19), we obtain

\[
(D^2 - a^2) \left[ \left( D^2 - a^2 - \frac{2K}{c_0} \right) + \frac{K^2}{c_0(1 + K)} \right] w = - \left( \pi^2 + a^2 + \frac{2K}{c_0} \right) \cos \pi z, \tag{20}
\]

whose general solution is given by

\[
w = (B_1 + B_2) \cosh x_1 z + B_3 \cosh x_3 z + A \cos \pi z, \tag{21}
\]
where
\[
A = \frac{\pi^2 + a^2 + 2K}{c_0 (\pi^2 + a^2 + K^2 + 2K)},
\]
and \(B_1, B_2, B_3\) are arbitrary constants. Further, \(x_1^2, x_2^2, x_3^2\) are the roots of the auxiliary equation of Eq. (20) and are given by \(x_1^2 = x_2^2 = a^2\) and \(x_3^2 = a^2 + (K^2 + 2K)/(c_0 (1 + K))\). Now, applying the boundary conditions on \(w\) as given by (4), we get

\[
B_1 + B_2 = \begin{cases} 
0, & \text{for Case 1 of (4)}, \\
-\frac{\pi A}{M_1 \cosh(x_1/2)}, & \text{for Case 2 of (4)}, \\
\frac{\pi A x_3}{x_1 M \cosh(x_1/2)}, & \text{for Case 3 of (4)}, 
\end{cases}
\]
and

\[
B_3 = \begin{cases} 
0, & \text{for Case 1 of (4)}, \\
-\frac{\pi A x_3}{\pi x_1 A}, & \text{for Case 2 of (4)}, \\
-\frac{\pi A}{M x_3 \cosh(x_3/2)}, & \text{for Case 3 of (4)}, 
\end{cases}
\]
where, \(M = x_3 \tanh(x_1/2) - x_1 \tanh(x_3/2)\) and \(M_1 = x_3 \tanh(x_3/2) - x_1 \tanh(x_1/2)\). Hence, using the values of constants \(B_1 + B_2\) and \(B_3\) in Eq. (21), we have

\[
I_2 = \int_{-1/2}^{1/2} wF \, dz = \frac{A}{2} [1 + L],
\]
where,

\[
L = \begin{cases} 
0, & \text{for Case 1 of (4)}, \\
\frac{4\pi^2}{(\pi^2 + x_1^2) M_1} - \frac{4\pi^2}{(\pi^2 + x_1^2) M_1}, & \text{for Case 2}, \\
\frac{4\pi^2 x_3}{x_1 M (\pi^2 + x_1^2)} - \frac{4\pi^2 x_3}{x_1 M (\pi^2 + x_3^2)}, & \text{for Case 3}, 
\end{cases}
\]

Now, using the values of integrals \(I_1\) and \(I_2\) respectively from Eqs. (18) and (22) in inequality (17), we get

\[
R_c \leq R = \frac{(\pi^2 + a^2)^3}{a^2 (1 + L)} \left[ 1 + \frac{K}{c_0 (\pi^2 + a^2) + 2K} \right],
\]
\(\forall a^2\) \hspace{1cm} (24)
Substituting the different values of \(L\) from Eq. (23) into Eq. (24), we can have the expressions for Rayleigh numbers for each case of boundary conditions, respectively.

Taking \(K\) to be large (i.e. \(K \to \infty\)), we can reduce the values in \(L\) for different boundary conditions to

\[
L = \begin{cases} 
0, & \text{for Case 1 of (4)}, \\
0, & \text{for Case 2 of (4)}, \\
\frac{4\pi^2}{a^2 (\pi^2 + a^2) \tanh(a^2/2)} > 0, & \text{for Case 3 of (4)}, 
\end{cases}
\]
Hence, for large \(K\), inequality (24) yields the following bounds for \(R_c\):
(a) for Case 1 and Case 2 of boundary conditions

\[
R_c \leq \frac{(\pi^2 + a^2)^3}{a^2} \left( 1 + \frac{K}{2} \right); \hspace{1cm} (25)
\]
(b) for Case 3 of boundary conditions

\[
R_c \leq \frac{1}{a^2} \left( \frac{a^2}{a^2} + 2 \right) \left( 1 + \frac{K}{2} \right), \hspace{1cm} (26)
\]
in view of the earlier defined positivity of the term in the denominator of the expression this yields

\[
R_c \leq \frac{(\pi^2 + a^2)^3}{a^2} \left( 1 + \frac{K}{2} \right), \hspace{1cm} (27)
\]
for all \(a^2\), for a particular value of \(a^2 = \pi^2/2\) at which the minimum of the above expression exists, we have

\[
R_c \leq \frac{27\pi^4}{4} \left( 1 + \frac{K}{2} \right). \hspace{1cm} (28)
\]

For small values of \(K\) (i.e. \(K \to 0\), a limiting case of classical Bénard convection as the microrotation is negligibly small), the roots of the auxiliary equation of Eq. (20), namely, \(x_1^2, x_2^2\) and \(x_3^2\) become equal and consequently \(L = 0\) for all combinations of boundary conditions and hence inequality (24) together with the minimum value of \((\pi^2 + a^2)^3/a^2\) yield

\[
R_c \leq \frac{27\pi^4}{4}. \hspace{1cm} (29)
\]

It can be easily seen from Ref. 7 that, when both the bounding surfaces are taken to be dynamically free, the value of Rayleigh number for lowest mode is given by

\[
R = \frac{(\pi^2 + a^2)^3}{a^2} \left[ 1 + \frac{c_0 (\pi^2 + a^2) K + K^2}{c_0 (\pi^2 + a^2) + 2K} \right]. \hspace{1cm} (28)
\]

Equation (28) for large value of \(K\) clearly leads to the following inequality

\[
R \geq \frac{(\pi^2 + a^2)^3}{a^2} \left( 1 + \frac{K}{2} \right), \hspace{1cm} \forall a^2. \hspace{1cm} (29)
\]

Combining inequalities (26) and (29), we get the exact value of \(R\), for large values of \(K\), as

\[
R = \frac{(\pi^2 + a^2)^3}{a^2} \left( 1 + \frac{K}{2} \right), \hspace{1cm} \forall a^2. \hspace{1cm} (30)
\]
Since, the minimum value of \((\pi^2 + a^2)^3/a^2\) with respect to \(a^2\) is \(27\pi^4/4\), therefore for large values of \(K\),
for Case 1 of boundary conditions, Eq. (30) yields the following exact value of $R$ as

$$R_c = \left(1 + \frac{K}{2}\right) \left(\frac{27\pi^4}{4}\right),$$  \hspace{1cm} (31)

a result as obtained by Ahmadi\textsuperscript{7} also for Case 1 of boundary conditions.

In the present analysis, PES is shown to be valid by using the Pellew and Southwell technique, which is uniformly valid for all possible cases of boundary combinations. The validity of this principle thus rules out the possibility of oscillatory motions of growing amplitude i.e. for the problem the onset of instability is through stationary convection only. The present work extends the analysis of Refs. 3 and 4 of variational formulation to a more general problem, namely, Rayleigh-Bénard convection in micropolar fluids.

The values of $R$ obtained by the method of variational principle are computed numerically by using the trial functions satisfying the essential boundary conditions. The upper bounds of $R_c$ for large and small values of $K$ have been derived which are uniformly valid for all combinations of boundaries. Consequently, for the case of both dynamically free boundaries, an exact value of the critical Rayleigh number for large $K$ has been obtained as $R_c = (1 + K/2) \left(\frac{27\pi^4}{4}\right)$. It is clear from the expression that the microrotation has stabilizing effect on the onset of stationary convection, a result also obtained by Ref. 7. Further, for small values of $K$ (i.e. $K \to 0$), inequalities (27) and (28) clearly imply that for the case of both dynamically free boundaries $R_c = \frac{27\pi^4}{4}$, the same value of $R_c$ as obtained by Ref. 3 for classical Bénard problem.

The effect of $K$ (the microrotation coefficient) on the Rayleigh numbers (given in expression (24)) for each case of boundary combination for positive range of values of $a^2$ (the square of wave number) and for fixed value of $c_0 (= 0.001)$ is depicted in Figs. 1–3.

It is clear from Figs. 1 and 2 that the value of the Rayleigh number firstly decreases to attain a critical value corresponding to certain wave number and then increases. In other words, the behaviour of $R$ with respect to $a^2$ for micropolar fluid layer heated from below is similar to that for Rayleigh-Bénard convection. However, Fig. 3 reveals that for the case of one rigid one free boundary combination, the Rayleigh number takes very small values corresponding to small value of wave numbers and increases for increasing wave number in comparison with other combinations of similar boundary conditions. Hence, Case 3 of boundary combination has destabilizing effect. In other words, the microrotation character of the fluid particles hastens the onset of instability in this case of combination of dissimilar boundaries, a physical explanation of which is very much desired.

The above analysis thus clearly and unequivocally establishes that the nature of bounding surfaces affects the onset of Rayleigh–Bénard convection in micropolar fluids.

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